Theory of Elasticity
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Lecture – 47
Complex Variable Method (Contd.)

Hello everyone, this is the 5th class of this week. Let us continue with what we have been doing in the last couple of let couple of classes. See if you recall in the last class we rewrote all the governing equations for elasticity in terms of complex variables and if you recall these were the equations.

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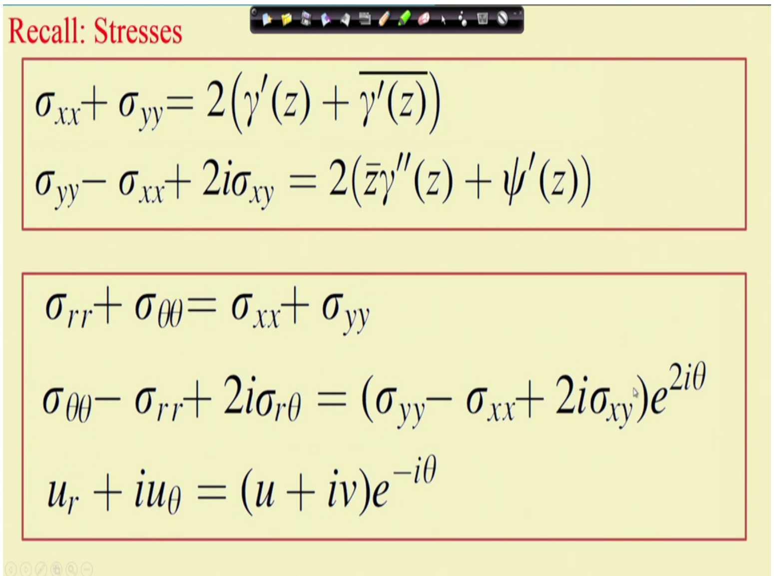

These stresses where gamma z and psi z they are km potential. So, gamma x plus gamma yy is equal to this and gamma y minus gamma x plus 2i sigma xy is equal to this. Now, again you recall we discuss also in the last class why this particular form is written because if you write this particular form then essentially you have three conditions, you can derive from this the first two equation. If you compare the real part in the first equation that gives you gamma x plus sigma x plus sigma y and then in the second equation if you compare the real part it gives you sigma y minus sigma x. And if you compare the imaginary part this is sigma xy.

So, essentially these are three equations though it is written in complex the real form it is two equation. Similarly, the same thing you can write in polar coordinate system; the first one the first this and these are the same equation and then you can do you can write the displacements are also in terms of complex variable representation.

Now, this was the equation stress equation and then similarly we can have this equation for displacement.

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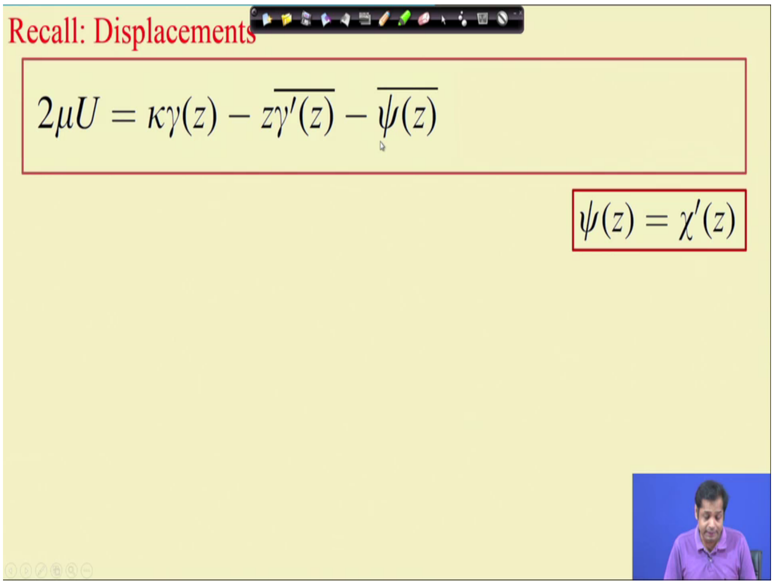

This is essentially the Navier's equation and these equations when written in terms of these km potentials then this equation gives you that. We also discuss then how to through one example, what is the use of these equations both the equations. So, the stress equation and also the displacement equation.

The general procedure is like this; we have to assume one potential, here in this case two potential. If you recall when we solve these problems in Cartesian coordinate systems, we started with an assumption of Airy stress function.

Now, here the Airy stress functions, now is represented in terms of two potential; one is gamma and another is psi. So, we have to start with if you any problem in elasticity that we need to solve using complex variable method once the problem is described, the problem is described means the all the equations are written on the boundary conditions are specified.

Then when you start solving in the first step would be to have an assumption of these potentials gamma z and psi Z. Once you have that potential then we have to substitute this potential in these equations the stress equation and the displacement equation then compare the real part and the imaginary part, we will get the set of equation in terms of constant and then we apply the boundary conditions to get those constants. That is the general procedure and that is we demonstrated in the last class through a very simple example.

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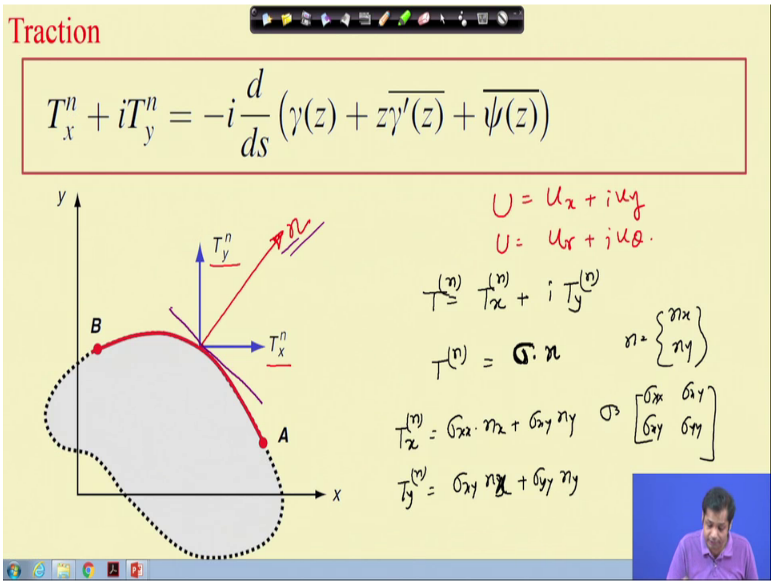

Now, one thing we did not discuss in the last class that is the boundary condition. See if you recall the boundary condition can be of two type; one is the traction boundary condition where the stresses are specified at the boundary traction is specified at the boundary not the stresses. And then another boundary condition is the displacement boundary condition where the displacements are specified at the boundary.

Now, like we like we write the displacement in terms of we write the displacement U is equal to U ux plus iuy or in polar coordinate system if you can write U is equal to ur plus iu theta. So, similarly the traction also can be written in terms of um in terms of in terms of this complex variable.

Now, suppose we have take any arbitrary domain and suppose AB is the boundary where the traction is specified and suppose the traction the component of the traction is in x direction is Tx and then component of traction y direction is Tn and n if you recall, n is essentially because these the surface at this point this is the outward normal. So, this surface is represented by at this point the tangent to the surface is represented by these outward normal.

And that is why this n is written here. So, it is essentially traction on a plane which is specified by this which is represented by this normal vector small n ok. So, if this is the traction then similar to your stresses displacement everything the traction can also be represented as say total traction is equal to Tx which is total traction in n on on that plane is equal to i into Ty n.

Now, recall we also know that traction is equal to the relation between traction and the stress at a given point that is equal to sigma dot n. The sigma is the second order tensor, stress tensor and n is this outward normal. So, if n is equal to say nx and ny which are the components of n and sigma is equal to if you recall sigma xx sigma xy and then sigma xy sigma yy.

So, this essentially gives you that Tx, Tx is equal to this gives you sigma xx nx plus sigma xy ny and similarly Ty and that becomes sigma xy ny plus sigma yy ny; this is nx. So, this is the traction that we know already. Now, sigma xx and sigma xy and sigma yy can be represented in terms of in terms of the Airy stress function. Again Airy stress function can be represented in terms of these potential if you recall. Now, if we substitute that potential in these expression and then the final expression will be final expression for traction would be something like this something This final expression for traction will be something like this.

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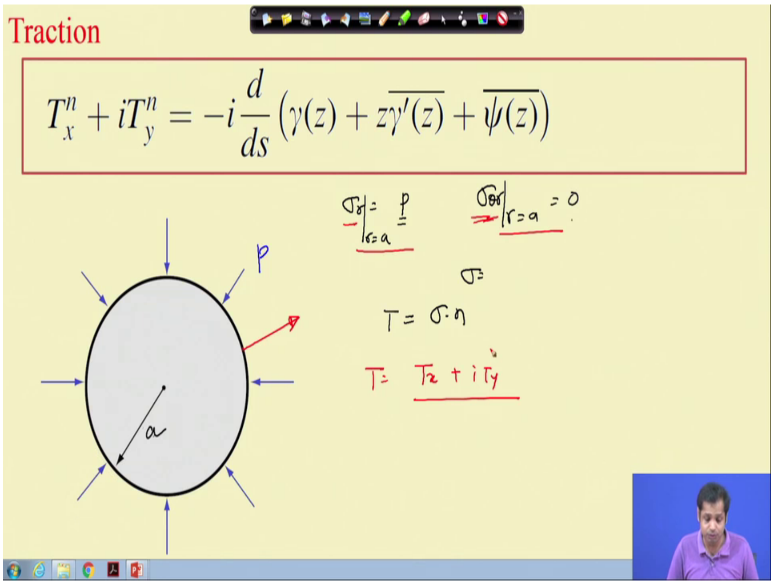

Now, this is the final expression for traction.

Now, just to give you an example suppose you take this example suppose this is the equation given ok. Now, if we have suppose we have something like this; look at the conditions given here suppose these value is P suppose these value is P and suppose this is a or take say unit value. Suppose this is U this is unit this is say 1 or a suppose a.

So, what are the conditions we have? Now conditions we have sigma r is equal to P sigma r at r is equal to a is equal to P and sigma theta, theta at sigma r theta at r is equal to sigma theta r or r theta r is equal to a is equal to 0 right. Now, then your stress tensor become sigma stress or at this point of stress tensor become sigma r is equal to P sigma r theta is equal to 0 and sigma theta theta is equal to 0.

Now, similarly if you substitute that on that on that any plane your traction would be then you can calculate T is equal to the sigma dot n, n would be the n would be any outward normal vector here and then if we substitute this your final traction will be Tx i plus i into Ty ok, this will be the traction on the entire plane ok.

This we can write in terms of polar coordinate as well if you if you if you write if you write this see if you sigma r and sigma theta theta then you can write the entire expression in terms of polar coordinate ok. When we solve an example probably this representation will be more clear. Now let us move on for the time being. Now, similar to traction like that what we have here we can also define the forces.

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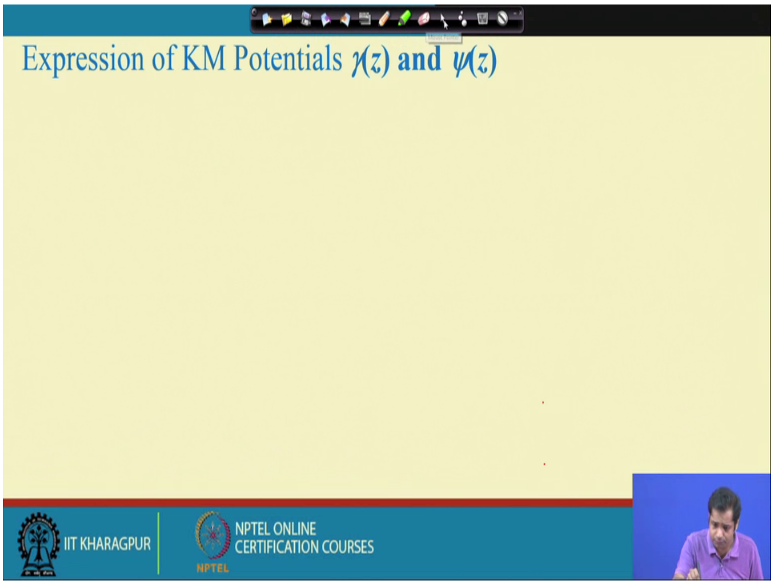

For instance, suppose we have we know the take this example ok. Suppose, the forces similarly total force, force will be how do we get the force? If the dist this traction at every point is if it is integrated over the entire surface then we can get the force.

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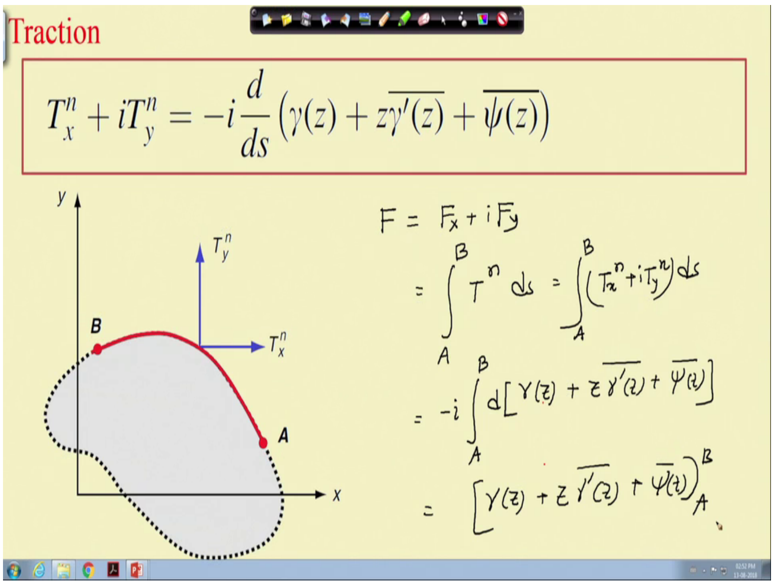

For instance, the force will be force total force which can also be written an in terms of Fx plus i into Fy. So, that will be the integration of total traction, the total traction that integration over A to integration over the entire domain right this and then ds and this will be you have traction is equal to this, this will be Tx n plus Ty n i into Ty n and this integration of ds over a to A to B.

Now, look at this expression is this and this can be written as. So, this minus i will come here minus i and then A to B and this entire thing becomes d of the gamma z and then plus z gamma dash z conjugate plus psi z again its conjugate d of this.

And this essentially becomes gamma z plus z gamma dash z plus psi z gamma z this is evaluated at A and B. These expression is important because both the expression this expression and this expression because sometimes what happen sometimes the boundary conditions are given in terms of forces, in that case we have to express the force like this.

Similarly, sometimes you can have boundary condition given in terms of say moments and in that case how do we write those boundary conditions?

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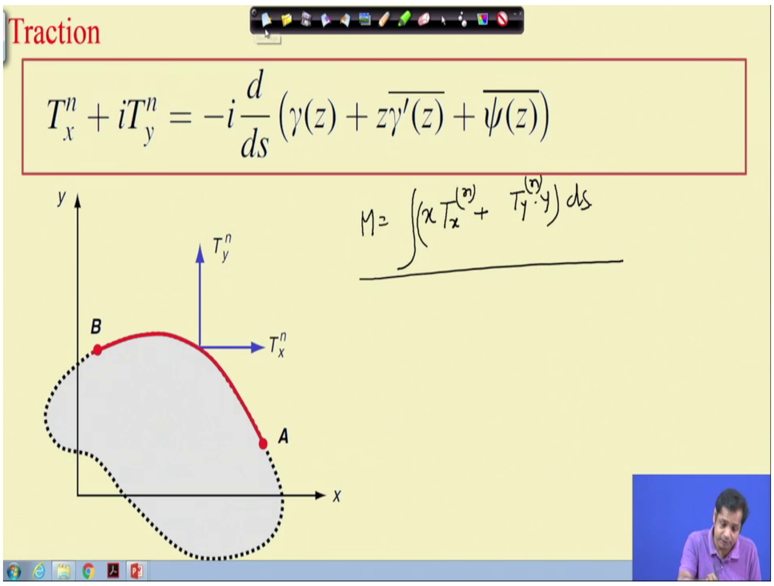

Suppose, the moments is equal to if you write moment is equal to the force multiplied by the distance. So, integration of integration of Tx which is this x into Tx plus then plus then i into your you get this thing that Ty which is n into y that integration over ds.

So, this if you can substitute you get the corresponding expression for moment right. So, this is important because as I said just now sometime for we may in count a different problem. So, the boundary conditions are given either in terms of traction or the boundary condition given in terms of forces or moment. So, how to take those forces and moments how to represent them in terms of complex variable this is an example of that.

So, let us move on. Now, you see. So, one thing is see if you look at the procedure; the procedure is there is nothing new in the procedure just we I have mentioned that many time the procedure remains same. Only problem the only thing which is important here and the previous methods the Airy stress function methods also that we have to start with an assumption of gamma z, we have to in this case gamma z and psi z.

Once we have some assumption once we start with some assumption rest of the thing is you substitute that expression of gamma z, the potential in the governing equations and the boundary conditions and then find out the associated constant. Now, but the only thing is here when we when we when we solve this purely Airy stress in term we wrote this in terms of x and y and phi, the Airy stress function the it was just the 1 function the phi we had to choose.

Now, in this case the Airy stress functions are written function is written in terms of two potentials, so, you have to assume two potential. Now, you see now these assumptions are not very straightforward unlike the unlike phi.

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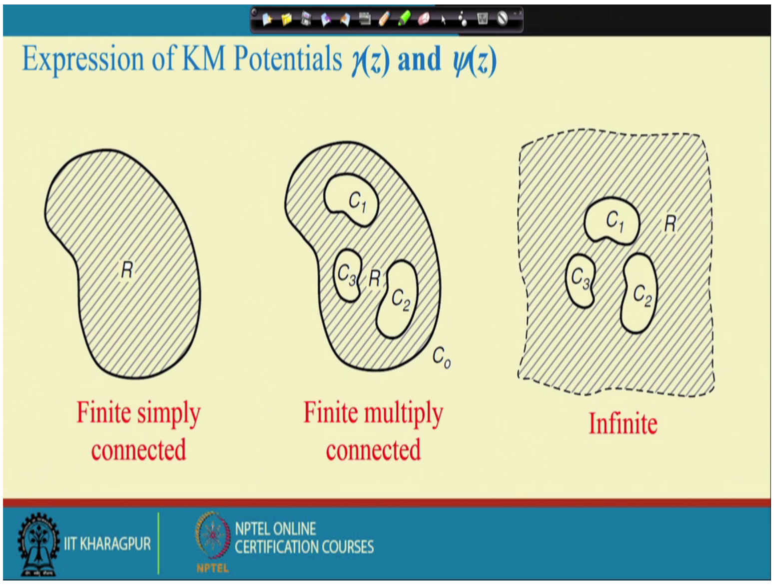

You see there are three different kinds of domain is given here; one is the simply connected domain, we already discussed what is simply connected domain and another thing is domain is finite domain, it has a finite boundary. Then the second one is a finite boundary domain and then is a multiply connected domain. You have these are internal these are sorry these are some holes in the domain and the third one is it could be simply connected or multiply connected the important thing is the domain is infinite domain. Now, the choice of these potentials gamma and psi they are different for different kinds of domain, the general choice of this potential.

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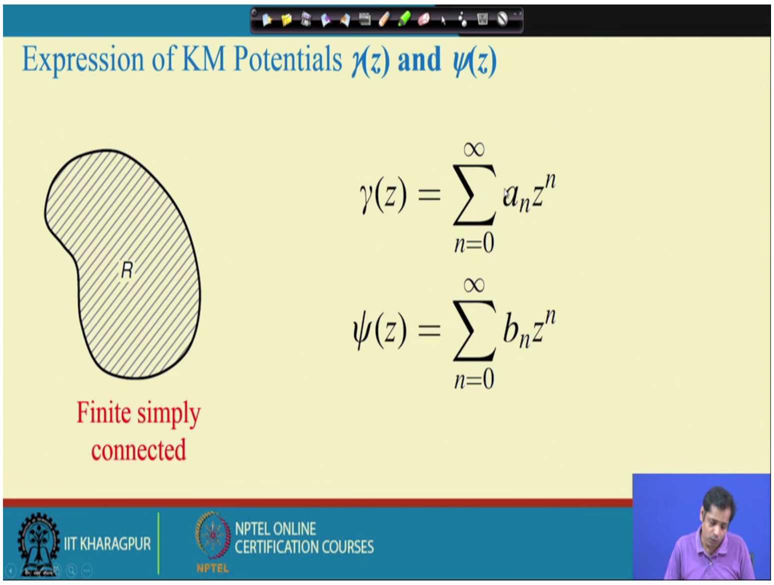

Now, for instance I only give you the detail how to get this how to guess or how to arrive at these equations. If you remember at the beginning of this week I gave you two references; one is one reference is by Muskhelishvili that if you go through that reference you will get all the details how to arrive at these potentials.

If the domain is a finite simply connected domain then what happens? It is very similar to phi we can have a polynomial assumptions like this; a state power series expression now gamma is a polynomial psi is a polynomial a and b are the associated constants and then those constant need to be determined. This are the general form, but it does not mean that we always we have to take the if you look at this it is a summation over infinity through it does not mean that you have to take see o we have taken finite number of terms, we have to take a finite number of terms in these power series ok; this is for this domain.

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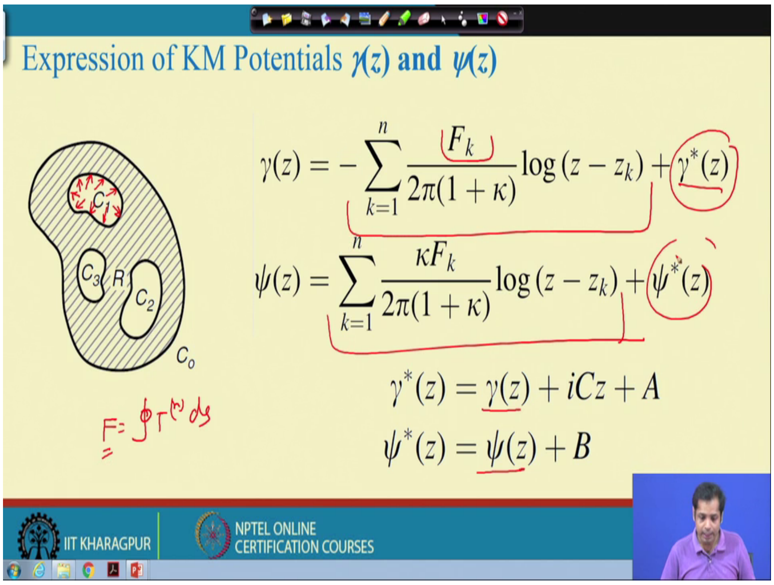

If it is multiply connected domain then what happens? Then your potentials are this and look at these F k is essentially this these F k; first of all this there are two things one is this summation when k is equal to 1 to n this summation is essentially your how many loops you have, how many these such holes you have. For instance, the given figure you have just 3 3 holes.

So, here k will be 1 2 3. Now, and what is F k? F k is a say for now each hole if you have some boundary conditions, for instance, for instance if you for if you add this hole you have some forces specified here. So, we know the, what are the traction at the boundary at this surface and internal loop this loop. Now, that traction can be converted into force just now we saw that force is equal to integration of this traction ds over this.

Now, it is a close integration to close integration. So, this will be Fk. So, F k is the force on this. Similarly, if you have several depending on the how many loops you have? These are the forces in different loop. So, this first term is for this multiply connected because of this holes ok. Now, the second term is very same as the previous one, the simply connected case. Second term is written in terms of gamma z and psi z where gamma z.

And psi z are the same as the you can have the power series representation of gamma z and psi z for this part. So, in this case the additional parties only the first one ok.

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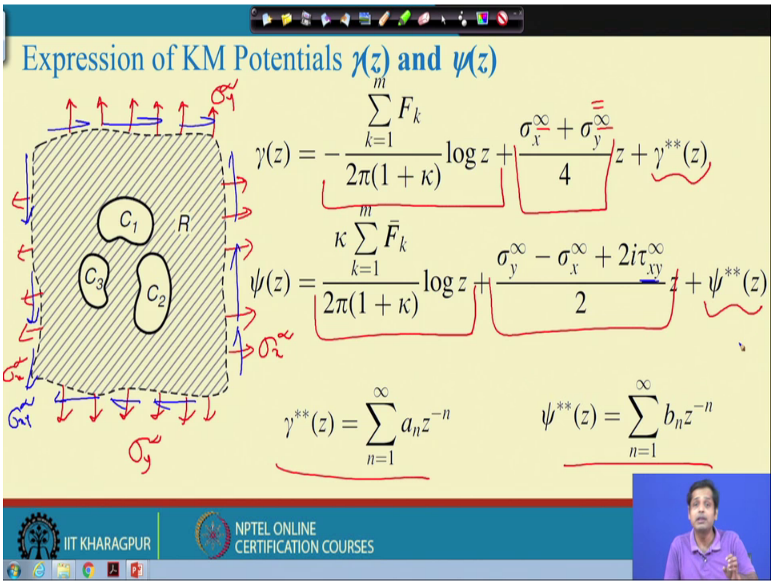

Now, if the domain is infinite the this one, now it is a very general expression. So, infinite domain and also you have some it is a multiply connected domain. So, this part remain same in the previous one this 1 this will be the same as the previous one.

This is because of these internal loops you have, the holes in the domain you have and m is the total number of hole and this is the summation over the entire into the number of holes and F k is the forces specified on the loop. Now, gamma star then the third part gamma double star and psi double star as the same as at this expression this is also written in terms of power series ok.

Now, but look at please look at one thing, one thing is very important here.

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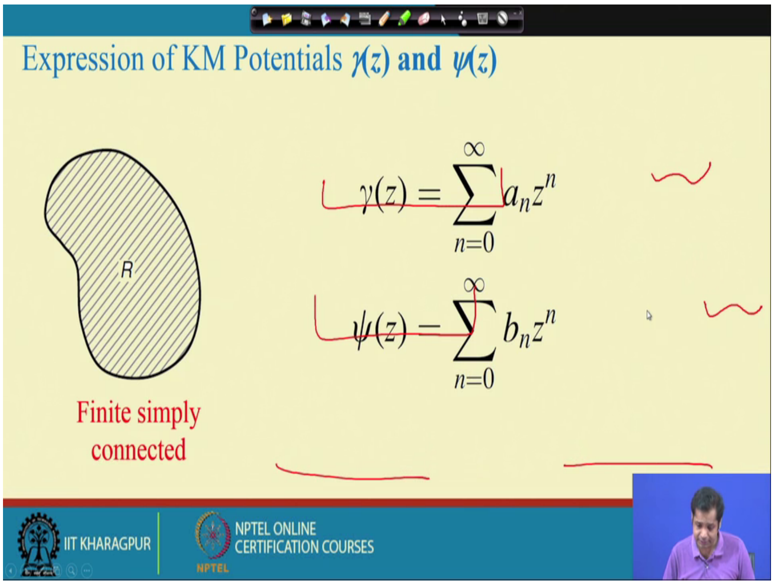

If you look at the expression of this; this is written in terms of z to the over n, whereas, in this case it is written z to the over minus n. So, it is essentially 1 1 by z 1 by z square and so, on ok.

Now and these are these are the stress values that is specified at the infine in that the at the boundary. The infinity means at the infinite boundary what are the stress you have. For instance, you can have you can have say normal stress here.

If this is normal stress, this is normal stress and these are the normal stress, these are sigma y infinity, sigma y infinity; this one. Similarly, you can have sigma x here, these are all sigma xx sigma x, this is sigma x infinity, sigma x infinity. It means that are the infinite if you fictitious boundary at infinity then what is the stress on that? Similarly, sigma tau xy or sigma xy is essentially if you have some stress like this these are these stresses these are stresses and these are these stresses at the this. So, this is sigma xy infinity or tau xy infinity in same thing. So, this is the expression of gamma z and psi z.

Now, once we have the general expression of gamma z and psi z, it does not mean that we all we have to take this expression because entire expression because it is a essentially infinite series, we have to take a finite number of term depending on the equations and depending on the boundary conditions specified, depending on the information that we have about the problem.

So, we have now we have seen three different cases and what are those three what are the what are the possible or the general this potential general form of these potentials for three different cases. Now, if you go through any book there are many examples are given and all the examples you can put either of this anyone of these three such cases and you choose gamma z and phi z accordingly.

Once we choose the gamma z and phi z next of the process is the same way that we demonstrated in the previous class. You write the constant in terms of the real and imaginary part and then substitute gamma z and psi z in the stress equation and then compare the real parts and compare the imaginary part you get the expression for stresses in terms of constants.

And then you have the boundary conditions, traction boundary condition specified, find out those constants. Similarly, you substitute these potentials in the Navier's equation and then compare the real part and imaginary part get the expression of u and v, you have in terms of constants and then boundary conditions are specified, apply those boundary conditions to get those constants ok, the process is exactly same. Now, next let us take one example. This example just I will give you the I will tell you these steps we have discussed. I will tell you the final results of this example.

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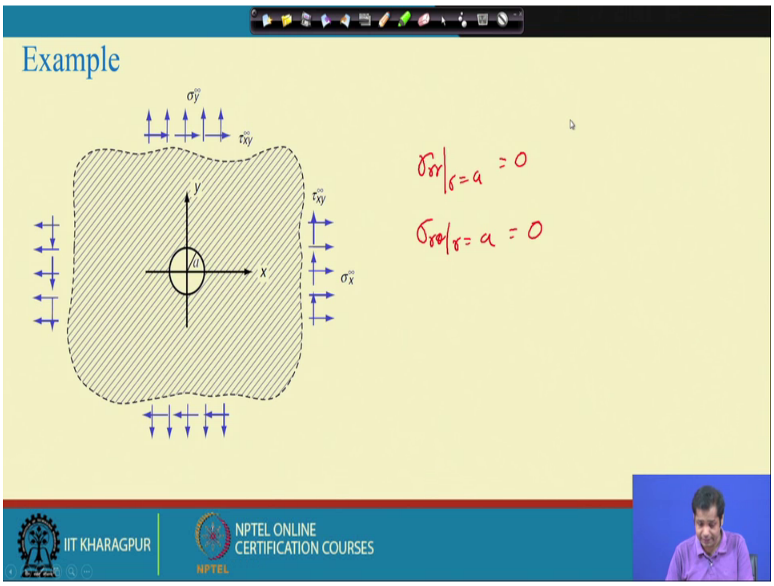

The particularly these examples the solution of this example we are discussing today because that solution is very important because form this solution we are going to make a very very important point that we will be doing in the next class ok. Now, you see these exam these is essentially the very similar to this is the guess in the third case. So, multiply connected domain and there is in finite domain. So, these are the general form of the potential.

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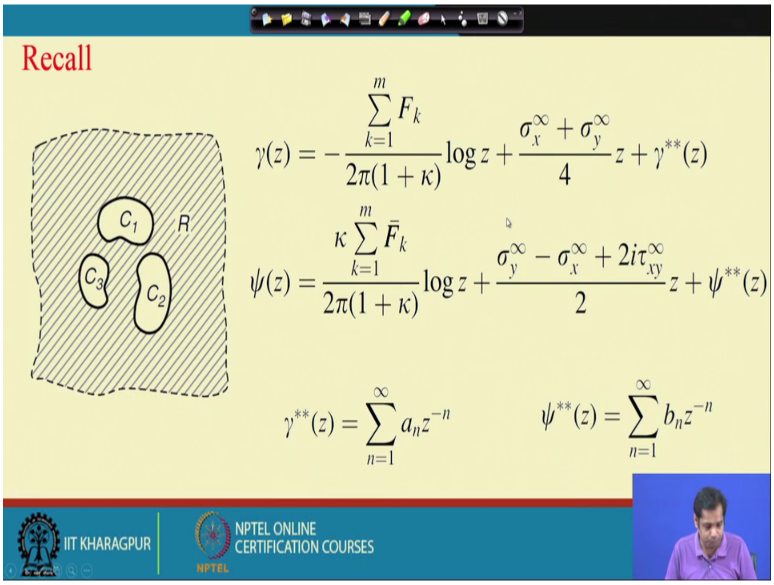

So, the potential will be potential will have.

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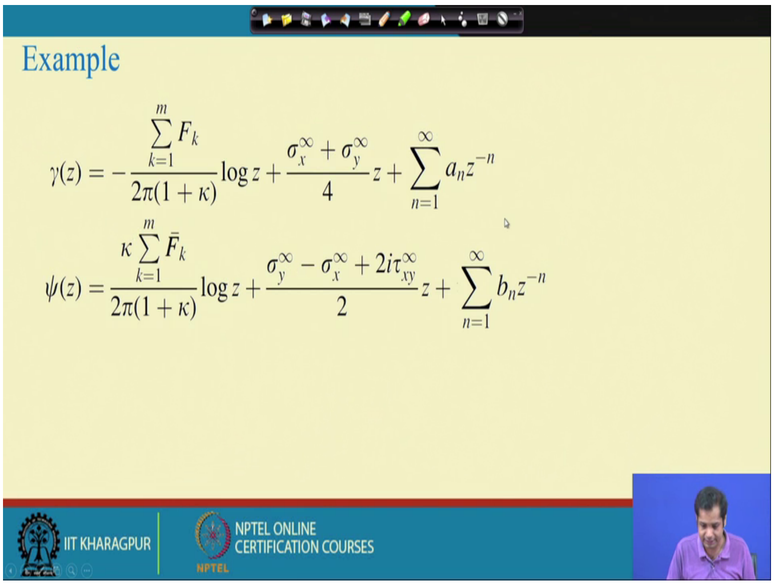

So, for this case we have the potential like this right. So, these gammas gamma double star psi star z is essentially this ok. Now, once we have that then you see what in this example what information we have? You recall that this is the way this is the expression for stress in terms of this potential.

And if you use the polar coordinate system this is the expression of this is the expression of different stress component in terms of the potential ok. So, this is essentially this part if you this part is essentially the same as this part. So, these essentially become this into 2i theta additional 2 i theta ok.

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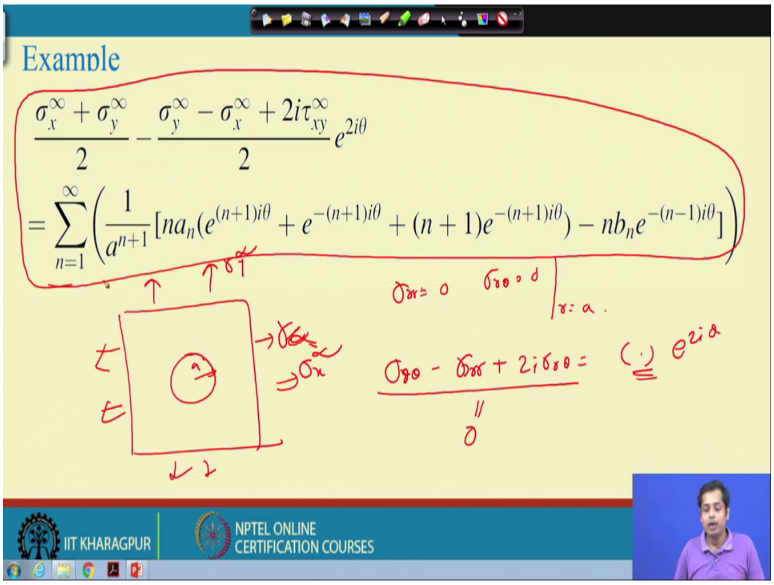

Now, so, if you do that; now what information we have about this problem? Now, you see in this problem what we have is at this boundary your sigma rr at this boundary what we have is here you look at what is the condition we have? Sigma r plus sigma r at r is equal to a that is equal to 0 and sigma rr and sigma r theta at r is equal to a that is also equal to 0.

So, if we substitute this condition here, in this expression and anywhere on this boundary sigma theta theta is not specified, so, if we substitute that expression in this here in this expression and then write this entire thing in terms of write the entire thing in term entire thing in terms of this potential.

And the potential substitute that substitute the expression of this potential into this equation and then we make them 0, we make them 0 because it is 0 in both the cases it is 0 right, in both the cases it is 0 in this example in both the cases it is 0. So, entire expression of stress will be 0. So, here your this part will be 0. If we substitute that this entire thing go to 0 then what we have is we have the final expression is this ok, this is the final expression.

Now, you can have these for a given problem this may be some value are the you may have there is no stresses are the I know stresses at that infinite that fictitious boundary at the infinity. So, this part will be 0. Now, once we have the please I will just repeat once again these the step. These step is very state these step is what? These step is you have the problem like this and then you have at the infinite at this boundary you have some stresses sigma x sorry this is sigma x infinity sigma y infinity and so on. Now, at the if this is a, so, at this boundary your stress is 0; sigma r is equal to 0 and sigma rr and sigma r theta is equal to 0 at r is equal to a.

Now, you have an expression of sigma theta, theta minus sigma rr then plus 2i sigma r theta is equal to something e to the power 2i theta right and this something this is written in terms of the potential gamma z and psi z. What is now this part is 0 this part is 0 at this boundary, if you satisfy this equation at this boundary this step is 0.

And substitute this potential the expression of potential just now we discuss here then you just slightly rearrange it and get this expression, get this expression entire expression ok. Now, once you have the entire expression then the rest of the thing is we have to compare the similar term, either the real term or then we compare the what is the in this case if you compare e to the power to the power ni theta term, if we compare to the power ni theta term so, then what happens or e to the power 2i theta term, here we are we have 2i theta term. If you compare those term then what do you have is you have expression for a 1 b 1 and all other expressions are like this.

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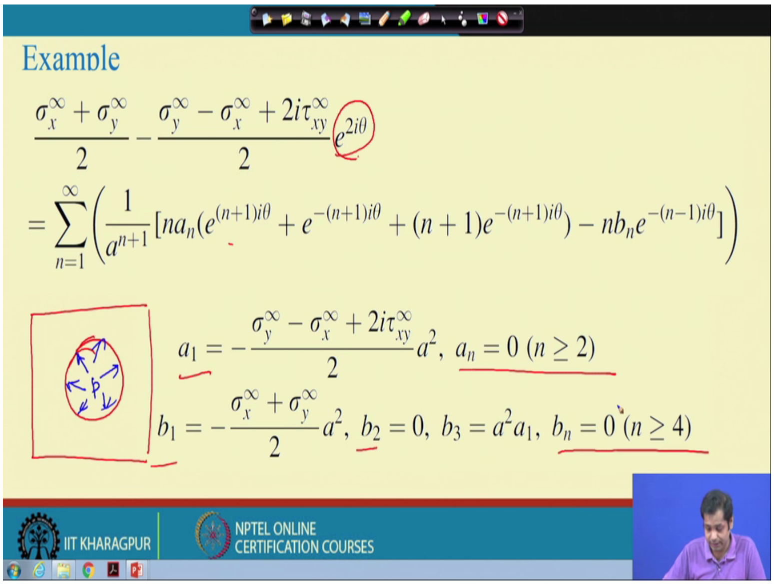

Now, this is important, yes. Now what how do we get it? Just look at this expression and if we compare whatever we have 2i theta term here and whatever you have whatever 2i, e to the power 2i theta term here their coefficient should be same, and if you compare this coefficients then you get these expression, expression of a 1 b 1 then a 2 I sorry all other an will be 0 if it is greater than 2 and all other bn will be 0 if it is greater than 4.

So, essentially even if you even if you started we started with infinite series. Essentially, what we have? We have a finite number of term for the given problem. If you apply the conditions for the given problem then you have a finite number of term, but recall, but remember.

If the if the problem is slightly different, problem is slightly different in the sense if they this is the internal this is the whole, but suppose this whole is not the stress free whole. So, you have some forces you have some pressure here, you have some pressure here. So, its sub that pressure is P then your this you may not get this equation because this equation you as you arrive by making by of substituting that boundary condition.

In that case that that force that stress will be we have to write that in terms of the traction that we discussed and substitute that traction value here and we will get the slightly different form and then if you again do the same exercise compared the term by term and get the expression for these constants ok.

Now with this what is important for now is now once we have this constant plus look at once we have this constant then we have the potentials ok. In these in these potentials what are the unknowns? Unknown variables are only these as these are the unknown variables.

Now, we know what are the boundary conditions we have, substitute that boundary conditions get these unknown variables. Once we have the unknown variables we have the potential, substitute that potential in the stress equation and at any given point we can compute the stress field ok. Next what we do is next this example particularly we are discussing, there are many example given in the book in order to because of the because of the restriction in time that we have.

Here we cannot we cannot do many exercise problems many example problems and demonstrate all these all these expressions that we derived through some example this is not possible. But here the purpose has not purpose has been to let we know what are those what are the concept what are these equations and how in a general way you can use these equations to if you want to solve any problem in elasticity using complex variable method.

Now, please convince yourself by applying this concept these equations that we derived in this week and try to solve some examples yourself and there are some examples given in the book until unless we you really implement these equations and solve them things may not be clear ok. The particularly this example we discussed because from this example, we want to solve this example.

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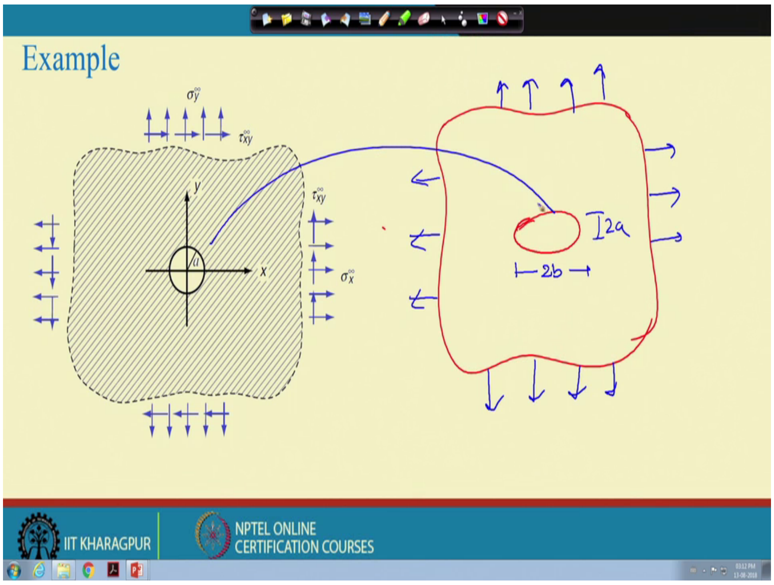

You have an infinite you take a domain and then this domain has elliptical hole and this is subjected to some forces like this ok, some forces like this. So, we use this solution of this equation, the solution of this equation just now we derived the solution of this equation is this is the solution of this equation.

Now, we use the solution of this equation to get the solution of this. So, this is elliptical hole, so, suppose this is 2 b and this is 2 a and these solution from this solution we will make a very important point. The point is so, important that that is essentially a starting point of a different branch in mechanics. Just an example this problem opens a door or provokes you to think some important concepts, some important thing that will open a door or that will that will open a door through to a new branch of mechanics that we will discuss tomorrow next class.

With this; so, I stop today. So, next class will be complex variable solution of elasticity problems using conformal mapping. The conformal mapping because these example we cannot these example we cannot it will be tedious to apply the boundary conditions here, but we define a map from this circle to from this circle to this ellipse and that maps will be the conformal map that we discussed in one of the classes this week. So, use that map to get the solution of this equation.

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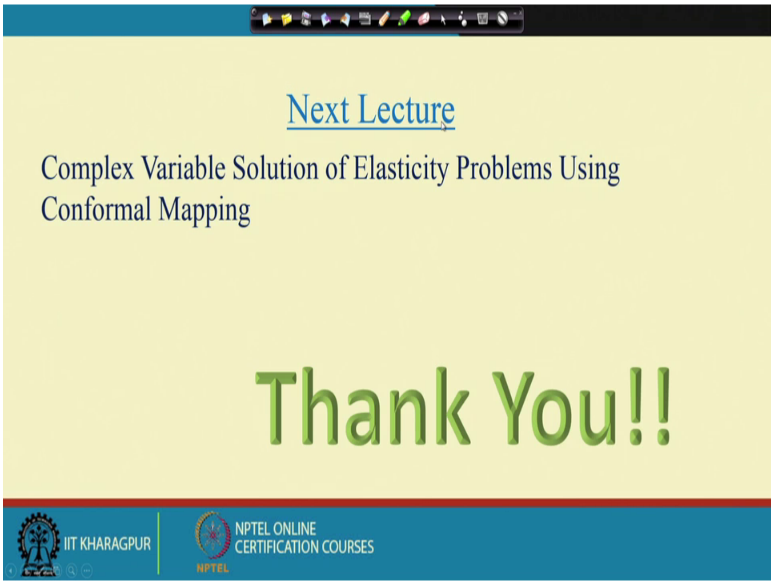

So, next class will be the solution of boundary value problems in elasticity using conformal mapping. By this I stopped today, see you in the next class.

Thank you.