Theory of Elasticity
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Lecture – 48
Complex Variable Method (Contd.)

Hello, everyone. This is the last class of this week and today's topic is Complex Variable Solution of Elasticity Problems Using Conformal Mapping. So, it is just an application of conformal mapping for elasticity problem.

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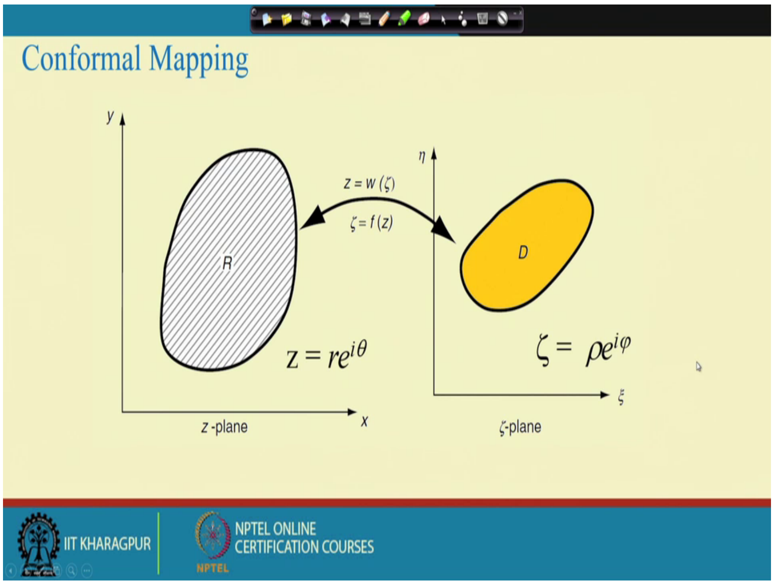

So, conformal mapping we already discuss what is conformal mapping. The mapping between two plane here for example, in this case the mapping between Z plane and zeta plane and the map is this one is the map, but with this map what happens if we take if we take two points here if we take up any point here and take any curve any two curve here say C 1 and C 2 they and at this point if we take two tangent then the angle between these two tangents one tangent is for C 1 and another in for C 2. And, if the corresponding projection of that point in here is these and these and corresponding tangent is this and this and this angle and this angle will remain same, but that is conformal mapping, ok.

Now, here this is Z plane; Z is expressed as Z is equal to r to the power i theta and zeta is equal to rho into the i phi. So, these is please note that and so, because when we write the formulation in terms of this conformal mapping will be using these notations. So, rho and phi, so it is in zeta plane and corresponding in xi Z plane is r and theta.

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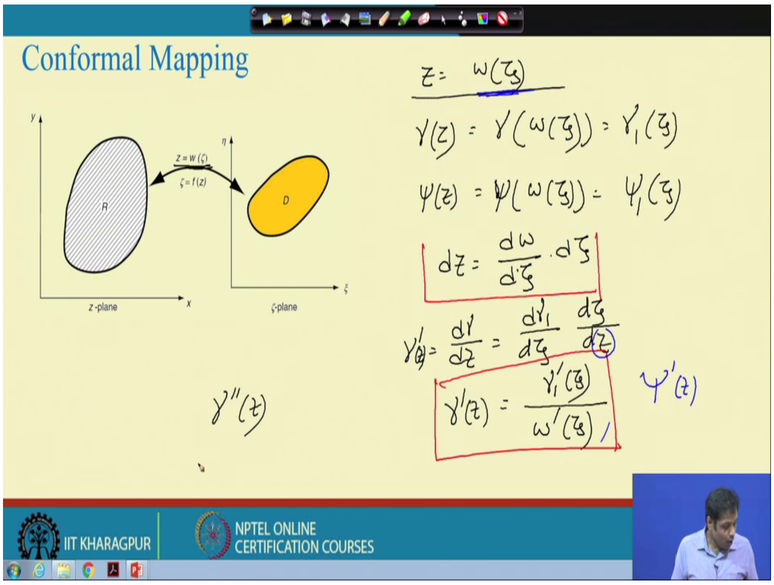

Now, so, first what we do is first will start with this map we assume that the this map exists this there is a map which is a conformal map this map and for a given problem, the problem that we will be considering what would be that map that will see. The expression for that map the relation between Z and zeta will see that for a specific problem suppose that map exists.

Now, what we have to do is now, we have to write all these equations that equations that we derived the stress equation then the displacement equations then also the traction vector; so, all these equations that we derived in terms of complex variable Z. Now, Z will be replaced by the map and then that we have to rewrite those equation in terms of that map. Let us do that.

So, what we have here is Z is equal to say you look at this Z is equal to W of zeta Z is equal to w of zeta that is the map we have ok. Now, so, we have two potentials gamma and psi two game potentials. So, let us write that potential in terms of zeta. So, gamma is the potential which is a function of Z.

Now, if I substitute gamma Z write Z as this. Suppose that is equal to gamma 1. Now, gamma 1 and gamma is essentially same when it is written in terms of zeta then we are representing it as gamma 1, but when it is written in terms of Z it is represented as gamma gamma. Similarly, psi Z will be psi Z is Z is replaced by this w of zeta and this is suppose psi 1 zeta. So, same as gamma when it is represented in terms of zeta it will be psi 1 and when it is represented in terms of Z it is psi.

Now, we also know from this that then del Z dZ is equal to dw from this map dw d zeta into d d zeta this one, it is a stator we can have this relation, right. Now, once we have that relation let us find out what is if you recall in the expression of stress the combination of stresses in that expression or also in the combination of displacement in those expressions we have gamma dash Z then gamma double dash z. So, let us find out the derivative of gamma.

So, gamma dash Z gamma dash which is essentially d gamma dZ this is in terms of Z which is d gamma dZ that will be equal to say d gamma 1 d zeta and then d zeta dZ, right. Now, d gamma 1 d zeta let us write it as gamma 1 dash. So, gamma 1 dash is essentially it is gamma dash it is the derivative of gamma 1 with respect to zeta. Now, what is d zeta dZ if we from this expression we have d zeta d zeta dZ is essentially w dash. So, this is gamma dash Z this expression is important. So, similar way we can have the expression for psi expression for psi dash Z, same way we can have the expression.

Now, so, please note when we write gamma 1 dash this is essentially derivative with respect to zeta because gamma 1 is represented in terms of zeta, w dash that is represented in terms of zeta. So, w dash is essentially derivative of w with respect to with res with respect to this with respect to zeta. So, Z is replaced by here Z is replaced by in Z is replaced by w of zeta.

So, this is the expression we can have in the same a similar expression we can have for psi d. We can have this similar approach, we can apply the similar approach and the chain rule we can also determine what is the expression of gamma double dash Z that exercise I would not do it here that I leave it to you I will just write the final expression, ok. Once we have this, now recall what was our expression?

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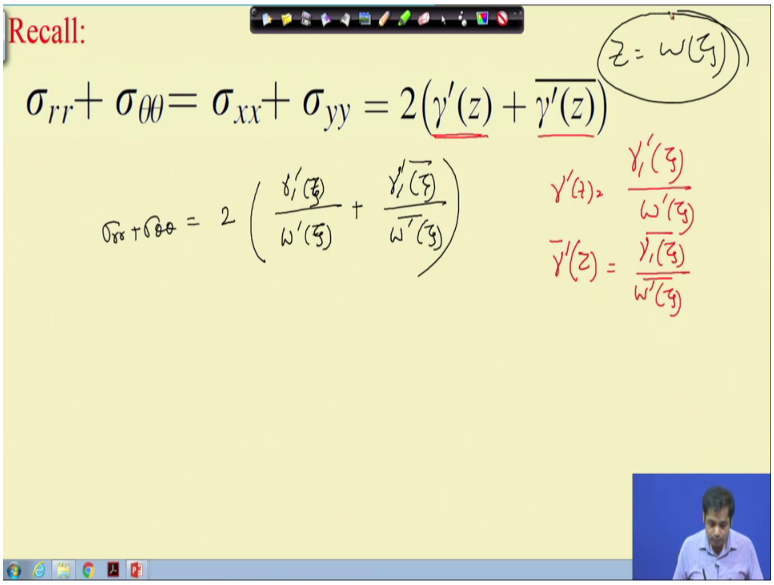

We had three expression we three expression we represented in terms of these potential km potential one is the combination of stresses this is the first combination of stress. So, this is the trace of the trace and this is written as in terms of gamma 1 and in terms of gamma dash is equal to this. And, just now we derive that gamma dash is equal to this gamma dash gamma dash Z is equal to gamma 1 dash zeta and w dash zeta, just now we derive.

Then, if we substitute that then what we have is so, this one is this and what only gamma dash the conjugate of this conjugate of this will be this gamma zeta and then w zeta conjugate of this and conjugate of this. So, if we substitute that in this expression then essentially what we have is we have sigma rr plus sigma theta, theta which is equal to sigma xx plus sigma yy as well, that will be equal to 2 into gamma 1 Z then w dash Z w dash zeta it should be zeta and then plus gamma 1 zeta conjugate of it and then w dash zeta conjugate of this is this expression.

Now, this expression is same, but now what we have done is we have written the expression rewritten the expression in terms of these expression was in terms of Z, now this expression is written in terms of zeta. Using the map when we write this expression in terms of zeta we use this map Z is equal to w of zeta and this map is a conformal map.

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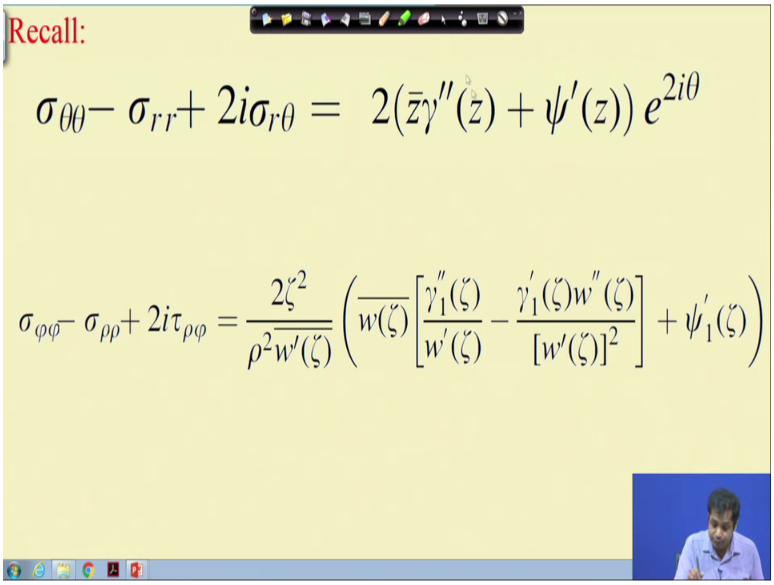

Now, so, we had another equation if you recall that equation was this was the another combination and similarly, if you substitute this double derivative gamma double dash Z and psi dash Z obtain one is following the same procedure and if you substitute that you will be having an expression like this. So, this 2 2 2 i theta the 2 i theta will term give you this part and then other rest of the things will give you this part in that part. So, you can do that exercise and then see, you are getting this expression or not, ok. This is all the stress equation, similar exercise we can do for displacement as well as for traction.

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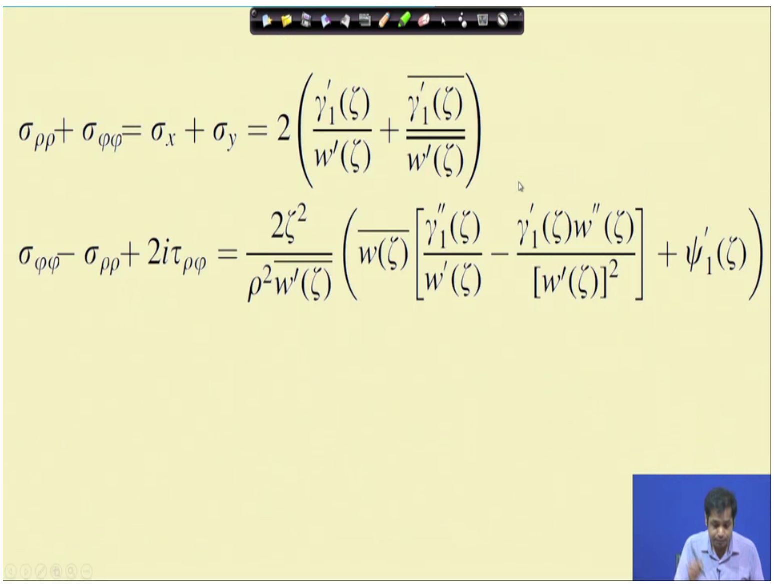

So, if you recall so, these are the summary of the equation that was the first equation this is the second equation, but the application of the equation will remain again same. You see if you compare the real part you will from the first equation you get sigma the trace. If you compare real part in the second equation we get sigma phi phi minus sigma rho rho and then compare the imaginary part you will get sigma rho phi or tau rho phi.

Please note another thing these are written in terms of these this is no longer sigma r and sigma theta theta. The reason is here now these the right hand side the expressions written in terms of zeta. So, the expressions are written with respect to zeta plane and that is why where the variable is defined as zeta is equal to rho e to the power i phi. So, that is why these are these are not rr and theta theta we have to use the corresponding symbol for this. So, this is the summary of the first equation.

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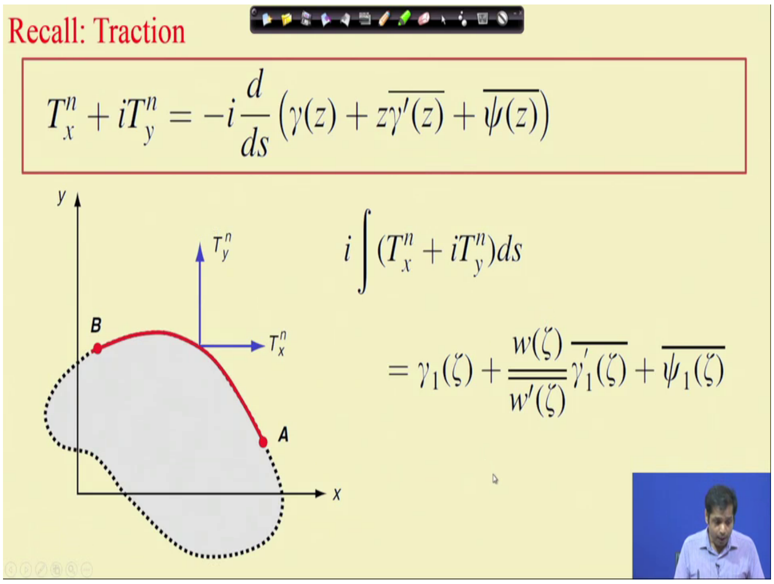

So, if you recall the traction vector, if we take traction at any point and then that traction is represented in terms of km potentials as this we know what is the expression for gamma 1 gamma dash and psi n gamma Z if we substitute that expression with this map Z replaced by w of zeta and then we have an expression like this these exercise you can do. Now, this is not the expression for traction, this is the expression of total force on this, ok, great.

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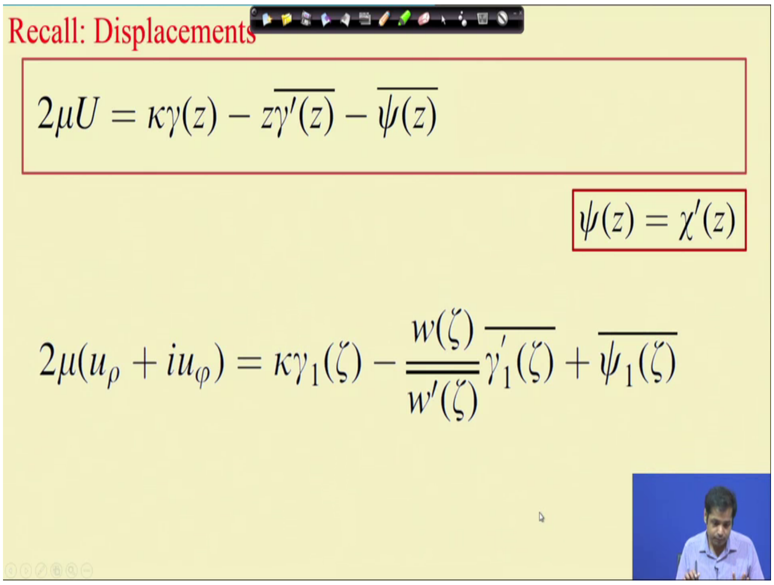

So, again similarly, if you recall the displacement was written in terms of K in potentials like this substitute the potential substitute this derivative of this potential and then final expression what we have is this. So, these equations we already derived now, what we have done is that same equation now represented in terms of zeta. Now, so, once we have that. So, we are now in a position to apply this for a problem.

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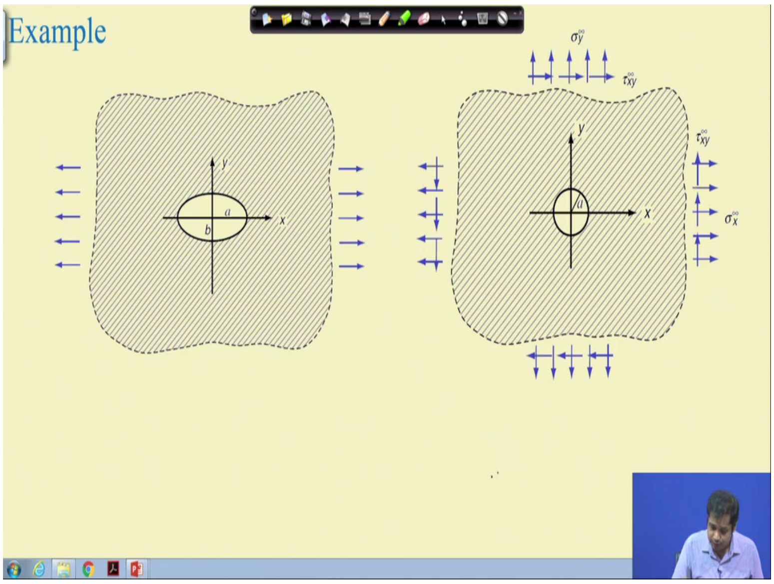

The problem that we consider is this we have a infinite plate with an elliptical hole and then which the axis are minor axis and major axis are given as a and b and we did subjected to in plane loading in one particular direction like in this direction, say for example. This example ok, this is the common example an any elasticity book if you take this example along among along with many other example the these examples are also these example is also given. These figures and all these things here is taken from the book by said the reference of that book I already gave you at the beginning of this course.

See, call recall now this is this we have this we know the solution of this equation is not a solution of this problem last class we had the solution of this problem, ok. Let me let me make one point here, in the last class we just we just discussed what are the how to solve this equation, what are the final expression for the constant as a function of these stresses sigma y infinity sigma x infinity and so on. We really did not determine what is the distribution of stress or component of stress at a given point within this entire within the body we have not done that please do that.

Now, once you have the potential with you and the rest of the thing is we have to differentiate the potential to get the expression of stresses. You do that and then see how these distribution of what how is the distribution of stress especially near these elliptic near these circular hole. Do you get the same stress at the near the circular hole as the stress far away from the hole or you get some or the stresses near the hole is more, that you should experience that is very important thing.

Let me tell you the whenever you have this kind of hole or opening the stresses at that point there will be will be higher as compared to the point which is far away from this opening, ok. This is called stress concentration. So, you must experience that you must solve that problem entirely and plot the stresses on this plane and then see how this stress distribution is ok.

So, this is the problem with us. Suppose, now this is subjected to subjected to a stress S, S, so, in one particular direction. So, if you compare with this example the example that we did in the last class then what we have is, we have sigma sigma x infinity that is equal to S. So, in this direction we have stress and then sigma y infinity that is equal to 0, and sigma xy infinity that is also equal to 0. So, only these stress you have all other stresses are 0, sigma xy or tau xy is the same. Now, let us see how to solve it, how to get the solution of this.

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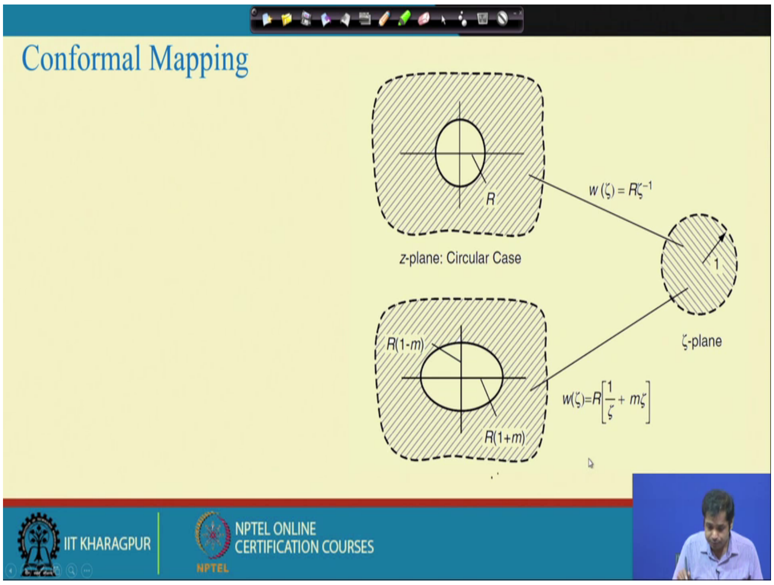

Now, we use the concept of conformal mapping. So, this is conformal mapping. Now, suppose you have a we have you have a circle in zeta plane let any unit circle, but now the radius is 1 unit circle in a zeta plane and then you if you map with the outer you look at the look at the outer look at this periphery it is a circle the dotted line.

Now, it is mapped with Z plane and this map is like this map is like this. So, Z is equal to so, Z is equal to this is equal to Z ok, this is w zeta. If you map it map it like this what you do is you if you map it take some values take some arbitrary value of R and m and then you we know that Z is equal to Z is equal to e to the power r e to the power i theta and zeta is equal to rho e to power i phi in this case rho is 1 because it is unit circle. So, rho will be 1 here.

Now, what you do is you try to plot it you take up you take points on zeta plane map it on project it through this transformation and plot it find out Z and then plot it. Then we will see that if it is this it is transformed through this true through this map through this expression what you get is you get an ellipse who is one axis is the minor axis is one minor axis is R into 1 minus m and the major axis is R into 1 plus m.

Now, if you take m is equal to 0, in this equation if you take m is equal to 0, then your major x major x is becomes r and minor axis become r. So, this becomes a circle of radius r. So, this is this map is this, this is the for m is equal to 0 what you get is this. So, essentially then in this case you are mapping in unit circle with a circle with which radius is R. If this radius is you want to map unit circle to unit circle put R is equal to 1 here. So, that you can check; so, this is the map we have, right. Now, let us use this map and then see how to solve and solve the example that we started with, ok.

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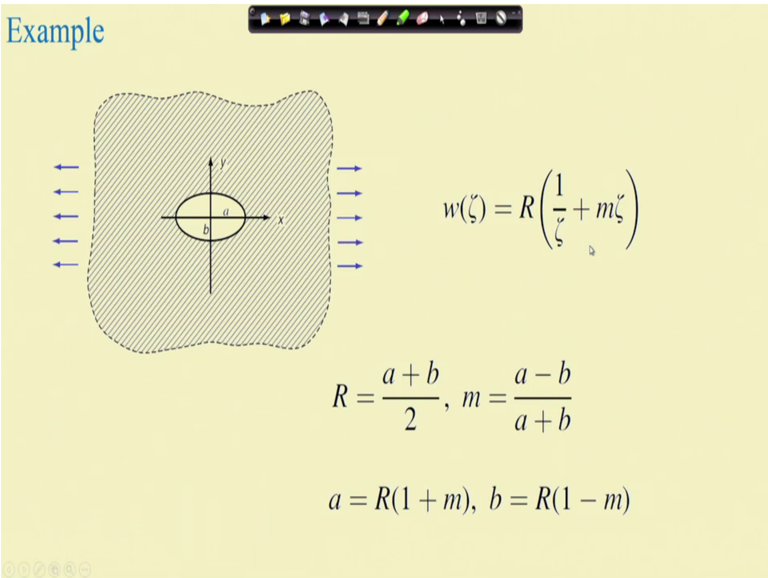

Now, take this example now. Just now we had this we have we discussed this map would discuss this map. So, this is the map we have this transformation we have. Now, in this case suppose these ellipse is your major axis is a and minor axis is b, then we just discuss here in this the previous example in previous slide, if you transform that through this through this transformation and then your major axis become R into 1 plus m and the minor axis become R into 1 minus m.

So, major axis is R into 1 plus m and minor axis R into 1 minus m, then if you can get it if you choose R is equal to this value and m is equal to this. So, you can get it. So, what we have is we have this is the problem and this is the map, which maps an unit circle in zeta plane and then we periphery is an ellipse another ellipse in Z plane and that ellipse the major and minor axis are a and b respectively which is related to R and m as this. Now, so, once we have this the next thing is we have to you recall the next thing is we have to substitute these all these expressions we have to substitute this map in that expression.

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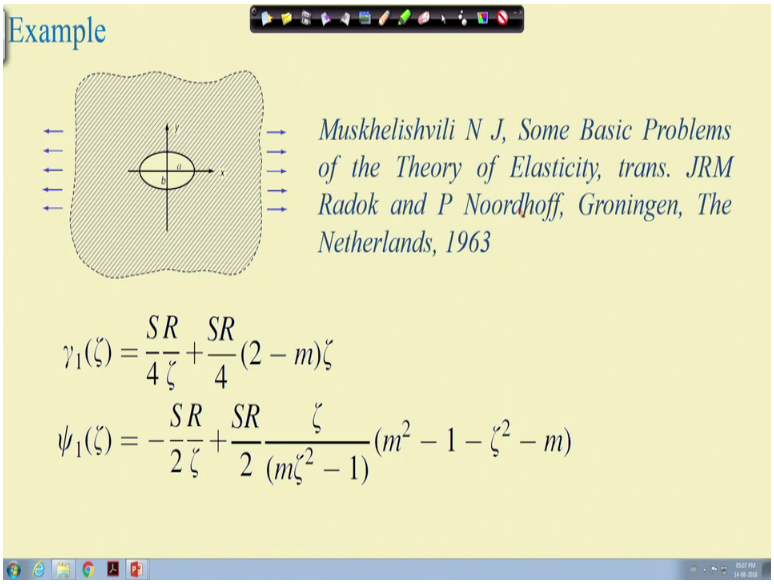

But, before that what we need is we need the we have to start with some potential right these came potential. So, it is start with now these potentials are this is your gamma, gamma, gamma 1 xi gamma 1 because if you recall when gamma is represented gamma is written in terms of Z, but when it is represented in terms of zeta it is coined as gamma 1 and similarly psi when represented in terms of zeta it was it was coined as psi 1.

So, these is that these are the potentials where S is the stress here S R is the radius of the of the R is the parameter that you use in the map and m is also the parameter zeta is an unit circle and then you use this use this potentials. Now, detail of these how to get this potential these tails of these are given in elasticity book and also the del is given in this reference. So, I will not discuss that detail, start with this potential ok. How to arrive this at this potential from the general expressions that you have to you have to find out, ok. Now, so, once we have this potential the next thing is this potential needs to be substituted on in that equation let us do that.

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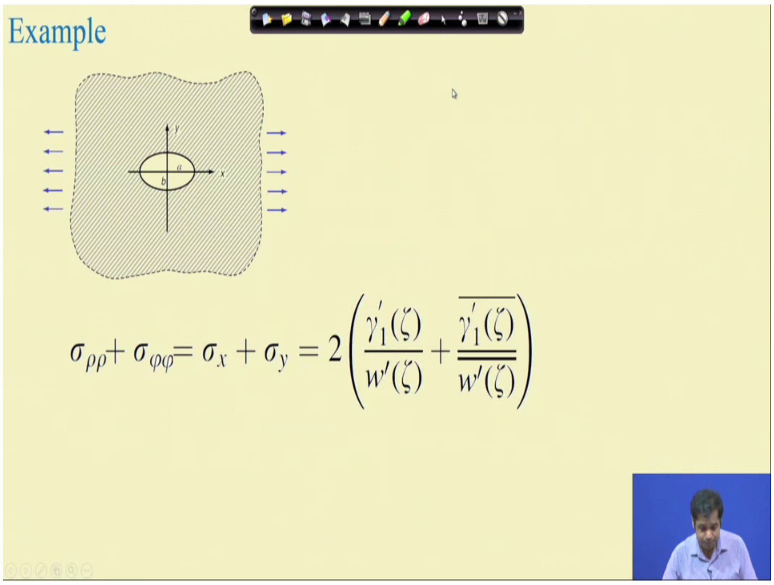

See if you do that this was the first equation, first equation written in terms of stresses and we derived it just now and then from we have gamma 1 and psi 1 is equal to this and then substitute gamma 1 and psi 1 in this equation and if you do that we get an expression we get an expression like this. Now, you see the since right hand side you have just the real part. So, this will be the real of this. In this case you have both imagine a real part, but if you compare the real part and real part. So, sigma sigma rho plus sigma phi or sigma rho rho plus sigma phi will be the real of this.

Now, what I want to know we want to find out let us find out what happens. We want right now we are interested to find out because we want to make a very important point and that is the reason why specifically we chose this example. Now, let us concentrate on the periphery of this, on this periphery and corresponding the on zeta plane the corresponding will be this that is in zeta plane, right on this period this is in unit circle this is in unit circle. Now, if we substitute if we know that here zeta is equal to zeta is equal to e to the power i phi because it is unit circle.

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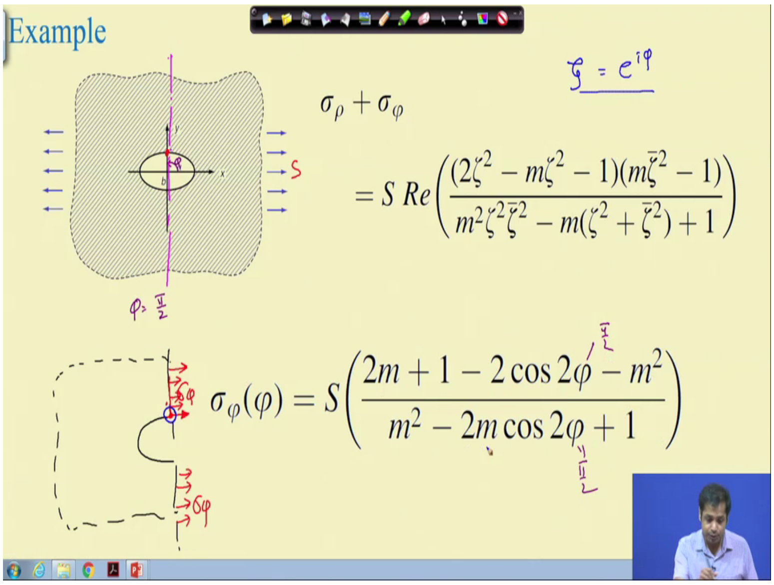

Now, if you substitute that in this expression in this zeta in this expression then what we get is we get we get an expression like this, ok. If you know if we substitute it this is the an expression we have not yet if you substitute that is your zeta is equal to zeta is equal to it is your i phi if you substitute that here we get an expression like this.

Now, here now suppose this load is acting in this direction load is acting in this direction. Now, we want to find out what is the stress at this point and stress at this point let us let us consider stress at this point. So, we want to find out stress only at this point the dot it this point and what is pi by 2 in what is phi in this case. So, this angle if this angle is phi, this angle is this angle is phi so; phi will be pi by 2, right.

So, let us substitute pi by 2 here if you substitute pi by 2 pi by 2 and pi by 2 what we get is the stress at this point. Which component of stress we get? See, look at you suppose you break it like this you cut you cut it at this point this is a section and then that section looks like this section looks like this. This is your outer periphery this is outer periphery and this is this is your this is the half circle and this is the section and we want stress at this point.

Now, what is sigma phi? Sigma phi is essentially stress in this direction sigma phi will be and sigma phi will be stressing not in this direction because just 1 minute. So, sigma phi you want to find out stress here to sigma phi will be stress in this direction. This is sigma phi. Now, so, if we take entire thing so, these will be sigma phi, this will be sigma phi sigma phi. Now, sigma phi at pi by 2 if we substitute at pi by 2 this gives you particularly stress at this point, stress at this point.

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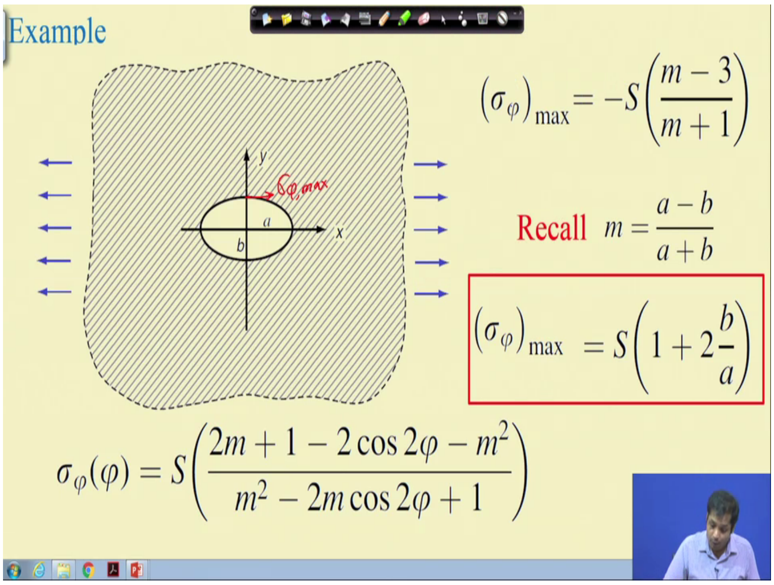

So, this stress if we substitute that this the value will be this. So, this is the value. Now, recall we have m is equal to a m is this we that is how we chose the map and if you substitute m is equal to here then this gives you sigma phi max; phi max means this is the stress at this point stress at this point means this is sigma phi max is equal to this.

Now, what it tells you? You see, now this expression is the stored this expression does not give you the distribution of stress. Please note, this expression gives you this stress at a given at a particular point and at a not all stresses, this gives you a particular component of stress sigma phi at particular point. So, this is the expression of the stress. Now, let us see what this expression tells, what story what is the story behind this expression.

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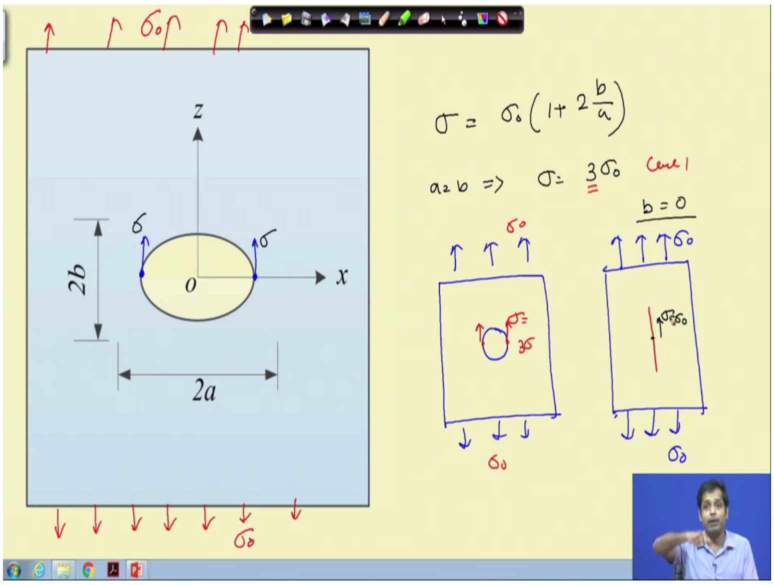

Now, now let us find out that expression. The expression is just now we have suppose if we have if we have a plane with elliptical hole like this and you apply some in plane load in plane load say which is equal to sigma in plane load sigma or S is used in the previous like this is sigma. Now, what this tells you then the stress if you take a point here at this point or at this point then these stress in this direction suppose that is sigma y or sigma let us write this is sigma 0 let us write. So, that we can this is sigma 0 this is sigma 0, then it tells you sigma this is sigma this is sigma, this sigma is equal to sigma 0 into 1 plus 2 b by a this is the expression.

Now, if you take if b is equal to if a is equal to b then what we have? If a is equal to b, then sigma is equal to 3 into sigma 0. It means that if we have a problem like this a problem like this a plate with in plane loading, but the plate is having a circular hole then these point is stress at this point these stress and these stress, if this is sigma and this is a sigma 0 this is a sigma 0 this sigma will be 3 into sigma. So, at this point the stress is three times the stress that you apply. So, this is called stress concentration, ok. In this case stress concentration factor is 3 mean the stress is increased. Now, the increased stress is three times the stress you have ok. So, this is one thing.

Now, if b is equal to 0, this is case 1 now, case 2 say b is equal to 0 b is equal to 0. If b is equal to 0 then what does it mean it means that if we have if we take a problem if we take a same problem like this we have a load like this like this and this is sigma, say sigma 0 this sigma 0 and then b is equal to 0 means this ellipse becomes a line and this what is the direction of the line this direction the line is this is it not?

So, we have this for case 1 it is a circular hole, for case 2 when b is equal to 0 means an ellipse becomes crack, but the what is the direction of the crack direction of the crack is along the line of application of the load. Now, if it is then at this point and at this point at this point if this stress is sigma this sigma is equal to sigma 0 means there is no change in stress. So, if that exercise you can do you take a paper piece of paper and in that piece of paper make some vertical cracks and then you apply a load like this and you will see that then what it tells you that is stress whatever stress you apply at this point you have the same stress. It means as if there is the paper will this plate in this case will be a as if there is no crack like this, stress distribution will be same as the case where there was no such crack. So, this is a case 2.

But, the most important thing is case 3 and what is case 3? So, let us write case 3 is case 3 is when once again let us write this sigma is equal to your sigma 0 into 1 plus 2 into b by a, right. Now, what is case 3? Case 3 is when a is equal to 0. What happens when a is equal to 0? It means that if you have a problem like this and which is subjected to a load like this load like this, ok. Just one point actually there is a there is a there is a swapping in the in the notation in the when we derived this expression in that case the ellipse was oriented in different in other way that is why we got 2 into b by a.

So, this case will be actually now in this case this will be a is equal to 0 this case will be a is equal to 0, not b ok. Now, then what happens then what happens. So, this is the case now what happens if you have if you take b is equal to 0, if you take b is equal to 0 then this becomes a crack like this. Please note, when we actually wrote those equations your ellipse is was in a different way. So, but now it is a different in different orientation. So, please when you when you interpret this what is the discussion please that that aspect keep in mind,, but the essence of the point that we want to make that will remain same.

Now, when your so, in this case in this case when a becomes 0 a becomes 0. So, the load is actually applying if we if we if we compare with the previous case, load is being applied like this the way we derived this equation let us consider the load in the same way load is applying like this, this is sigma 0.

Now, if it is then a is 0, then if we look at in a in other orientation then your this becomes a crack and then the stress at this point stress at this point and this point which is in this direction and in this direction, this becomes this stress that stress becomes infinity. Now, it means that if we have if we have a crack and if we have if we apply a load which is normal to the crack line then, at the tip of the crack these are the tip of the crack, at the tip of the crack this becomes infinite stress, becomes infinite. Just to make the point once again clear let me give you 2 2 let me give let me let me demonstrate you.

One problem was when you have when you have the second case was second case was this. You have a crack this is the crack and you are applying a load like this a load being applied in this direction in this direction. And, what happens in this case the stress at this point stress at this point and at this point the vertical state this stress at this point will be same as the apply stress as if there it will be a as if there is no opening. But, if your crack is like this if your crack is like this you look at this if your crack is like this and you apply a load this is the crack this is the crack and you apply the load. Then what happens, that at this point and at this point the tip of the crack the stress becomes infinite.

Now, this happens irrespective of the amount of load you apply, if there is a crack in a tiny amount of load across the perpendicular to the crack direction makes you makes the stress fill here at this at the tip of the crack causes infinite stress. Anytime any value of any nonzero value of sigma will cause infinite stress at the crack tip. Now, this is a very important observation and these we obtain from the from the this is the elasticity solution. Now, you see material has all the material has some finite strength, right. So, when a when a material fails if we apply a load on a material and as a consequence as the consequence sort of the of that load the stress develop. If you recall that stress is the response of the material to external threat. So, as a response to that threat, response to that load stresses develops in the material.

Now, the material has got some strength if the develops stress is more than that strength then the material fails. The material is safe when the stress, developed stress is less than the strength of the material right this is as simple as that, but then the observation that we have here is very very interesting and very very alarming as well. These observation tells you that if any tiny amount of load any tiny amount of load can cause infinite stress in material, if material has a crack and the load is perpendicular to the crack then the at the tip of the crack this load will cause in finite stress, but material cannot sustain any material cannot sustain in finite stress, then the material will fail, is it not? That is what the interpretation of this result means.

If we apply a load then this material fails. This material fails because they add this point and at this tip the stress induced stresses is very large material has got finite step expense right, but. But, if that is the criteria for failure that if the stress developed in the material is more than the strength and then fail if that is the criteria of the failure then as per this solution every material will fail if it experiences a tiny amount of crack, is it not?

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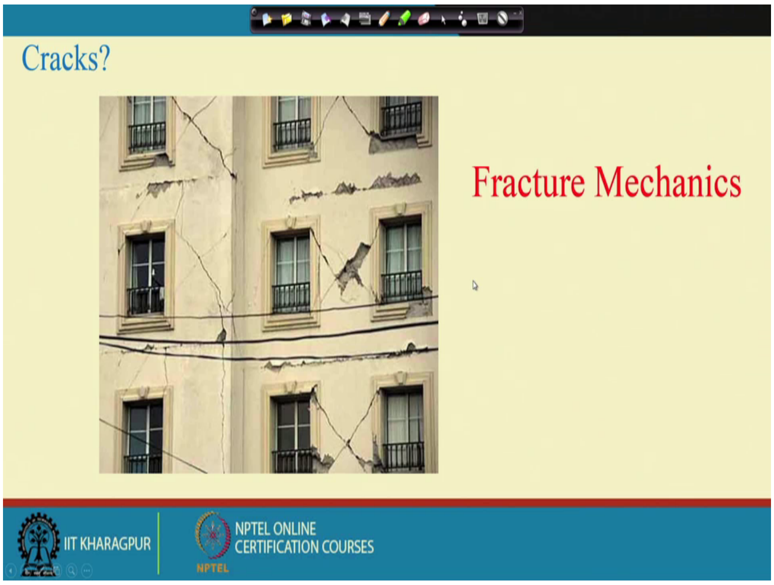

But, you see look at the real structure. You go out and look at the real structure just I give you an example you look at this building there are so many cracks in this building and in the building is still safe. The building is still serving its purpose. If not particularly for this building, you look around you see many things many things in the many things you see at your surrounding which has cracks a different orientation different lengths of cracks, but they are still taking loads, they are still safe. So, there is a contradiction, right.

The contradiction is the elasticity solution tells you stress is infinite at the tip of the crack and then if we go by the failure criteria stress is more than the strength then the material fails and then these two contradict with each other. There is something which is which is not properly reflected, right. Now, elasticity solution is there is no problem in the elasticity solution that is also correct and probably in certain circumstances the stress as a failure criteria this may be valid, but we cannot really say that stress can be taken as a failure criteria, is it not ?

Because, if we take stress as a failure criteria then there is a contradiction between the solution and the failure criteria and the real world structure which all do which all these structures have cracked, but they are still safe. Now, here we have we introduced a new topic called fracture mechanics. The motivation of this is so, what is the final outcome through this example is, see you have to you have to do these examples you have to follow these steps, if there is some as I said while deriving the equations there was a orientation different orientation taken for the ellipse and at the end of the day for demonstrating or the interpreting the results we took different orientation. So, those things you keep in mind while interpreting the entire discussion.

But, the essence is this slide that this takes us or this contradiction gives an opportunity to look beyond the stress as failure criteria, and there we have who introduced and it opens a new opens in window or door to a new topic or new branch in mechanics that is called fracture mechanics. This fracture mechanics is an tells you if there exists a crack in a material the way we have a crack here if there exists a crack in a material then how that crack behaves what is the mechanics of that crack how that crack propagates and is a an eventually leads to failure that entire discussion is given in this in this branch of mechanics, fracture mechanics. And, fracture mechanics require a separate course and I believe then you must be having course in fracture mechanics in your curriculum.

Now, this as precise all for this week; here the topic here the purpose of this week's lecture has been two purpose; one purpose is to introduce the complex very well approach and see that complex variable a complex variable approach is a very very powerful tools to solve many problems is elasticity and through one examples through a salt through solving one example another purpose of this week's lecture is to take you to a new topic called fracture mechanics.

So, with this I stop here today, from next class or next week we start a new topic which is Photo Elasticity. See you in the next week.

Thank you.