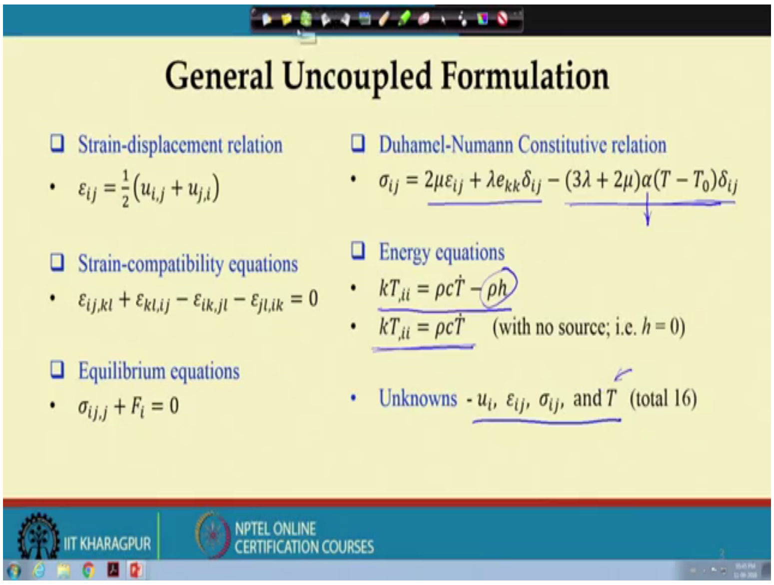
Theory of Elasticity
Prof. Biswanath Banerjee
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture – 51
Thermoelasticity (Contd.)

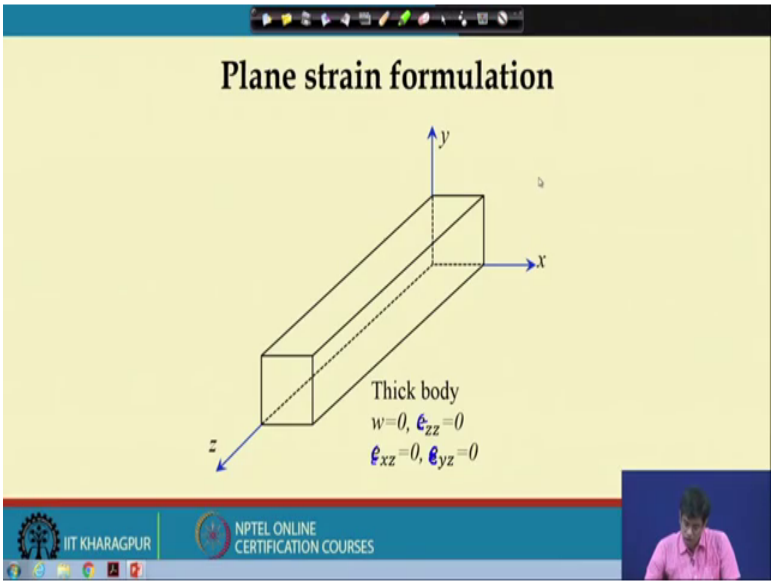
Welcome, so, we are discussing in the module 10 the Thermo elasticity. So, in the last class what we have derived essentially is what is the form for uncoupled thermo elasticity and specifically the Duhamel-Neumann relation for the thermo elastic material. So, the essentially the stress strain relation and we have discussed in terms of the strain energy which is not only dependent on strain now, it is dependent on the temperature also. So, this we have discussed in the previous lecture. So, now we know what are the thermo elastic deformation.

(Refer Slide Time: 01:05)



That is for an uncoupled formulation, what are the thermo elastic deformation and what are the stress strain relationship we know in terms of Duhamel relation.

(Refer Slide Time: 01:13)



So, we know the thermo elastic relation for instance the strain displacement relation and then we know the strain compatibility equation which is we studied earlier. For instance in a 2 dimensional case we know what is the strain compatibility equation for a, we also know the equilibrium equation. Here I am describing the body force in terms of F. So, and then the Duhamel- Neumann constitutive equation, in the constitutive equation we have this form, this is the mechanical stress and this is the thermal stress part and so.

Now, here the basic idea is your you have the additional material constant for a thermo elastic material which is the coefficient of thermal expansion of solid which is denoted in terms of alpha. For and this relation is for isotropic material, but for anisotropic material in different direction coefficient of thermal expansion can be different, for instance alpha x in the x direction, alpha y in the y direction, alpha z in the z direction, this is possible. Certainly, this will be then a diagonal matrix so, instead of a single coefficient alpha.

Now, for an energy coefficient that also we have seen in the last class that this is due to the thermodynamic theory we can now express it in this form, where the internal the rate of change of internal energy equals to rate of change of this c into temperature derivative. So, this is also we have seen the previous class.

Now, if there is a and this h rho h term is due to the heat source in the body. So, if there is no heat source this will be this and this comes from the Fourier law of heat conduction. So, that also you have seen in the previous classes.

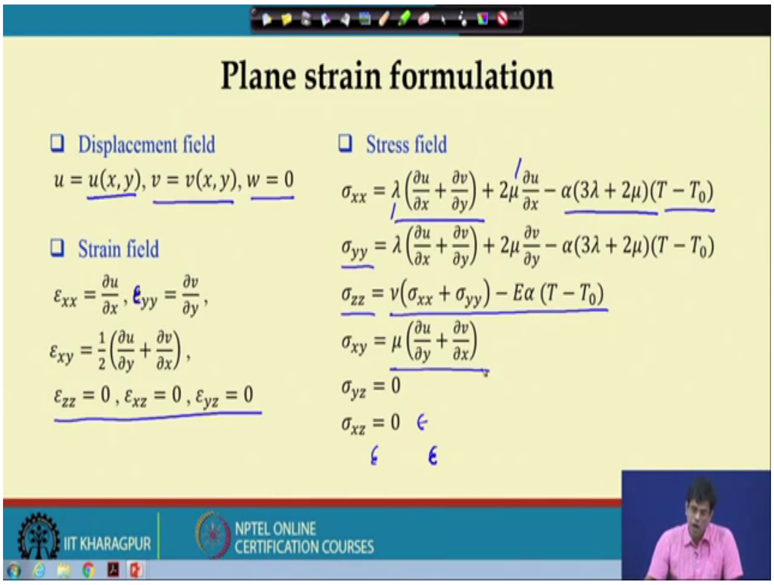
So, finally, the unknowns are these unknowns. So, one of them is temperature which is additional due to thermo elastic deformation. Now, since we are forming the uncoupled elasticity so, uncoupled elasticity means the heat conduction can be solved in a separately.

And the deformation or the Navier's equation of elasticity can be solved separately once we know the temperature change or the specifically delta T. So, this delta T is T minus T 0, T 0 is the initial temperature and T is the increasing temperature and this increasing temperature T in terms of x and y and z will be specified to you.

So, this becomes an input to the thermo elastic system thermo elastic problem. So, now here we will consider some simple cases like in the previous elastic purely elastic deformation where we have consider plane stress, plane strain case. So, in a plane strain case we know what is called plane strain by now. So, this is this will be epsilon so, this will be epsilon. So, here we know what is the meaning of plane stress so, this is a long body in a z direction.

So, we know that epsilon z z is] 0 epsilon x z and epsilon y z is 0 y z is also 0, but sigma z z is not 0 that we know from the plane stress assumption and also the w deformation is also considered as 0.

(Refer Slide Time: 05:11)

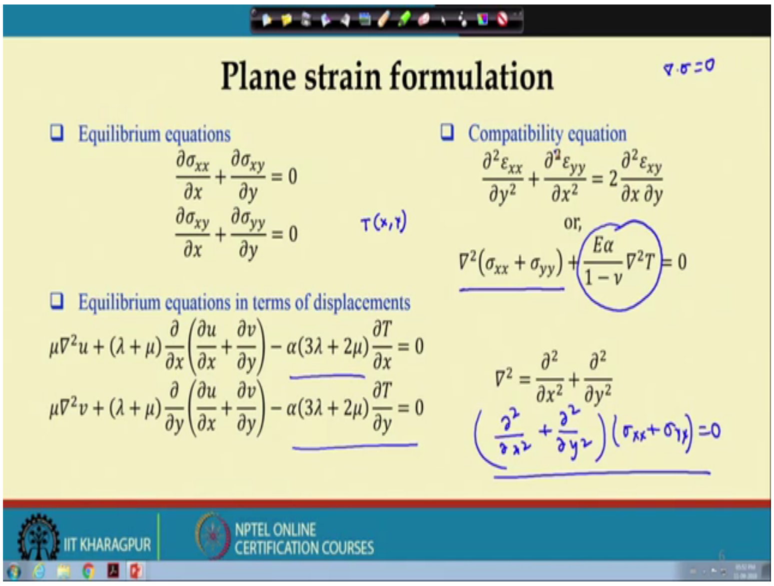


So, now this case what we can write is the displacement field. So, the displacement field I can write in terms of x and y term and then this displacement field is this u is a function of x y, v is a function of x y and w is 0. So, our usual strain field will be this so, del u del x is my epsilon x x, del v del y is epsilon y y and epsilon x y is the engineering shear strain is half of del u del y del u del v del x and this these are the plane stress assumption.

Now, similarly, to that if I now the do the stress field so, just I have to substitute to the Duhamel Neumann relation if I substitute. So, then this is my first lame constant and this is my second lame constant. So, lambda into this del u del x plus del u del v del y essentially epsilon x x plus epsilon y y and then 2 mu del u del x and then alpha 3 lambda plus 2 mu into T minus T 0.

Similarly, sigma y y and sigma z z for a plane strain is not 0 due to the Poisson's effect. So, this term will be there and this is your the sigma x y is the shear stress which is in terms of shear stress del u del y plus del v del x.

(Refer Slide Time: 06:51)



So, now in a plane strain formulation we will see the equilibrium equation. So, equilibrium equation again we know which is our basic Navier's equation of elasticity which is del dot sigma divergence of sigma equals to 0 for a static case if we do not consider the body force. So, that is del dot sigma equals to 0 if there is no body force.

Now, this quantity for a 2 dimensional body we know del sigma x x by del x plus del sigma x y by del y and so on. So, the equilibrium equation we can write in terms of displacements from the previous this thing. Because, we know the expression of stresses in terms of strains and then strains we can substitute the displacements. The expression of strains through the strain displacement relation we can substitute this in the stress relation and then substitute take the derivative. So, this is the final form of the equilibrium equation this equation in terms of displacement.

The important thing is here you see that del T del x and del T del y appears here that is temperature as a function of x y. So, here T is actually a function of function of x y. So, del T del x that is temperature distribution within the body will be space dependent so, this can be disappear.

Now, since we are following the uncoupled elasticity that is T is known to us. So, this will acts as a body force in the system. Because, you see T is known to you and T once it is known to you this will act as a body force in the system.

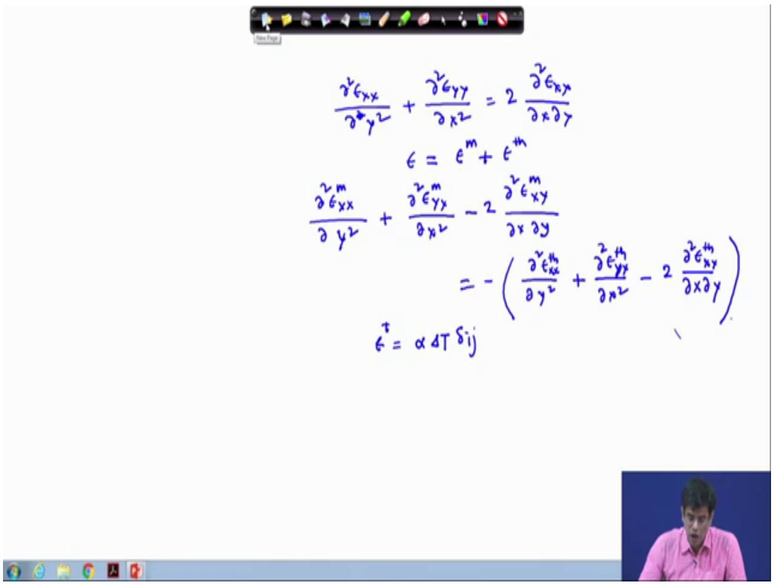
So, in a thermo elastic case even if you are following uncoupled thermo elasticity, even if you do not have the body force explicit body force thermo elastic deformation will due to this thermal change in temperature that is T since, T is a variable here. So, variable means it is especially vary so,; that means, it is not constant so, this quantities will be non zero. So, that means, this will give you additional body force.

Now, the compatibility equation, the compatibility equation already we know for a elastic case which is this. Now this can be easily seen that if I write this epsilon so, in terms of stresses so, there is this term is the compatibility equation for the purely elastic or the mechanical case that is del square by del x square plus del square del y square sigma x x plus sigma y y equals to 0. This is my compatibility equation for purely mechanical case.

Now, if we have the thermo elastic case if you remember we are also discussing this compatibility equation when you have a body force, explicit body force in the body. So, if you remember that then if you can easily derive this or you can just substitute this, this can be this extra term will be there in case of a purely thermo elastic uncoupled thermo elastic deformation.

Now, this can be this epsilon since we know the epsilon. So, this can also be written in terms of thermal strain which can be which I can show you in a detailed manner.

(Refer Slide Time: 10:51)



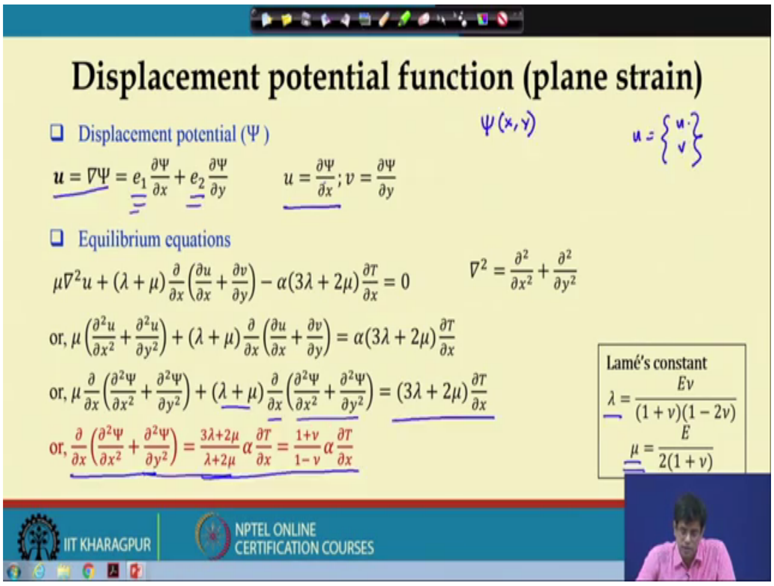
For instance, if you have this del square epsilon x x del square del y square plus del square epsilon y y by del x square equals to 2 del square epsilon x y into del x del y. Now, epsilon is actually your epsilon elastic strain or the mechanical strain and the thermal strain epsilon t h. So, I can just explain it to this in this form.

So, final the form will be del square epsilon x epsilon mechanical x x del y square plus del square epsilon y y mechanical by del x square plus or minus 2 del square epsilon m x y del x x y del x del y will be minus this quantity del square epsilon thermal del y square plus del square epsilon thermal by del x square minus 2 del square epsilon thermal by del x del y. So, this is epsilon x x epsilon y y and y y and this is epsilon x y.

So, now this equation you can modify if you know the thermal strain. For instance thermal strain we know that is epsilon thermal T is your alpha delta T. So, alpha is your you know that alpha here you know and so, epsilon and this is delta i j so; that means, this quantity will be 0, this quantity will be 0 finally. So, this is the compatibility equation in the thermal range.

Now, this we can finally, convert this thing into stresses and then stress we can write the compatibility equation in terms of stresses and this is the final form of the compatibility equation.

(Refer Slide Time: 13:39)



Now, here we have followed stress function approach in case of a plane stress problem and then also we can form the displacement function approach or displacement potential approach in the plane strains problem.

For instance, if I now assume that displacement potential so, for instance this displacement potential if I now write that my displacement potential is u is del u del x. So, I consider a function del psi which is a function of x y. So, which I am calling as similar to the stress function this I am considering the displacement potential.

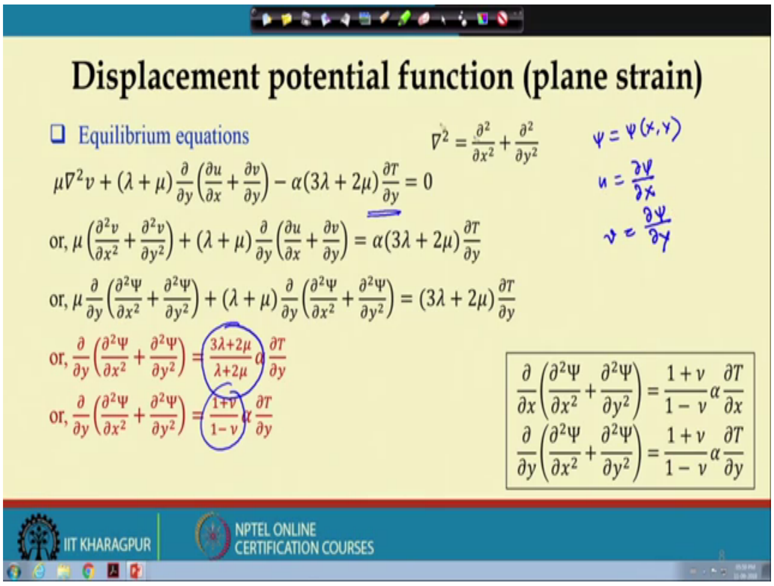
And now, this psi if I now assume that my u is actually del psi by del x and v is my del psi by del y then my displacement potential I can write in terms of e 1 and e 2 are the basis vectors in Cartesian co ordinate system. So, this is essentially u and v. So, essentially u is u 1 and u 2 or u v so, these are the thing.

Now, if I use this form then I get the equilibrium equation already we have seen in terms of displacement. So, the mu into del square u plus these quantities. So, u is essentially your del psi del x.

So, I will now finally, after some rearrangement I will substitute this psi this capital psi here. So, here if I substitute the psi so, my del square u by del x square plus del square u by del y square will be just second derivative of the psi. So, del square psi by del x square and so on so del del x of that. And then lambda plus mu is del del x of this quantity and then this is my equilibrium equation.

So, now if I write it in a proper manner then I can simply write it this quantity is actually lambda plus mu I can write it here and then I can arrive this form. So, where this lambda and mu is my first lame constant. So, now, once we know the displacement potential we can solve this problem, we can solve this equation to get the similar thing that we have solved the for stress function approach.

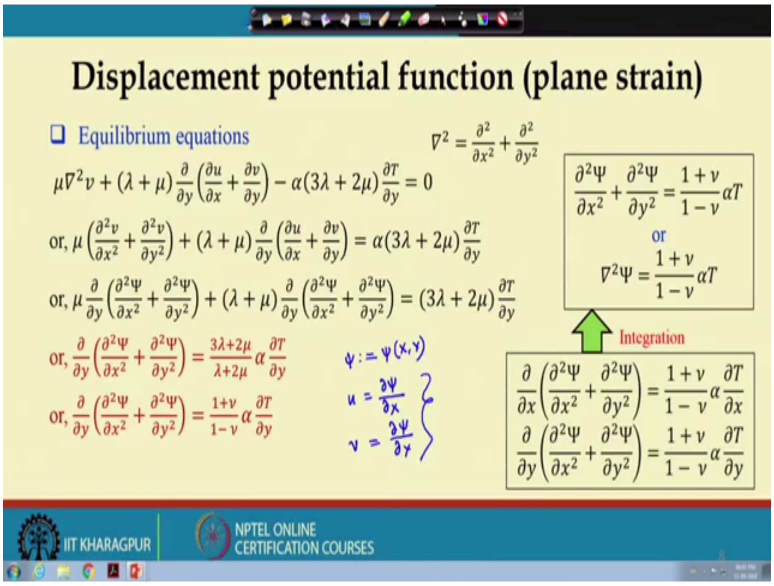
(Refer Slide Time: 16:35)



So, finally, this is the second equation, in the second equation again we can have this form so, this is. So, del t del y will be there so, that is a change and then this will be del square v. And then after some rearrangement I put the definition of psi that is I will write as u as del psi so, psi of x comma y and then del psi by del x and then v is your del psi by del y.

So, if I just now put this then I will have the similar to this form. So, after some modification or manipulation I can write the final form. So, my 2 equation essentially this is converted into this. So, if you write it in terms of e and mu this lame constant we know in terms of e and mu. So, this quantity can be achieved. So, this is my final form, this 2 equation in the previous slide we have seen this equation and this slide is this equation. So, this is my displacement function form of the this equation.

(Refer Slide Time: 18:03)



Now, so if I; now, write it in a if you see that this both the equation is actually very simple. So, I can integrate this both this 2 equation and after integration I will have a single equation which is del square psi equals to 1 minus nu by 1 minus nu 1 plus nu by 1 minus nu alpha T. So, this integration this is the function of x. So, this is the constant will be function of y.

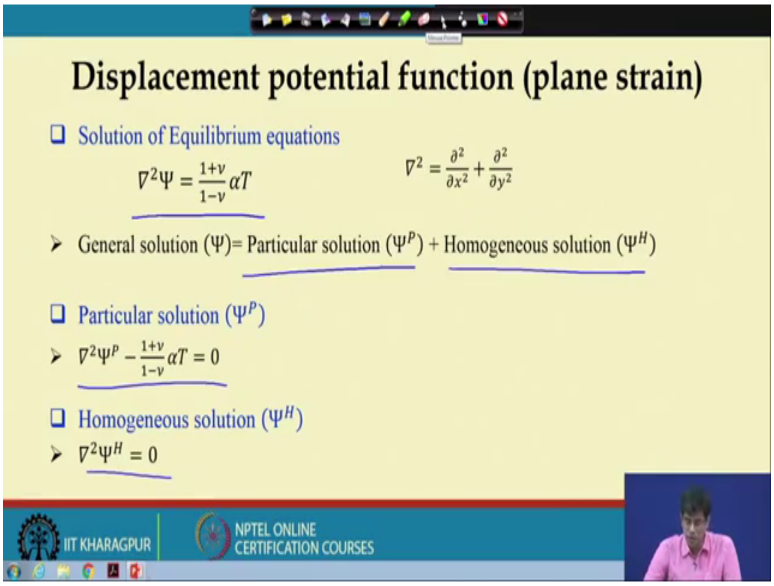
And then similarly function of x so I can this I left to you because we have seen several integrations in this course. So, this I left to you and this will be the final form of this thing, this you can derive very easily you just integrate it and you get this expression.

Now, you see this is a very simple equation. So, del square psi you know what is del square so, it is a Laplacian operator. So, this thing with this 1 plus nu by 1 minus nu alpha T, T is your temperature this is. So, this is a Poisson's equation Poisson's equation which we get from the displacement potential function. So, psi is a displacement potential function, if we know psi we can just take the derivative of psi and then get the displacement u and v.

So, the main advantage of this formulation is you basically arrive a single equation instead of 2 equation you will end up with the one Poisson's equation. And this poisons equation once you know the solution you essentially find out the displacement u and v, but displacement u and v as the specific form where the psi is a function of x y.

So, psi is a function of x y, but displacement u is actually del psi by del x and displacement v is del psi by del y. So, this specific form we have to this specific form have to be followed. Now, to follow this actually you need to satisfy the boundary conditions also. So, without boundary condition you cannot construct this potential function psi so that we will see when we will solve some of the examples. So, now this can be finally summarised.

(Refer Slide Time: 20:37)



So, this is a Poisson's equation so, in a general Poisson's equation you have 2 type means once I have a partial differential equation you have generally particular solution and the homogeneous solution. So, homogeneous solution you know you how to find out the homogeneous solution, you put the right hand side 0 and then find out what is the homogeneous solution. And then with the you introduce the right hand side and find out what is the your particular solution and the total solution will be a homogeneous solution plus particular solution.

But, the important thing here is if you look carefully that there is a if you use displacement potential function. So, you can essentially convert a single equation which is essentially a Poisson's equation with a right hand side or force vector which is essentially the thermal a part, due to the thermal part. So, T is a function of x y now so, this equation with the boundary condition we can solve for the general solution with the help of particular solution and homogeneous solution.

So, this is one approach of solving the thermo elastic problem or this approach is a mostly the analytical approach, but analytical approach may not be useful for all problems that we face in our system or society. So, we must follow the numerical approach and numerical approach we can use the finite element, finite difference or mesh free methods all those methods we can use to find out the even Fourier series we can sometimes use too semi analytical approach, Fourier series also we can use to find out the solution of such system.

So, the today's lecture we summarise this. So, we have seen what is the plane strain condition will lead to, what are the differential equations in terms of displacements, what are the equilibrium equation in terms of displacement. And what are the Duhamel Neumann constitutive relation in terms of constitutive relation and compliance relation we have seen and then what is the compatibility equation.

The compatibility equation will have a specific right hand side due to the thermal component then what we have done? We have just introduced the displacement potential function. So, which is essentially psi here and this psi is essentially del u del x and this u is actually del psi del x and v is essentially del psi del y.

Now, once we put this form into the displacement form of the equilibrium equation, we arrive a single equation for 2 separate equation by integrating it. Because, we have we are solving a 2 dimensional plane strain problem so, you have 2 equilibrium equation. So, once you write in terms of displacement of those 2 equilibrium equation and then you substitute this displacement potential function then you arrive finally, after integration arrive finally, a single differential equation or a single partial differential equation or 2 dimensional equation of which is this.

So, which is essentially a Poisson's equation with a right hand side and then the solution of the system with the boundary condition is particular solution and homogeneous solution. So, you know from your basic engineering mathematics how to solve these equations. So, we will solve some problem and we will show how it we can solve some problems of practical interest. So, I stop here today and in the next class actually we will consider the plane stress case.

Thank you.