Theory of Elasticity
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Lecture - 46
Complex Variable Method (Contd.)

Hello everyone, this is the fourth lecture of this week, in the first 3 classes what we did is we briefly revisited some of the intuited concept of complex variable which we require to formulate the problems in elasticity, that formulation that exercise we will do today. So, today's topic is the Complex Variable Formulation of Elasticity Problems. You recall when we discussed the formulation of boundary value problems in elasticity at that time we had seen there are 2 kinds of formulation, one is the stress based formulation. Whether your primary variables are stresses, you have to determine stresses from the equations. And there is another formulation displacement based formulation where the equations are written in terms of displacement.

We will do the same exercise we will write those equations both the formulation. But, in that time those equations were written in terms of x y or in polar coordinate say r theta, now the equations will be written in terms of z and z bar. These are the 2 variables that we used in complex in complex method. Now, let us start with the stress based formulation, if you recall if we combine the governing equations and then the compatibility equations and then write define a potential a stress function phi which is called airy stress function as this then the final equations we can have this.

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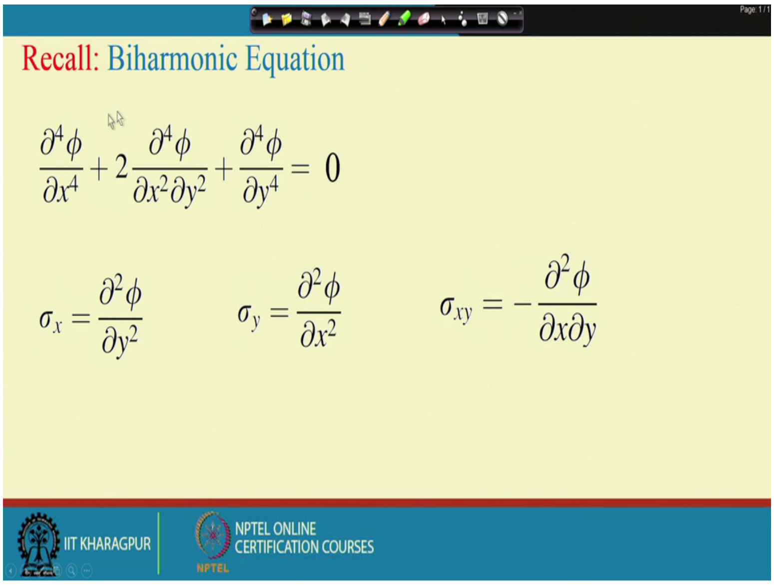

Biharmonic equation, if there is no body force, if we ignore the body force neglect the body force then this equation the form of these equations as equation is same for both plane stress and plane strain problem. So, let us now write this expression this is the Biharmonic equation, then let us now write these same Biharmonic equation in terms of in terms of complex variables.

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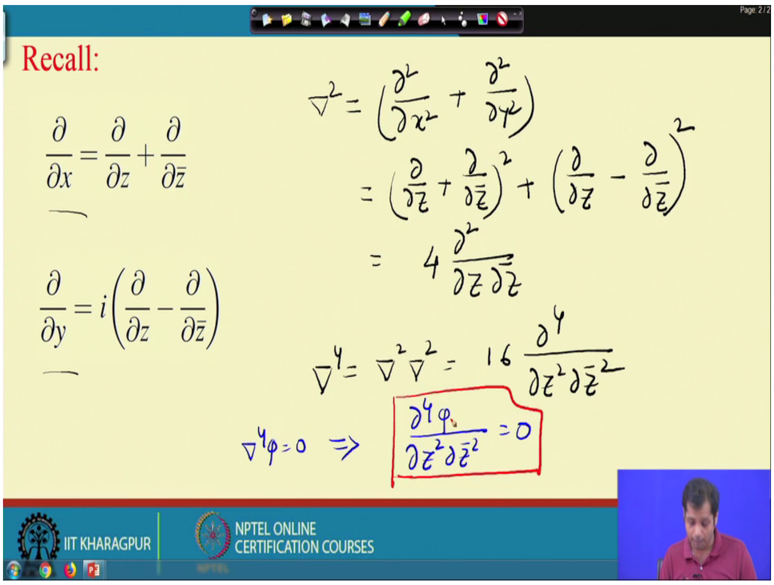

Now, you see let us the Biharmon equation is essentially can be written as like this right. So this equation is essentially, this equation is essentially del 4 phi is equal to 0 and del 4 phi is essentially your del 2 of del 2 by del 2 is the Laplacian operator right. Now, so, you recall these are the things the relation between del del x and del del z and del del z bar we derive this relation in one of the classes in this week. Now, what is the let us find out del 2 operator. Del 2 operator in del 2 operator is del 2 del x 2 plus del 2 del y 2 right.

Now, let us write this del 2 operator in terms of z and z bar using this relation. Now del x so, what it is? It is essentially if we substitute del del x from this expression, we have del del z plus del del z bar and then square plus del del z. You recall we discuss that though if we know z z bar z bar is essentially conjugate of z, but here we create z and z bar as independent variable. Equations are written in terms of z and z bar.

So, now this is square so, this if we combine this, this eventually becomes 4 del 2 del z del z bar. So, this is the Laplacian operator. So, del 4 operator, del 4 operator then del 2 of del 2 del 2 del 2 and this essentially becomes the square of this. So, 16 del 4 del z 2 del z bar 2 ok. Now, the Biharmonic equation was del 4 phi is equal to 0 del 4 phi is equal to 0. So, this gives if I write now this del 4 operator in terms of z and z bar. So, these equation becomes del 4 phi del z 2 del z bar 2 is equal to 0. So, this is the Biharmonic equation, the equation that we had in case of in terms of x y and an r theta coordinate we already derive this is the equation; the same equation written in terms of z and z bar.

Now, the solution of this equation, then what we did in the case of in xy plane or in r theta plane? Within we try to found out the find out the solution of that Biharmonic equation, here also we do the same exercise. The solution of this equation the phi once we have the phi then the phi will be related to stresses and then stress. So, with the knowledge of phi we can determine stresses. Let us see how it is to be done in the case of complex variable approach.

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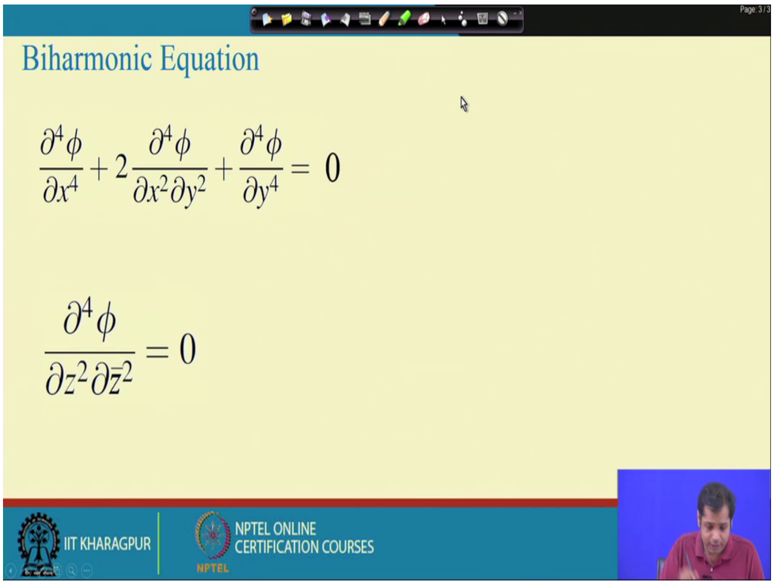

Now, so finally, equation we have is the this is the equation the Biharmonic equation the same equation now becomes this now.

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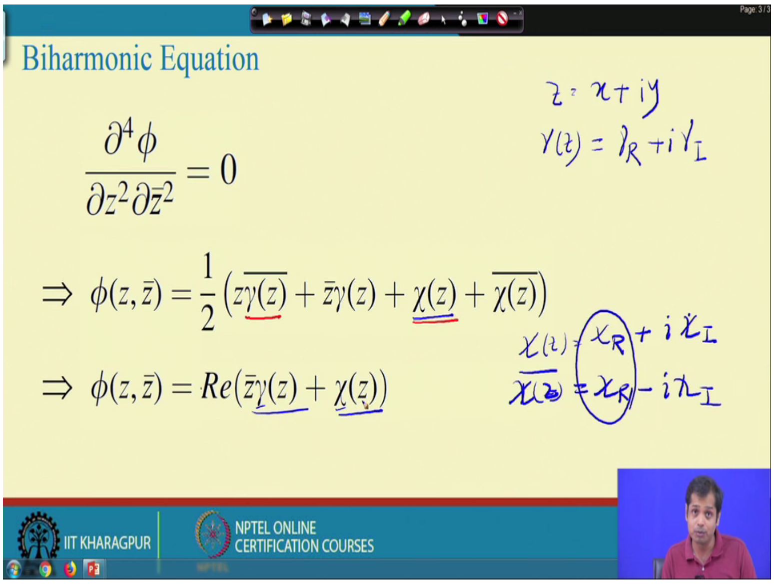

If we integrate this equation directly if we integrate this equation then that integrated after the integration the equation becomes the phi can be written in terms of 2 functions gamma 2 function gamma and chi ok. You can directly just integrate it, once first you integrate with respect to z and then integrate with respect to z bar and then you get this expression. So, that exercise that derivation I leave it to you that is also given in the book.

Now, so if we now you see z is what? Z is equal to x plus iy z is equal to x plus iy. If gamma is the gamma is the gamma is the function of z so, gamma is the function of z. So, gamma also can be written as gamma real plus i into gamma imaginary right. So, that is how the complex function is written real part and the imaginary part. So, if we assume if we say that if you if phi has to be real then there in this expression there will be some real part and imaginary part. If phi has to be real, then we can ignore the imaginary part and then the phi will be real of the entire thing, real part of the entire thing.

Now, so, you see the real part of the entire thing if you see gamma z gamma gamma z bar if chi chi z if chi z can be written as chi real plus plus i into chi imaginary. Similarly, chi of chi of z bar chi of z bar can return as chi real minus i of chi imaginary. So, if you add them so, only left with chi real part that is why it is a chi real part this is the real part. So, imaginary part goes and then you have essentially the real part say phi has to be real. So, this is for phi, we will we will discuss we will these 2 potential gamma and chi they play a very important role here. So, essentially what we have done here is unlike the unlike the previous cases where the phi was written in terms of x and y and r and theta it was just one function phi, but now what we have done here is the phi is written in terms of 2 functions gamma and chi.

So, essentially the problem is reduced to finding these 2 constant where the in earlier cases the problem was to find out only phi. Here the phi is a combination of chi and gamma we have to find out these 2 functions and their associated concern is to be determined based on the boundary condition. We will come to this point shortly. Now, once we have this so, this is for the stress formulation.

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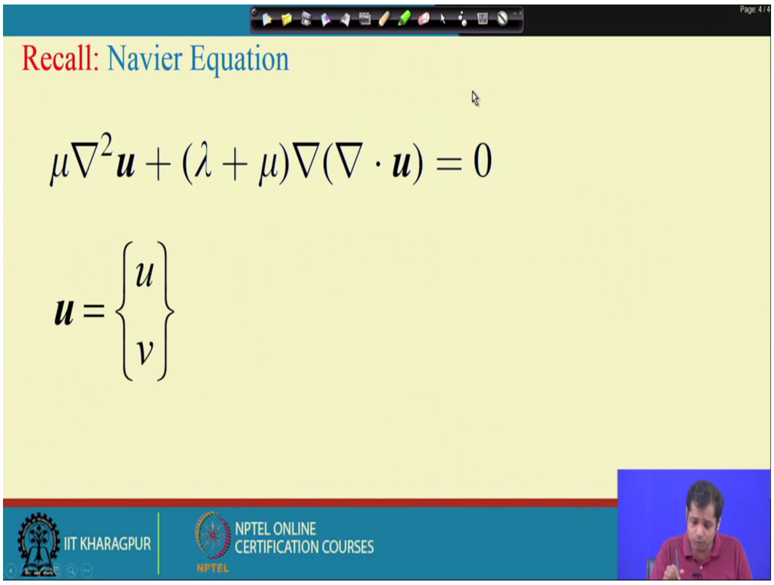

So, similarly if we have to do the similar exercise for displacement formulation if you recall this was the expression which is called Navier's equation that is the equation written in terms of displacement where u in this case u is a vector. So, u is a vector u is a vector which has 2 components u one v. Now, we have to write the same expression in terms of complex variable. And del operator was if you remember del operator was del del x and then del del y del del y right this was the operator.

Now, u let us define u complex displacement is equal to u plus i into v. So, u and y if you take u is in this direction, this is the direction of u, this is the direction of v. That was u and v defined u is the displacement in say x direction v is the displacement in y direction and then if we define a complex displacement which is which has to be defined on a complex plane in this case uv plane. So, this is U is equal to u plus i into v. So, for us equations to be written in terms of capital U not their component; now, so let us do that exercise.

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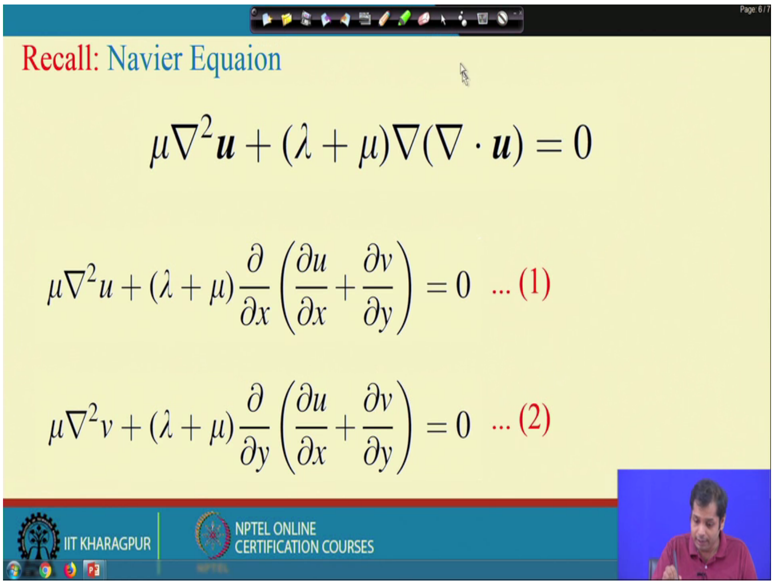

So, if we do that if you recall this expression this is the Navier's equation and then if I write this expression in x direction and again in y direction, this will be the this will be the relation right. So, since we are talking about plane problems we have 2 equations x and y direction. So, this is equation 1 and equation 2.

Now, let us take equation 1 equation 1 plus i into equation 2. The reason is in the previous case if you in the previous case if you see. Just now we wrote that u is equal to u plus iv, now we have to write this equation only in terms of capital U. So, these 2 equations which have in 2 directions that equation needs to be combined and get just one equation. Now, this is how you let us combine these 2 equation. So, write equation 1. So, equation 1 plus i into equation 2, let us see what happens. Now, if I write that.

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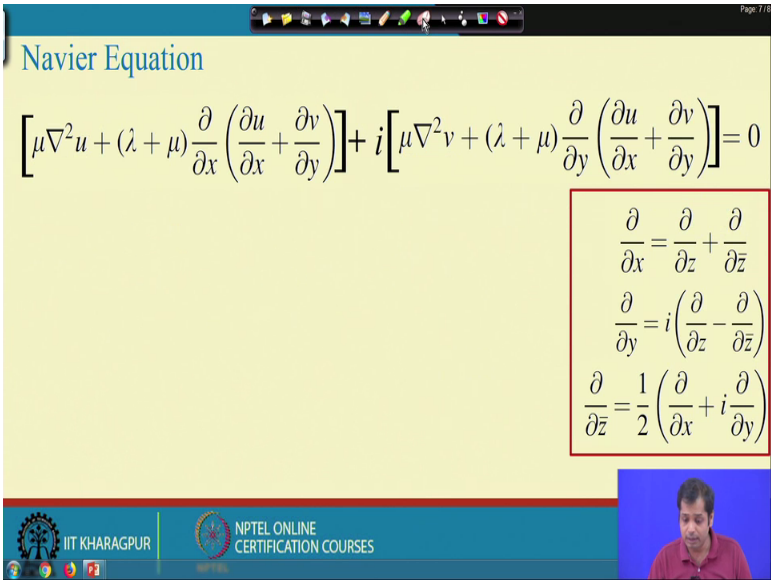

So, this is the equation 1 in the previous slide and then i into equation 2. Now, if you do that then what we have here is you see. So, let us consider first this term and this term, this term and this term. This becomes what? This becomes mu into del 2 del 2 of u plus i into v right. So, these 2 term becomes this, let us take the other 2 term. Other 2 term becomes lambda plus nu lambda plus mu and then del del x of del del x plus i into del del y and then del u del u del x plus del v del y that is equal to 0. So, this is we have right.

Now, what is del 2 operator? Del 2 operator just now we discuss that del 2 operator was what? Del 2 operator was 4 into del 2 by del z del z bar right. So, this becomes mu into 4 del 2 del z del z bar and u plus iv is u that is how u is defined plus. Now, then we have this term if you recall if you recall this expression where this the relation between del del x and del del y and del del z del del z bar this is the relation we had.

So, these expression del del x del del x plus i into del del y that gives you 2 into del del z bar. So, this becomes lambda plus mu 2 into and this becomes del del z bar. Now, similarly if you substitute del del x from this equation and del del y from this equation and combine them. if you combine them then you get an expression final expression like del u del z plus del u del z, this is the bar. So, when we say del u del z bar it means that del u bar del z bar it is to enter the this is the conjugate of the entire del u del z that is equal to 0.

So, this expression you get by combining Navier's equation in both the directions and then this is the final equation in written in terms of z and z bar. So, the final expression is finally.

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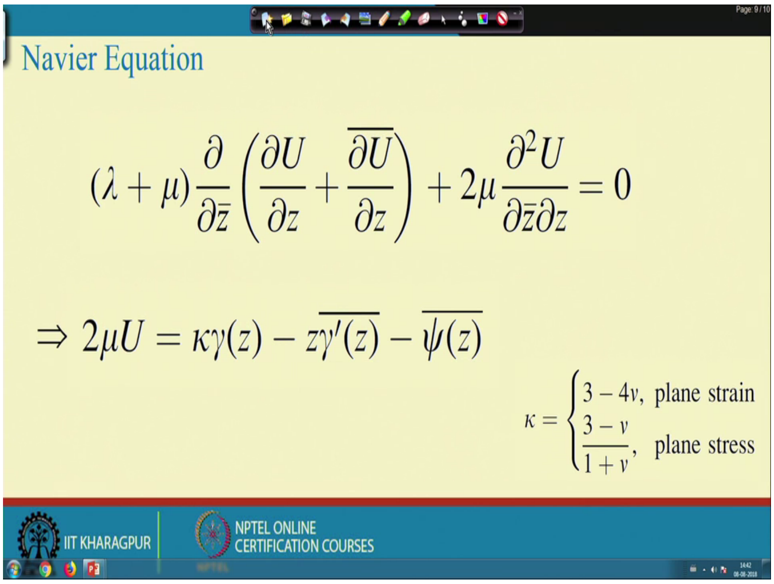

That this is the expression that you have just now we derive. Now, now similar to the previous case where the equations were written in terms of stress and we directly integrated it let us do the same thing for this. And if you do that exercise if you integrate that then the expression that you get here is this is the expression we get.

Now, you see the gamma z here this gamma z is the same gamma z that is it is not a separate function it is the same gamma z that we have in case of stress formulation. Now, then we have one more additional function which is psi z there it was chi z. But, this psi z and chi z are related to each other and this relation is ok. Before that there is a chi here there is this kappa here kappa is for play whether it is a plane stress problem or plane strain problem depending on that you have this kappa.

So, kappa these values for this for plane stress and this is for plane this is for plane stress and this is for plane stress problem. And then the relation between psi will shortly come psi and the chi that we discussed in the in the stress formulation. But, again the same thing the entire u is written in terms of 2 functions.

So, essentially the problem whether you use displacement formulation or stress formulation what we have seen this entire thing is a essentially reduced to finding these 2 functions gamma and psi. So, these 2 functions we have to assume and these functions may have certain constants and then this constant need to be determined based on the boundary conditions. Now, we will demonstrate that through few examples.

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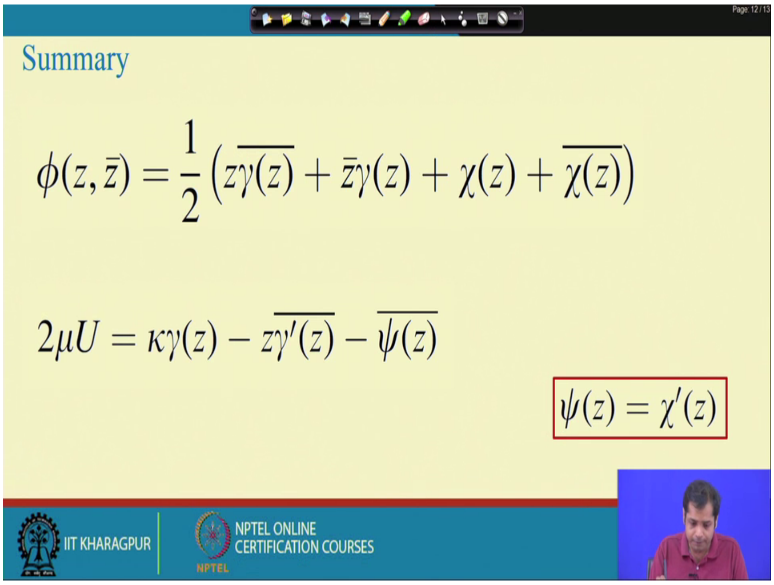

So, summary you see this is the summary if you recall this if we write the entire expression of phi not just the real part. This is the entire expression of phi and for stress formulation that this is the entire expression for u. And the psi and here the this gamma z and this gamma z this gamma z this gamma z they are same. Only thing is this chi and this psi are related to each other and that relation is this ok. So, once you do that divisions integrate it and you get that you can get this relation.

So, now then ok so, then what we have? So, these are the things we have right. Now, but still we are not in a position to before we apply these equations to solve problems. Let us let us let us take this equation further and then see what this equation tells you about the stresses and the displacements.

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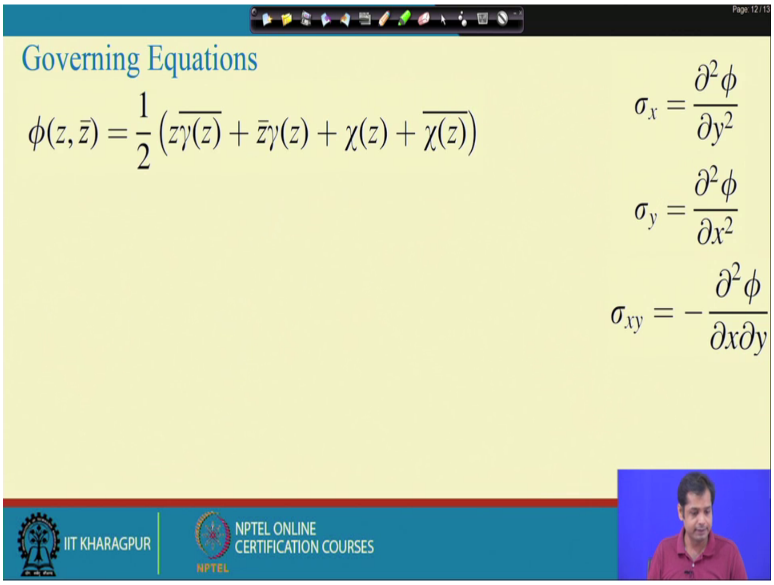

Now, you see let us take this governing equation this is the phi. You recall this was this was the relation of stress function with the corresponding stresses. Now, if I write sigma x plus sigma y sigma xx plus sigma yy ok. Sigma x plus plus sigma y mean this plus this essentially becomes del 2 del x 2 plus del 2 del y 2 of phi right. So, this is del 2 of phi.

Now, what it is essentially? The del 2 operator is what del 2 operator we have just now we have seen for complex variable del 2 operator is 4 into del 2 phi del z del z bar this is the del 2 operator. Now, phi expression of phi is given here and if you substitute that expression in this equation then what you get is sigma xx plus sigma yy that is equal to 2 into you can do this exercise 2 into gamma dash z plus gamma dash z is conjugate this is the expression.

So, this expression is expression number 1, you may be thinking that what is the use of this expression, why suddenly taking sigma xx plus sigma yy the, why it is done it will be clear shortly. Now, let us do this let us take one more combination that combination is say sigma yy sigma yy minus sigma xx plus 2 i sigma xy ok. Now, this is what sigma yy is this and sigma xx is this. So, essentially this become del 2 phi del 2 phi del x 2 minus del 2 phi del y 2 and then plus 2 into this. So, this becomes minus 2 i 2 i del 2 phi del x del y.

Now, we know the relation of del del x and del del y, where del del z and del del z bar with that relation. If we substitute if you write the entire expression in terms of z and z bar in terms of derivative with respect to z and z bar and then substitute the expression of phi and do the derivative, you get the final expression and that expression becomes.

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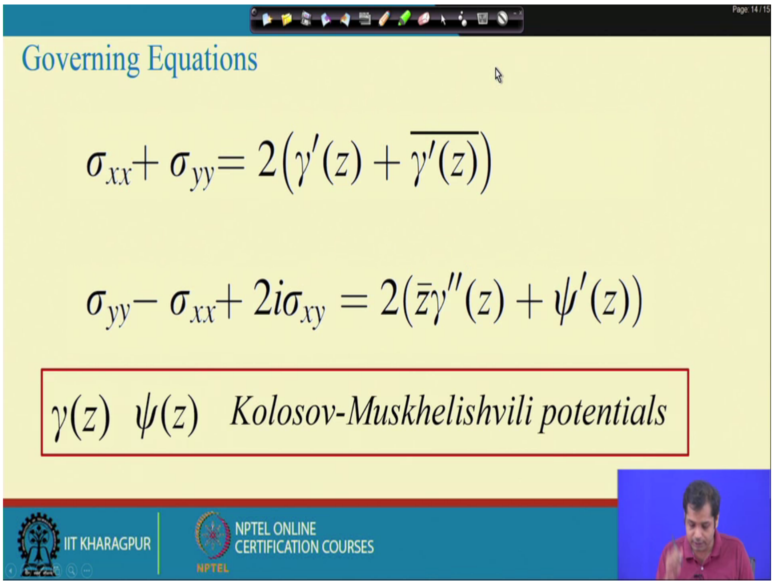

That expression become this, this is the first expression which we which we derived and this is the second expression for this. Now, so for the time being just for a moment so, this is equation number 1, this is equation number 2. For a moment we will application of this equation why suddenly in this form we write this equation it will be cleared shortly through an example. Now, if we do the same exercise for polar coordinate system then what will happen?

Now, if you recall in Cartesian coordinate system the stresses are related stresses are written as this right. Sigma xx then sigma xy sigma xy sigma yy; the sigma x y and sigma y x m that is why that because, of the symmetry the off diagonal terms are written same. Sigma r r sigma r theta and then sigma r theta sigma theta theta, that is written in terms of polar coordinate system.

Now, what is the trace of this if you recall when we discussed about invariant stress invariant the first invariant was the trace of the stress tensor. Trace of the stress tensor is sigma sigma xx plus sigma yy. In this case trace of the stress tensor is sigma rr plus sigma theta sigma theta theta.

Since this is i 1, this is i 1 the first invariant first stress invariance. So, therefore, sigma xx plus sigma yy equal to sigma rr plus sigma theta theta. So, this straight away we can write. So, if we have if you have to write this expression in terms of a polar coordinate system. So, this expression the first expression remains same, but only thing is only thing is that z is instead of z we need to write in terms of r and theta and if you do that exercise the expression that you get is ok.

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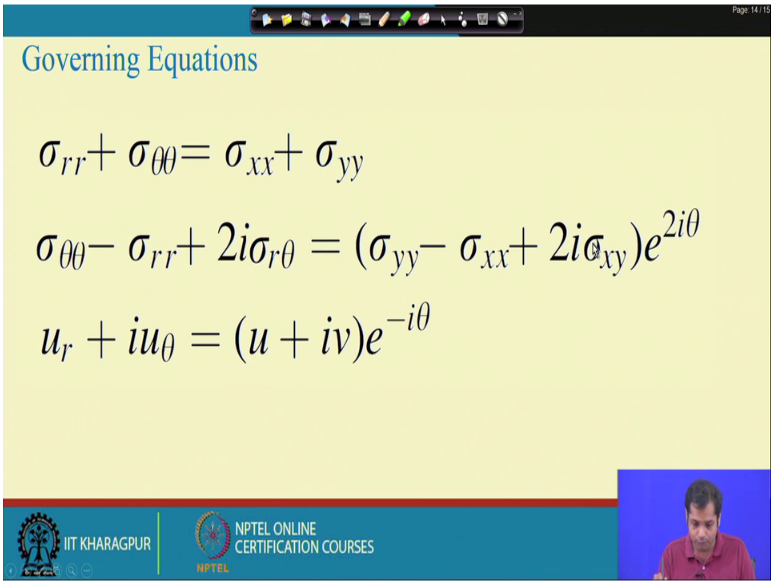

Before that the expression that you get is this. So, this is the expression for just now we have seen this is the first invariant and then this is the second expression. Second expression also become this and this is how we write the displacement if it is written in terms of r and theta. Now, these x these 2 potentials this first the gamma and this psi these 2 potential is called Kolosov- Muskhelishvili potentials ok.

So, essentially our problem is to find out the start with some assumption of this potential and substitute that potential in these governing equations. So, that and these potentials will have some constants and then find out this constant. Finding the relation between this constant through these equations and determine those constant. in addition to that you have boundary conditions and determine those constant through these boundary conditions or any other conditions that you have in the problem. Now, before we actually solve any elasticity problem or detail elasticity problem just to get a flavor of the use of these form use of this equation let us take one example suppose.

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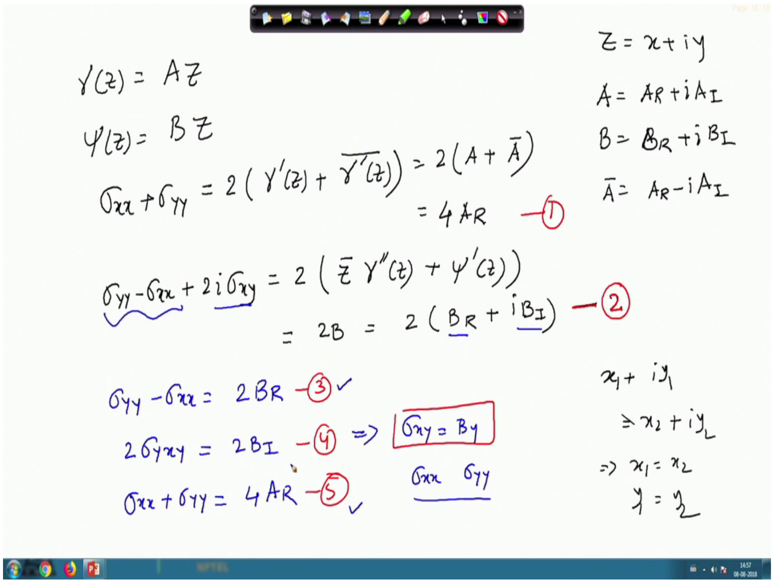

A potential is given to you, now our job at the end is to assume the potential and then find the constants in that expression, But, before we come to that point come to actually solve boundary value problem just to get a flavor of use of this equation. Suppose, the potential is given and that potential is say gamma z is equal to gamma z is equal to AZ, A is the complex constant and psi Z is equal to say psi Z is equal to say B into Z.

Now, Z is x plus iy a is complex constant. So, A also have a real part which is AR plus i into imaginary part. Similarly, B is a constant; B has a real part real part plus an imaginary part. So, essentially we have 4 constants here AR, AI, BR and BI the real and imaginary parts of both A and B; so, for 4 constants.

So, these potentials are given. So, let us write these stresses and displacement in terms of these constants. Then apply the boundary conditions of stress and the boundary conditions of displacement to determine this constant. For a given problem suppose this is the potential given. Now, the first equation if you recall the first equation was sigma x plus sigma y sigma xx plus sigma yy that was equal to that was equal to 2 into gamma dash Z plus then gamma dash Z conjugate. This was the equation right now gamma Z is gamma Z is given here.

So, this becomes 2 into A plus A bar the conjugate of A bar. Now, A is AR plus i y. So, A A bar will be AR minus i into A conjugate. So, essentially this becomes 2 into or 4 this becomes this becomes 4 into AR 4 into AR. So, this is our equation number say 1 equation number 1.

Now, right then sigma x is second equation second equation if you recall it was sigma yy minus sigma xx then plus 2 i sigma xy right. And this was 2 into 2 into Z bar and then gamma double dash Z plus psi dash Z. So, this expression probably we did not write in the previous slide we did not show in the previous slide. But, if you substitute if you substitute the expression of del del x del del y in terms of del del z and del del z bar essentially you will be getting an expression similar to this.

Now, what is this becomes? You see this becomes if you if you this will state away will be 0 because, this is first order it is a linear equation. So, essentially what you have psi psi psi psi the first derivative of this. So, this becomes first derivative of this becomes B. So, this becomes 2 B. So, this is essentially 2 of B real part plus plus i into imaginary part ok. So, this is equation number this is equation number 2 right, equation number 2.

Now, you see now look at this we know this property of complex variable that if we have x1 plus I into y1 is equal to x2 plus i into y2, then if we compare the real part and the compare the imaginary part this gives us x1 is equal to x2 and y1 is equal to y2. So, real part should be same and the imaginary part should be same. So, in this equation in equation 2 if we compare the real part what are the real parts? So, this is here left hand side this is the real part and corresponding real part is this. And this is the imaginary part and corresponding imaginary part is this. So, compare the real part and the imaginary part what we have is sigma yy minus sigma xx that is equal to 2 BR.

And if we compare the imaginary part this becomes 2 sigma xy that is equal to 2 BI. And from first equation if you compare the real part and real imaginary part there is no imaginary part here. Here also there is no imaginary part. So, we have sigma xx plus sigma yy equal to 4 AR. Now, you see if we so, this is equation number say eventually. So, this is equation number say 3, equation number 4 and equation number 5. So, if we solve equation these 3 equation number 3 and equation number 5 then what we get is we get sigma x. If we solve them then we gets expression for sigma x and expression for sigma y and from equation number 4 we can get expression for sigma sigma xy.

Now, we see what advantages we had here, if we write this expression in terms of that is the reason why this particular combination was chosen. Now, from the first expression if we compare the real part and real part we get sigma x plus sigma y and if we come from the real from the second equation if we compare the real part we get sigma y minus sigma x. And these 2 gives us 2 equations from where we can determine sigma x and sigma yy.

And then from the second equation if you compare the imaginary part we get sigma x y. So, this if you solve them 3, 4 and 5 these equations straightaway give you sigma x y is equal to B y. So, this is first equation this and then if you solve this equation number 3 and 4, similarly you get expression for sigma xx and sigma yy in terms of AR and BR. Now, once you have expressed sigma xx, sigma yy and sigma xy in terms of this constants in order to determine this constants you need boundary conditions. There will be some stress boundary conditions given. Suppose for instance if we if we say that it is all if it the stress is constant everywhere in the domain.

If the stress is constant then sigma xx sigma yy all are constant and that value if you substitute here you get these constants for BR, BI and AR. So, whatever information you have about this stress this stress boundary condition you had substitute that and get this relation of BR, BI and AR. Now, do the same exercise for displacement if we do the same exercise for displacement you recall what was the displacement formulation. Now displacement was the expression for the Navier's equation was something like this.

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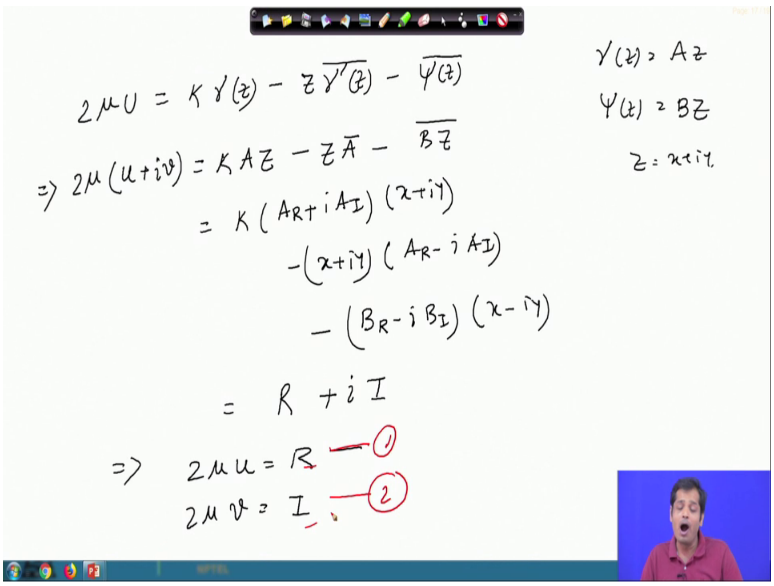

2 mu 2 nu u that was equal to kappa which depends on plane stress or plane strain formulation gamma Z minus Z gamma dash Z conjugate of this minus psi Z conjugate of this. And then here what we have is we have gamma Z is equal to gamma Z is equal to AZ and psi Z is equal to B psi Z is equal to BZ. Now, if we substitute this here then what expression we have now let us write this expression. So, this is equal to 2 mu u can be written as u plus i into v. And then you substitute gamma Z here and then if we substitute gamma Z here then what we have k gamma Z is equal to A into Z.

So, this becomes A into Z and then minus Z into gamma dash Z conjugate. So, this becomes A bar conjugate of A bar and then minus psi Z s psi Z is this. So, this becomes BZ. So, entire this thing. So, if you write a in terms of A bar and B bar. So, essentially if you write Z is equal to x plus iy. Then eventually what you have is just I am writing the expression you can you can check it a AR plus i into AI this is A written and then Z becomes x plus iy so, this is the first expression.

And then the second expression becomes minus x plus iy is Z and then A bar A bar is AR minus i into A imaginary. This is the second equation and the third equation become your B bar B bar is BR B real minus imaginary and then conjugate of B and conjugate of Z minus i y. So, if you simplify this expression then you get some real part and then some imaginary part i into imaginary part. Right, then what is left compare the real part and real part and compare the imaginary part imaginary part. So, this gives you 2 mu u that is equal to real part and then 2 mu v is equal to the imaginary part. So, these 2 are the equation equate these 2 are the equation number 1 and equation number 2 for displacement.

Now, in real part and imaginary part what you have? That real and imaginary parts are written in terms of some constants. Now, you have some boundary condition the boundary conditions specified now if we substitute that boundary conditions we get this we get this constant. So, you see the objective of this exercise has not been to really solve any boundary value problem. The objective of the exercise that just now we have done here to tell you or to try to convince you why this particular form the sigma xx plus sigma yy and then sigma y minus sigma xx plus 2y sigma xy that particular form is written that particular form is used. Because, this is the application of that this is how this particular form can be used and can ease our problem in finding or in solving this equation.

So, next class what we do is next class next class we will start with this and then we we use all this formulation that we have written today derive today. And then we do some examples of boundary value problems. Where we see how to assume that in this example the potential is given that gamma z and psi z. We will see what are the possibility, how it can be assumed, whether it is a polynomial or different kinds of function.

And then for a given problem we assume a potential and then that potential will be having some constants will determine the do this exercise and then determine those constants based on the boundary conditions specified. So, next class the topic is complex variable solution for boundary value problems in elasticity with this I stop here today see you in the next class.

Thank you.