Theory of Elasticity
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Lecture – 08
Concept of Stress and Strain (Contd.)

Hello everyone. Welcome to the second class of this week. You have seen last week what we discussed is, in any engineering design selection of material is the important step. And selection of a of a material for a particular design which is subjected to particular environment, the selection depends on the comprehensive understanding of a material or rather how a material behave when it is subjected to certain kind of environment, a certain kind of agitation or certain kind of threat.

So, understanding of that response is very important. Now, when a material is subjected to any kind of threat like human being, it also becomes stressed, right this is one way of responding to that threat stress. Now, in addition to that stress there is another kind of response, which is strain which has which is measurable which we can visualize unlike stress. And then what we were discussing in this week is, how to define this stress, how to define strain, how to represent them.

And then the discussion, we started in the last week is the concept of stress. We started with the definition of a definition that we had in our school days that stress is equal to force by area. But then we discussed we have to look beyond this definition. And because now we are not dealing with just one area or one force there are different different areas, different planes and different forces we have.

So, there are different components of stress. We also discussed that the stress is a point wise description of the response of a body. We cannot say that stress in a body is this is this statements is vague in the sense, that stress may be uniform may not be uniform. So, the correct statement would be stress at a point in a given body. And then so, stress is a point wise description.

Then we also discussed that a point can be idealized by a by an area or a volume depending on the dimension we are dealing with, at the infinitesimal area infinitesimal volume. And then what we discussed today is, we will start with fixing our coordinate system, and the sign convention, and then we will see: what are the different stress components we have at a point, and how to represent those stress components schematic through a schematic diagram. And then we also see that some of the properties of this some of the properties that there is stress at a point. Now, so, today's topic is the stress as a tensor, we will we will see that you have a brief description or tensor in the very first week of this course, ok. You see what we have what we done so far is, suppose we have any arbitrary 3 dimension any arbitrary body, suppose a body like this.

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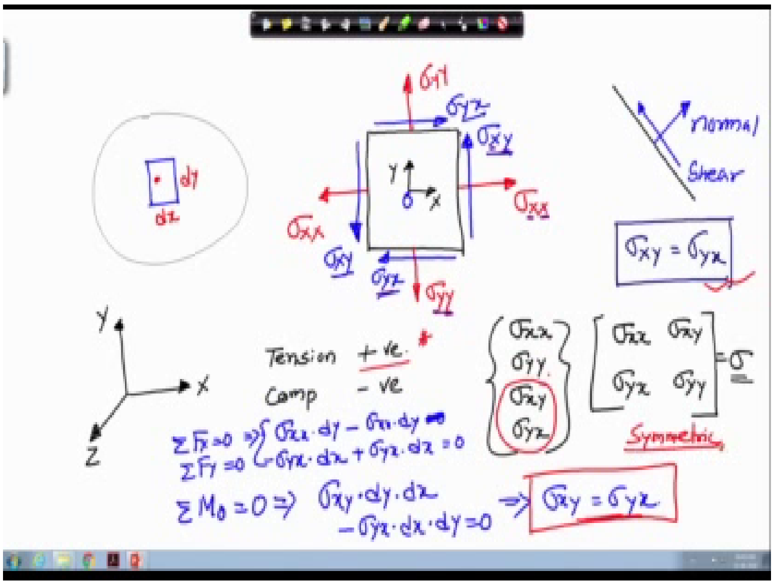

Any arbitrary body you take slightly because, any arbitrary body, and in that arbitrary body suppose we have a point here any point at this point, ok. Now, around this point we can have a small an infinitesimal area like this. And, then this is the representation of, this area is the representation of; this area is the representation of this point. So now, the stress, there are different components of stresses. So, stress at this point we can show graphically or schematically through this diagram.

Now, let us first fix our sign convention. The coordinate the sign convention that we use is this is the coordinate system. This is x, this is y. And if it is in 3D, then we have a z axis as we used, we will be using right hand thumb rule for this coordinate system. Now, once we have this coordinate system, we have to fix the sign convention for the stress, we have if it is tension we take tension as positive, and compression as negative, compression is negative. Now, you put a star mark here, because we will see that in addition to compression and tension there is one more deformation mode. We will not discuss that right now next class when we discuss about strain, then we will come to this point once again, ok.

Now, you see we also discussed last week that if we have a plane, if we have any arbitrary plane, if you recall if you have any arbitrary plane, then on this plane we have we can have a normal, we can have a normal component of stress, and we also have a tangential component of stress. So, this is normal component and this is shear component. Now, let us see then now if this is a point in 2 dimension, and this is the representation of the point on this diagram we will represent the 4 stresses. So, on this we have a stress component, and on this surface also we have stress component. It is a normal stress component. Now since this direction is x direction, we call it sigma x x, this is sigma x x.

Now, what this double x means will come shortly. Now then on these direction we have a normal stress, this is normal stress sigma y y on this also we have a normal stress which is sigma yy, ok. So, these are the normal stresses we have. As tension is positive as per our sign convention the tension positive means, the forces will be in this direction in x in along this in x direction, and the force will be along this in y direction.

So, the normal stress will cause elongation in the elongation in the body in that particular direction. That is the sign convention here. Now then we have shear component. Now shear we have sign convention is this. Now we have this is a shear component. And this shear component let us write it as sigma x, this is sigma xy. Now, again I will come to this point why it is sigma xx: what is the meaning of sigma xy and so on. And then we have here also we have the same sign same sigma x y. On this plane we have another shear component which is sigma y x, and on this plane also we have a shear component which is sigma yx, ok.

Now what is sigma xx and sigma yy means, sigma xy means? You see this is the let us draw the, this is the x direction and this is the y direction, right this is the y direction. Now sigma xx it is acting on this plane, right it is acting on a plane which is normal to x direction. So, this first x, the first x is stands for the plane on which this stress is acting. So, this x means that it is acting on a plane which is normal to x direction. And that the second x here is, it is acting on the plane, but on that plane what is the direction of the stress. On this plane the direction of this stress is x, ok. That is why this is sigma xx. Now what is sigma yy first y is the first y defines the on which plane this stress is acting.

For instance this is acting on this plane which is normal to y direction. So, this first y stands for that. And then the second y is on this plane the stress is in the y direction so, this is sigma yy. Now why it is sigma xy? Because the first x as I just now said it tells you on which plane it is acting it is acting on a plane, it is acting on this plane. So, it is acting on a plane which is normal to x direction, that is the first one is x. And then on this plane in which direction, this what is the direction of the stress, direction of this trace is sigma y.

So, similarly this is sigma xy and you see this is sigma yx. And why it is sigma yx? Because, the first one is y which is acting on up which is the depth which is which is the plane on which it is acting it is perpendicular to y direction and on this plane it is acting in x direction it is acting in x direction. That is why it is sigma yx, ok. There is we will we will see shortly that the sigma xy sigma sigma sigma xy is equal to sigma yx, ok.

But just bear with me for some time to come to this point, ok. So, this is this representation of stress at this point. Now depending this is a schematic representation of this stress. So, how many stress components we have now at this point? So, we have 2 normal stresses, we have 2 normal stresses which has sigma x sigma x and sigma y sigma y. There are 2 normal stresses and then 2 shear stresses, sigma x y and sigma yx.

Now so, this so, these are the 4 stresses we have. So, we can have we can represent this. See this is the graphical representation of the stress at that particular point. And this is the same stress reached written in a vector form. Now another way this stress can be represented and this representation is say tensile representation sigma xx, and then sigma xy, sigma xy, sigma yx and sigma yy, this is.

Now, the diagonal term in it written in terms of matrix; now sigma that the diagonal term of this matrix are the normal stresses sigma xx and sigma yy. And then the off diagonal terms are the shear stresses. Now so, this is sigma. So, this is the graphical representation of the stress at a point. This is all the stress components written in vector form. And then these stress component written in your matrix form in the tensile form.

Now, these are the stress components at a point in 2 dimensional space, now coming to this quickly at this point. You see, now we all we all are aware of a moment equilibrium equations, right equilibrium equation says that the body has to be equally the body is in equilibrium then every point in the body is in equilibrium. Every point the body is in equilibrium means static equilibrium; means that, the summation of forces in all direction should be equal to 0, and the summation of force summation of moments about any point should be equal to 0.

Now this body has to be in equilibrium, and this point is this point also has to be an equilibrium and if these are the stresses at this point let us see, whether this stress the distribution of stresses and the different components they satisfy the equilibrium condition or not. Suppose the distance suppose the this distance is dx, and this distance is dy. dx and dy you do not have to write. When you when you when you represent this stress the stress at a point in a graphical form dx and do I require to convert these stresses to forces, right.

Now, so, equilibrium conditions this the first equilibrium condition says that summation of forces, summation of summation of force in x direction is equal to 0. Let us see whether it is 0 or not. The summation of force this is this plane the length of this plane is dy. And sigma xx is the normal stress on this. So, sigma xx into dy, we can assume it is unique thickness sigma xx, and then minus this is in the negative in the other direction sigma xx into dy which is equal to 0.

So, the summation of x is equal to 0. Similar way you can show that summation of y that is also 0. Now you know sorry there is one more thing. Now this is this is another 2 forces we have. Another 2 forces we have; one is, this force in this direction and this force in this direction. So, sigma yx sigma yx this is acting on delta x minus and then plus sigma yx which is acting on delta x. So, that is equal to 0. So, summation of fx is equal to 0. Similarly, you can show that summation of summation of fy is equal to 0. Now we need to let us see whether the summation of moment is equal to 0 or not.

Now, if I take if this point is o, if I take moment summation of summation of moment about o, and equilibrium condition says that that has to be equal to 0. So, let us see if this is o, then these normal stresses will not cause the normal stresses will not will not contribute, because the this is taken from this is this moment is taken from this point the center point. And this normal stress multiplied by this area the force in x direction will pass through this center point will pass through the center point.

Therefore, this normal stress will not contribute to this to this equilibrium equation. Only the shear stresses will contribute to this equilibrium equation. Now you see now sigma xy this one and sigma xy this sigma xy will produce a produce an anti-clockwise couple. And this anti clockwise couple is sigma xy, the this is the stress and then this is acting on dy. So, multiply it by dy which becomes force, and then multiply it by the distance between these 2 force, these 2 forces which is dx this becomes an clockwise couple. Now similarly an anti-clockwise an anti clockwise couple similarly a clockwise couple will be created by sigma yx right. This is also sigma yx acting on dx and the distance between these 2 forces dy. So, there has to be equal to 0.

So, this becomes you check sigma xy is equal to sigma yx, right. Just now that is the thing we wrote here. So, this comes directly from the momentum moment balance equation right. So, if sigma xy is equal to sigma yx, then these representations is a vector representation it essentially becomes this becomes one. So, we have just 3 component of stresses, sigma xx, sigma yy and one shear stress; either you can write sigma xy or sigma yx.

And then in a tensile in that in this representation, this is diagonal with the normal stresses and the off diagonal which is the shear stresses. Now we have seen there is a off diagonal terms are same. So, these matrix become symmetric matrix. So, this is a symmetric symmetric max matrix, this is symmetric. So, this is the representation of stress in 2 dimensions. The same thing we can do in we can do in higher dimension. If you recall, that if it is higher dimension then any material any material point is now in the in the 2 dimension it was just a square.

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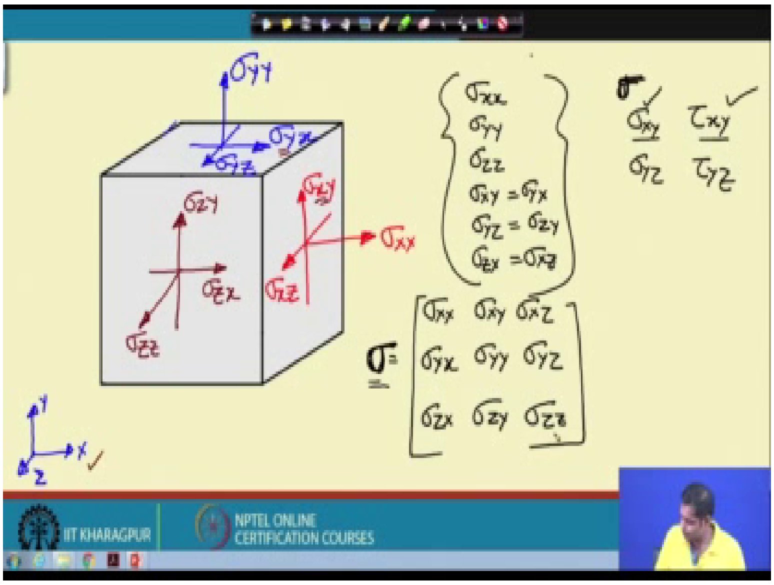

And in this case or rectangular; in these case, this representation is in 3 dimension. So, we have 3 planes on this. So now if it is in 3D this is the diagram. We write our sign our coordinate axis; coordinate axis is just now we showed. This is, let us use a different color for this, ok. So, coordinate axis is this is our x, this is y, this is x direction, this is y direction and this is our z direction. Now, so, let us show these stresses on this plane first. Now on this plane, we have one normal stress, one normal stress is this is sigma xx, why it is sigma xx? Because it is acting on a plane which; is perpendicular to x axis, and then in the direction of x.

So, sigma xx right, and then we have one more for one more stress a tangential stress this stress this is sigma xy sigma xy. Why it is again sigma xy? It is acting on a plane perpendicular to x axis and the direction of the stress is y in the y direction. And then we have another stress component like this. This is sigma xz. So, this is acting on a plane perpendicular to x direction and the direction of the stress is in z direction. So, this is on a plane which is perpendicular to x direction.

Let us do it for other plane. So, on this plane we have first a normal stress, normal stress is sigma yy. And then we have we have one tangent one shear stress which is sigma yx. Sigma yx and then another shear stress we have this one which is sigma yz, sigma yz. Sigma yz it is acting on a plane on the normal perpendicular normal to the z axis normal to the y axis and the direction of the stress is in the z direction.

Now, then on another force another stress on this on this plane we have we have a normal stress which is sigma zz, sigma zz, then we have a shear stress like this which is sigma zy, and then we have a shear stress like this which is sigma z zx, ok. Now similarly so, for clarity these stresses are shown only on this plane. Other 3 planes also we can have similar kind of stresses. Now, only thing is that your sign convention, your sign convention is tension is positive compression is negative this is your direction coordinate system. And then the clockwise couple is these for sigma xy the shear stress, if it is produced clockwise couple anti clockwise couple which is taken as positive.

But as I said please put a star mark here we will when you come to this strain different deformation mode will the sign convention will be clear. So, this is for the timing you take this sign convention for granted, ok. So, now following the same concept the previous. So, this is the graphical representation of the stress at a given point in 3 dimension. We have how many components of stresses we have, we have total 9 components of stresses.

3 normal stresses and then 6 shear stresses. 2 on each plane total 9 components, but again. If you, if we apply the symmetry the the moment equilibrium we can show that the shear stresses are same sigma xy and sigma yx they are same. So, essentially then we can have 3 normal stress and 3 normal stresses and 3 shear stress set a total 6 stress components. Now, these stress components are written in written in this form. So, first is normal stress sigma x x and then sigma yy and sigma zz. So, these are 3 normal stresses. Then we have sigma xy sigma yz and sigma zx, but remember sigma xy is equal to sigma yx is equal to sigma zy and is equal to sigma xz.

So, the symmetry and all will be discussed in detail in next 2 weeks. There we will see that there are in addition to this there are depending on the material there are many other kinds of symmetry exist. So, this is in this is in in a vector form. Now if I have to express this in a matrix form this sigma becomes. So, this is sigma xx, then sigma xy and then sigma xz; sigma yx, sigma yy, sigma yz, sigma zx, sigma zy and sigma zz, ok.

So, this is sigma written in this is the stress written in matrix form. Usually when you it might when you see any book, there are two things you can observe; one is whenever we write tensor, second order tensor then it is always written with a bold letter. So, when it is sigma is written in a bold letter. It is not possible here, but if you if you see book if it is it in a bold letter implicitly you can you can assume it is written in a tensorial form, ok. Another important thing is, in book you will find in some of the book instead of sigma xy it is written as all the shear components are written as tau. So, a normal components are sigma x sigma yy and sigma zz, but the shear components instead of sigma xy it is written tau xy instead of sigma yz it is written tau yz. They are not exactly same.

But what is the difference? What is the difference between sigma xx notation and tau xy notation again? We will discuss next class when we define different kinds of strain, specially the shear strain. So, this so, if we apply the symmetries of the if we apply this then this matrix is also become symmetric. And how many components we have here we have total a 6 component of stresses 3 normal stress and 3 shear stresses.

So, this is how stress at a point can be represented. Now let us try to understand there are some of the properties this stress tensor has. These properties are important, and what are those properties? But before coming to that property, you see if you recall that if we have to, if we have a coordinate axis like this, if we have a coordinate axis like; if you have a coordinate axis like this, and x xy and zz.

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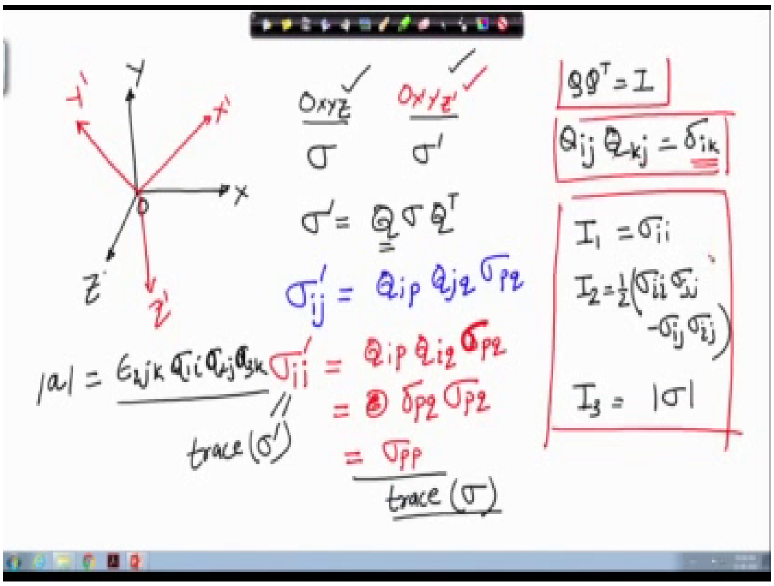

And then which is written as say Oxyz, ok. This is the origin o. And then if you have another coordinate axis, for instance, here itself we can write this is a this is x dash, this is y dash and this is z dash, ok. And, which is written as o x dash y dash z dash. So, if you if something is defined with respect to xyz coordinate system, xyz coordinate system means o this coordinate system. And then the same thing when we define, when we, when you define with respect to another coordinate system the prime coordinate system. So, we can transfer the we can we can have a transformation, we can define a transformation matrix which transform any quantity any tensor, any order of tensor from one coordinate system to another coordinate system, right?

So, if we have for instance, if we have if you have stress, say sigma is the stress which is defined in this coordinate system. And sigma dash which is the stress with prime which is defined with respect to the prime coordinate system. Then sigma dash then sigma if you remember the sigma dash can be represented as Q sigma Q transpose; where Q is the transformation matrix. And Q this transformation has certain properties, and the properties is Q Q transpose is equal to y. So, this transformation is orthogonal transformation.

Now, if I same thing if I write in indicial notation, it says that Q Q Q ij and Q kj kj that is become delta ik, when delta is the kronecker delta. So, this is the, this is this is written this is this property is the orthogonal property; the same thing written in this shear form as this, ok. Similarly, if I if I have to write this transformation in indicial form, this will be that sigma sigma ij dash, the the I th j ij components of tensor sigma dash. That will be equal to Q ip, please verify this do not take this for granted. Q I/ Q Q, and then sigma sigma pq, right sigma pq.

Now, this becomes this is j, ok. So, this is your, this is written in a initial in indicial form, right. Now why we are talking about transformation here? You see there are certain properties; for instance, in this case when we write sigma as this, when we write sigma is this. The stress at a given point is this. What does it mean? It means that stress at a given point expressed with respect to this coordinate system, with respect to this coordinate system. But if I change the coordinate system, then what will happen? Then my, these stress the representation of the stress with respect to a different coordinate system still. It can be represented it if it can be represented it in a tensorial in a tensor form.

But the components of this tensor will be different, because the reference, frame the reference axis with respect to which these components are calculated that is now different. Now so, it is very important when we say that 6 stress at a point is this. Or stress at a point either you either graphically or in a in a tensor form when you write stress, at a point it is very important to mention that stress is represented with respect to which coordinate.

If you change the coordinate that representation, due to the components of that tensor will be different. Now there are certain properties, where which do not do not change if you change your coordinate system. For instance, if I say that if I say that say volume of this object. Now, whether I take this as coordinate system or this as coordinate system or this as coordinate system, the volume of this object will remain same, isn't it ? So, the volume is now volume does not vary if we vary the coordinate system the volume remain unchanged. So, this property is called invariant. This property means something which is remain unchanged even if you change the coordinate system.

Now, what we have seen here? If we change the coordinate system, then the representation of this stress, this representation of the state the components wise everything will change. But there are certain properties in this stress tensor which remain unchanged. Even if we change the coordinate system and those properties those are called stress invariant. The invariant of stress tensor, there are 3 invariant of stress tensor. But before coming in coming to that point what are the 3 invariants of stress tensor, let us, suppose we know that consider the stress of this stress tensor. What is the stress of sigma?

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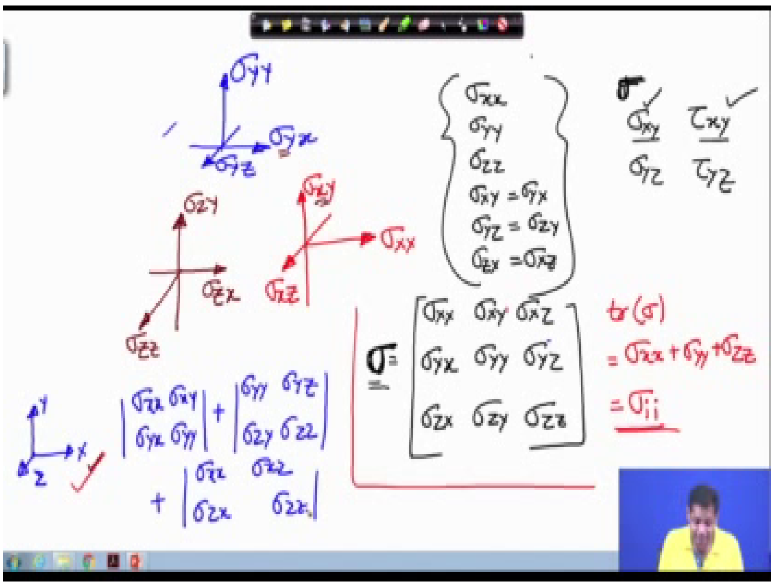

The stress of sigma stress of sigma is equal to sigma xx plus sigma yy plus sigma zz, right. Now, if I write that trace in initial notation, then the stress become this become sigma I sigma sigma ii; ii is the repeat repetition. So, the as per Einstein convention, this is summation. So, it means sigma 1 1 plus sigma 2 2 sigma 3 3 or sigma xx plus sigma yy plus sigma zz. So, the stress of this stress of a matrix can be written in indicial form is this. You are aware of this.

Now, let us see what happened. By intuition we can say that if we change the coordinate system the components will change, ok. But let us see what happens to this stress a stress of this stress tensor, whether they change or they remain same. Now to do this, let us write this is the initial this is the, this is indicial form of the transformation. Now, then with this indicial form; if I sigma ii dash which is the stress of the stress tensor written with respect to prime coordinate system. This will be Q I p Q j will become I here, because this is instead of j we are using I so, these become I then q delta p p q. Now, look at this property. This is the orthogonal property, if Q ij and Q kj it becomes delta ik.

So, ip and Q iq this become Q. So, this becomes delta pq, delta pq this is sigma, delta pq into sigma, pq. Now what is delta pq? Delta pq is the kronecker delta. This is 1 when p is equal to Q, and this is 0 when p not equal to Q. And then so, only if we take the non-0 term other terms will be 0 then it is p is repeating and the Q is repeating. Means, it will be summation over p and summation over Q. P takes value 1, 2, 3 or x yz Q takes value 1, 2, 3 or x y z. So, similarly we have many components here, but all will not be non-zero only non-zero will be when p is equal to Q. Means, it will be sigma 1 1 plus sigma 2 2 plus sigma 3 3 or sigma xx plus sigma yy plus sigma zz which can be written in indicial form as sigma pp, when p is equal to Q.

You see so, this is what? This is the, this is stress of stress of sigma, isn't it? And what is this? This is equal to stress of sigma prime, right. So, even the stress tensor the component wise, they are different now, with when they are expressed in different coordinate system, but if we take stress, the stress of sigma with respect to xyz. This coordinate system and the stress of sigma with respect to this coordinate system they all are they are same. Means if we change the coordinate system the stress of a stress tensor does not change it remain same.

So, this is a this is an invariant. So, this is called this is the first invariant of stress tensor which is written as I 1. The first invariant is sigma or ii written in indicial form stress of sigma is the first invariant. Similarly, other 2 invariants are I 2 other the second invariants are the cofactor the sum of cofactor of this matrix. What are the sum of cofactor? If we check the sum of cofactor will be; in this case, see if you take first this cofactor, the if we take this then cofactor will be if you write it here, then this will be write sigma xx, sigma xy and then sigma yx sigma yy. So, determinant of this plus another will be take this sigma yy, and then sigma yz sigma zy and sigma zz.

So, another one another one is this. And last one is plus if you take the first and third row and first and third column, this will be sigma xx and sigma xz sigma zx and sigma zz. So, this is sum of cofactor you check that. And you will see that sum of the cofactors of sigma is also an invariant, ok. Now this is can be written in indicial form as this; half of sigma I I sigma jj minus sigma ij sigma ij or sigma ji.

Since it is symmetric, that is why I can write sigma ij sigma iz. This is the second invariant of stress matrix. The third invariant is I 3 will be the determinant of sigma. Now determinant can be also written in terms of initial notation as if we if you recall the determinant of any matrix, suppose if a any matrix is a is determinant is written in terms of indicial notation as epsilon ijk, epsilon ijk which is the permutation sigma 1i sigma 2 k 2j, and sigma or sorry, a in this case a and a 3k.

So, this is how you can write the determinant of any matrix. So, this is 3 invariant. What are the physical interpretations of these 3 invariants? We will discuss. Now what we have done so far is, we have seen how to represent a stress at a point in 2 dimension as well as in 3 dimension. We will also have seen in 2 dimension there are 4 components of; we also discussed what are the sign convention and how these stresses are written, what are the meaning of when we write sigma xy sigma yx sigma yz and so on. We also discussed that that for 2 2 dimension there are 4 component of stress to normal stress and 2 shear stress, but 2 shear stress are same which can be shown from the equilibrium moment equilibrium.

So, essentially there are 3 component of stress. And similarly in 3 dimension there are 6 component of stress. Then hence, this comp this stress can be written in a graphical form or in a vector form or in a matrix form. And if you are in a tensor form, if you write in a tensor form then in a 3 dimensional a 3 dimensional space the stress is essentially the 2 or the second order tensor. Now, these second order tensor the stress has certain invariant. Invariant means, if you change the coordinate system then the component wise the stresses will be different, but there are certain combination of these components which remain same and these are common these combinations are called stress invariant there are 3 such combination. The first one is the stress of the stress tensor, second one is the sum of cofactor and third one is the determinant of stress.

So, these are 3 invariant those invariants can be written in indicial form as well. We stop here today. What we do in the next class is, next class we will continue this discussion on stress. And then see how to compute traction at a point, what is the meaning of traction, whether the traction at a point is again valid order or not, what is the difference between stress at the point and attraction at the point.

And then we will move on to define strain different strain components and then we also discuss when we define when we write the strain components. We will discuss the another sign convention that I asked you to put a star mark, the sign convention for the shear stress.

So, I will stop here today. See in the next class.

Thank you.