NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 02 : Snell's law, Plane wave reflection and transmission
Lecture 05: Snell's law, Transmission and Reflection Coefficients, Fermat's Principle, Huygen's Principle

CONCEPTS COVERED

- Snell's law
- Take-off angle
- Transmission and Reflection Coefficients
- Fermat's Principle
- Huygen's Principle
- Summary

Snell's law

Snell's law can be applied to more complex earth structure where it can be considered as it is formed from stack of layers. Earth may be approximated as onion shell where the properties may change as you go down but do not change laterally.

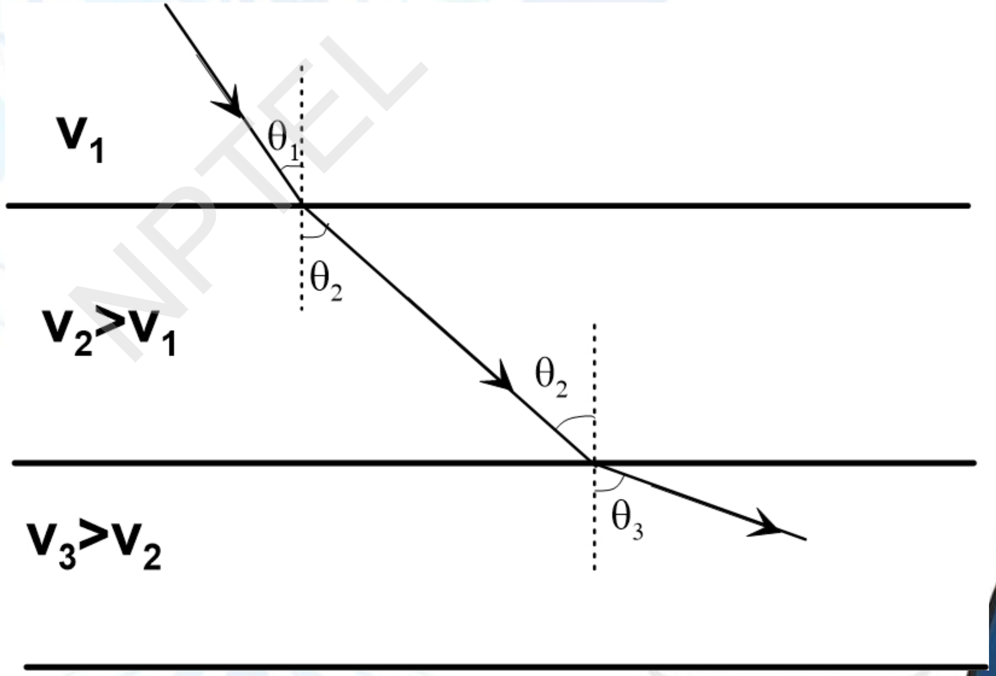
Consider an Earth that can be modeled by a series of isotropic homogeneous layers.

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} = \frac{\sin\theta_3}{v_3} = p = \frac{\sin\theta_i}{v_i}$$

also,

$$\frac{\sin\theta_{i+1}}{v_{i+1}} = \frac{\sin\theta_i}{v_i}$$

$$or \quad \theta_{i+1} = \sin^{-1}\left(\frac{v_{i+1}}{v_i}\sin\theta\right)$$



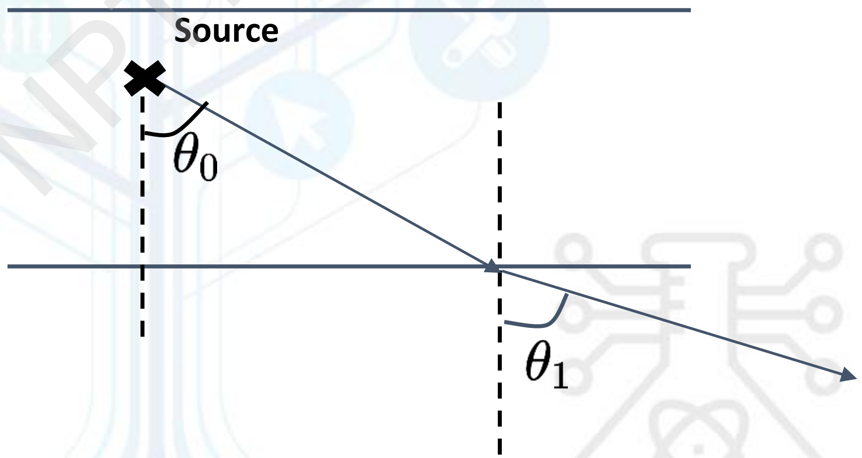
Cases:

1. $\sin\theta_i \dfrac{v_{i+1}}{v_i} = 1$ i.e., $\theta_{i+1} = 90^o$

2. $\left(\sin\theta_i \dfrac{v_{i+1}}{v_i}\right) > 1$ i.e., θ_{i+1} is undefined and all the energy will reflect back and no transmission will occur

Take-off angle:

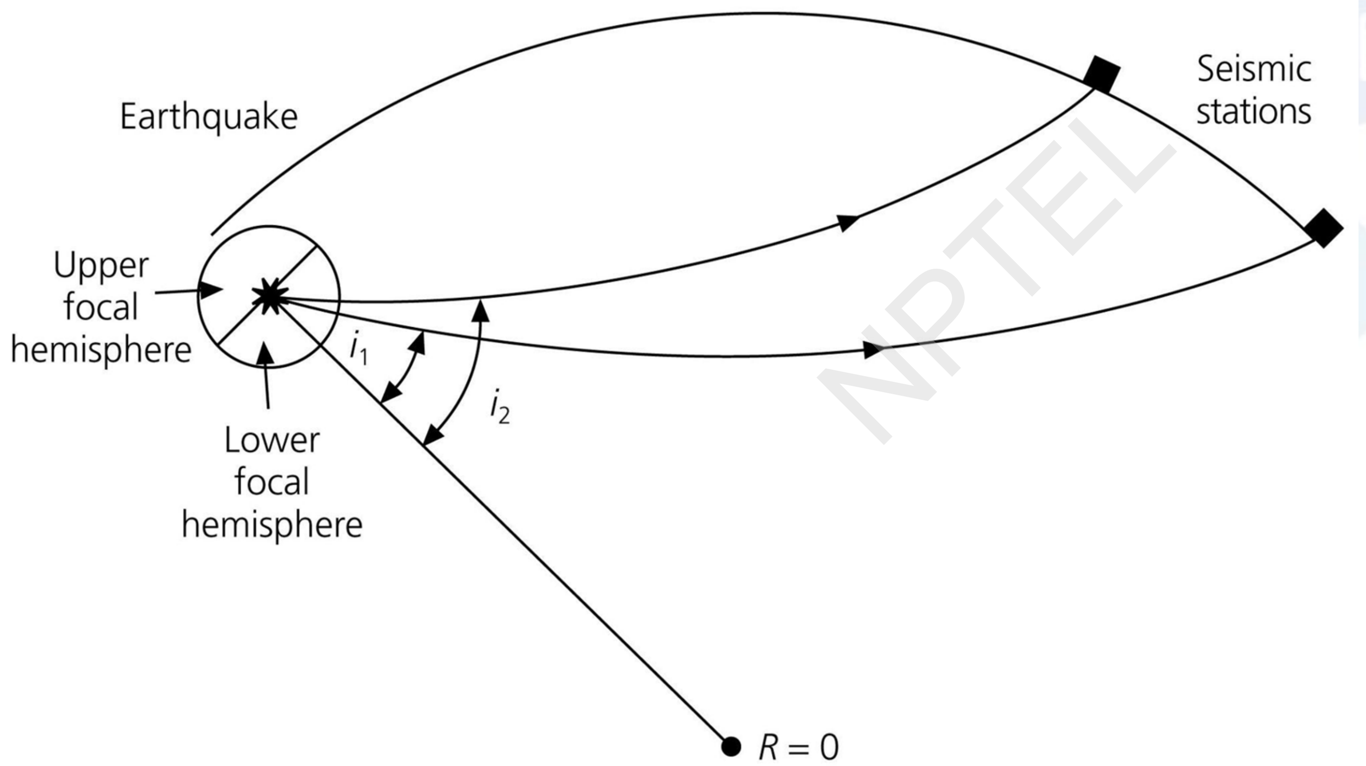
It is the angle at which ray leaves from the source measured in the plane containing source and receiver



θ_0 is take off angle

Take-off angle:

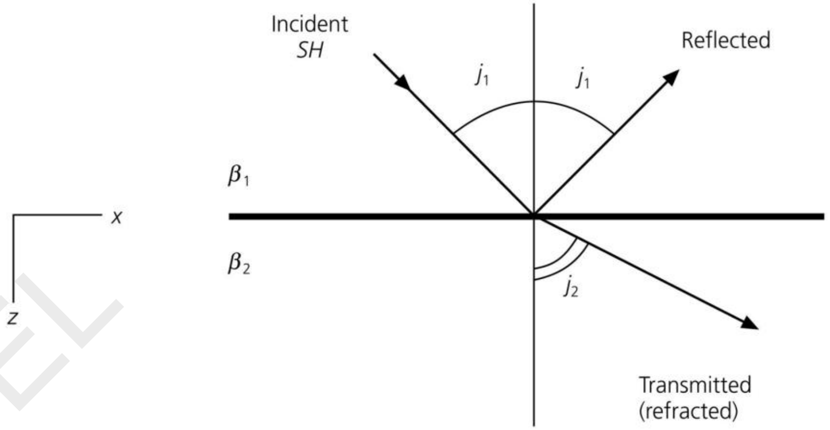
Note: How far the ray travels depends on its take-off angle; rays with larger take-off angles leaves the source closer to the horizontal and travel shorter distance than with smaller take-off angle



Reflection and Transmission

Let us consider a downgoing SH-wave which has displacement in Y-direction

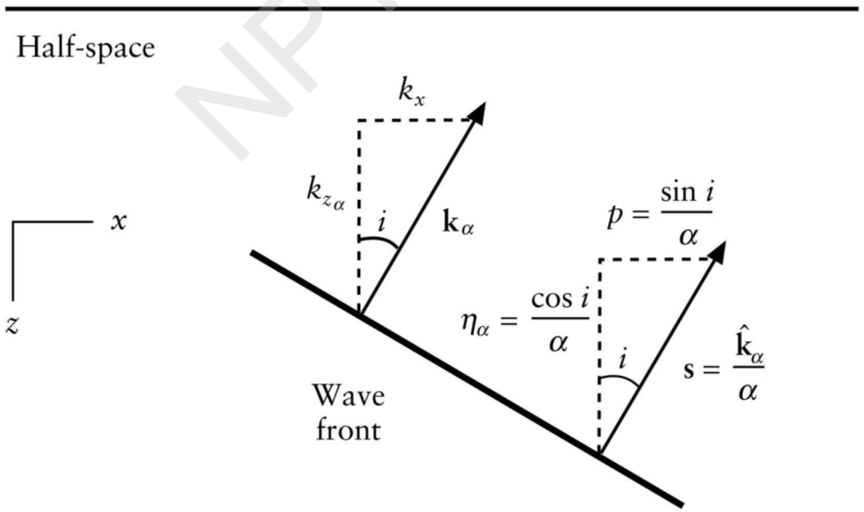
$$u_x = 0,\ u_y = u_y(x, z, t),\ u_z = 0$$



There are two boundary conditions need to be applied in order to see the waves behaviour at the interface (z=0)

1. **The wave is continuous**

1. **Tractions are equal**



Reflection and Transmission

Recall that: $p = \dfrac{\sin\theta}{v} = \dfrac{k_x}{\omega},\ \eta = \dfrac{\cos\theta}{v} = \dfrac{k_z}{\omega}$

The harmonic wave propagating in X-Z plane, can be described as:

$$
\begin{aligned}
u &= Ae^{i\left(\omega t - \vec{k}.\vec{x}\right)} \\
&= Ae^{i(\omega t - k_x x - k_z z)} \qquad \text{(downgoing)} \\
&= Ae^{i\omega(t - px - \eta z)} \\
&= Ae^{i\omega(t - \vec{s}.\vec{x})}
\end{aligned}
$$

So, for SH-wave vibrating in only Y-direction

$$u_y = Ae^{i(\omega t - k_x x - k_z z)}$$

Reflection and Transmission

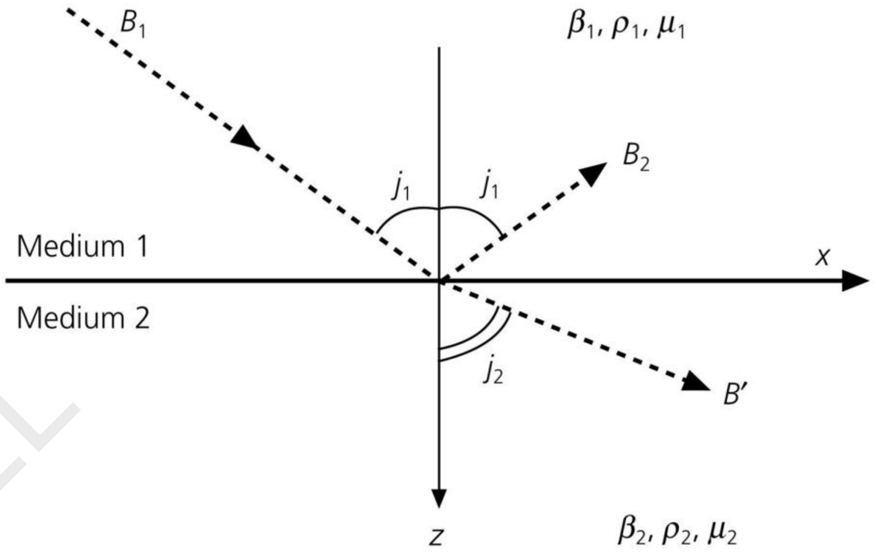
We can define one more parameter, the ratio of the vertical to horizontal wavenumbers

$$r_{\alpha,\beta} = \frac{k_{z_{\alpha,\beta}}}{k_x}$$

Vertical wavenumber $k_{z_{\alpha,\beta}}$ will change with the velocity of the medium while k_x will remain unchanged even though the medium change

In the top medium

$$u_y^-(x,z,t) = B_1 e^{i(\omega t - k_x x - k_z z)} + B_2 e^{i(\omega t - k_x x + k_z z)}$$
$$= B_1 e^{i\left(\omega t - k_x x - k_x r_{\beta_1} z\right)} + B_2 e^{i\left(\omega t - k_x x + k_x r_{\beta_1} z\right)}$$



In the bottom medium

$$u_y^+(x,z,t) = B' e^{i\left(\omega t - k_x x - k_x r_{\beta_2} z\right)}$$

As per the boundary condition,

$$u_y^-(x,0,t) = u_y^+(x,0,t)$$
$$(B_1 + B_2) e^{i(\omega t - k_x x)} = B' e^{i(\omega t - k_x x)}$$
$$B_1 + B_2 = B'$$

Also, $T_i^-(z=0) = T_i^+(z=0)$

At the boundary, $\hat{n} = (0,0,1)$ **and** $T_i = \sigma_{ij} n_j$

Since the normal is
$$\hat{n} = (0,0,1)$$
$$n_x = n_y = 0$$

This reduces the traction equivalent to

$$T_x = \sigma_{xz} n_z = \sigma_{xz} n_z$$
$$T_y = \sigma_{yz} n_z = \sigma_{yz}$$
$$T_z = \sigma_{zz} n_z = \sigma_{zz}$$

The SH-wave is vibrating in the Y-direction, the only non-zero stresses will be σ_{yz}, but let us show this. Now solve for traction at the interface

For an isotropic medium

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$$\sigma_{ij} = \begin{bmatrix} \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_x}{\partial x} & 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_y}{\partial y} & 2\mu\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & 2\mu\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_y}{\partial y} \end{bmatrix}$$

$$As\, u_x = u_z = 0 \; and\, u_y = u_y(x, z, t)$$

$$so,\; \frac{\partial u_x}{\partial x} \neq 0 \;\; \frac{\partial u_y}{\partial u_z} \neq 0 \;\; and\;\; others\, would\, be\, zero.$$

$$so,\; \sigma_{ij} = \begin{bmatrix} 0 & 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 0 \\ 2\mu\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 0 & 2\mu\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ 0 & 2\mu\frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & 0 \end{bmatrix}$$

$$T_x = \sigma_{xz} n_z = \sigma_{xz} = 0$$

$$T_y = \sigma_{yz} n_z = \sigma_{yz} = \mu \frac{\partial u_y}{\partial z}$$

$$T_z = \sigma_{zz} n_z = \sigma_{zz} = 0$$

Reflection and Transmission

$$\sigma_{yz}^{-}(z=0) = \mu \frac{\partial u_y^{-}}{\partial z} = i\mu_1 k_x r_{\beta_1}(B_2 - B_1)e^{i(\omega t - k_x x)}$$

$$\sigma_{yz}^{+}(z=0) = \mu \frac{\partial u_y^{+}}{\partial z} = i\mu_2 k_x r_{\beta_2} B' e^{i(\omega t - k_x x)}$$

$$\sigma_{yz}^{-}(z=0) = \sigma_{yz}^{+}(z=0)$$

$$\implies \mu_1 k_x r_{\beta_1}(B_2 - B_1) = \mu_2 k_x r_{\beta_2} B'$$

$$\text{and } B_1 + B_2 = B'$$

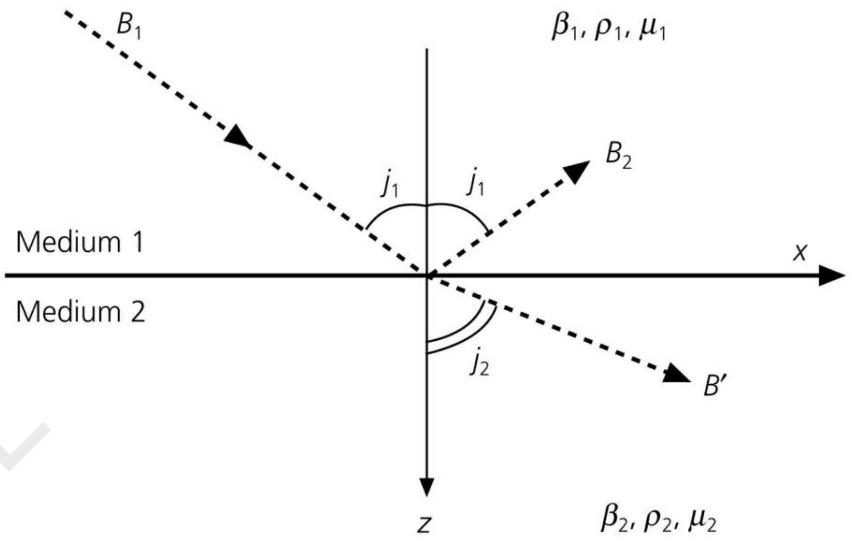
$$\text{If Reflection Coeff. } = R = \frac{B_2}{B_1}$$

$$\text{Transmission Coeff.} = T = \frac{B'}{B_1}$$

we get,

$$R = \frac{\mu_1 r_{\beta_1} - \mu_2 r_{\beta_2}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}}$$

$$T = \frac{2\mu_1 r_{\beta_1}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}}$$



It can be simplified by noting $r_{\beta_i} = \frac{c_x \cos j_i}{\beta_i}$

and we know that $\beta_i = \sqrt{\frac{\mu_i}{\rho_i}}$ or $\mu_i = \rho_i \beta_i^2$

then,

$$R = \frac{\rho_1 \beta_1 \cos j_1 - \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2}$$

$$T = \frac{2\rho_1 \beta_1 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2}$$

P-SV Waves

You can follow similar steps for P-SV waves which will give you something called Zoeppritz Equation

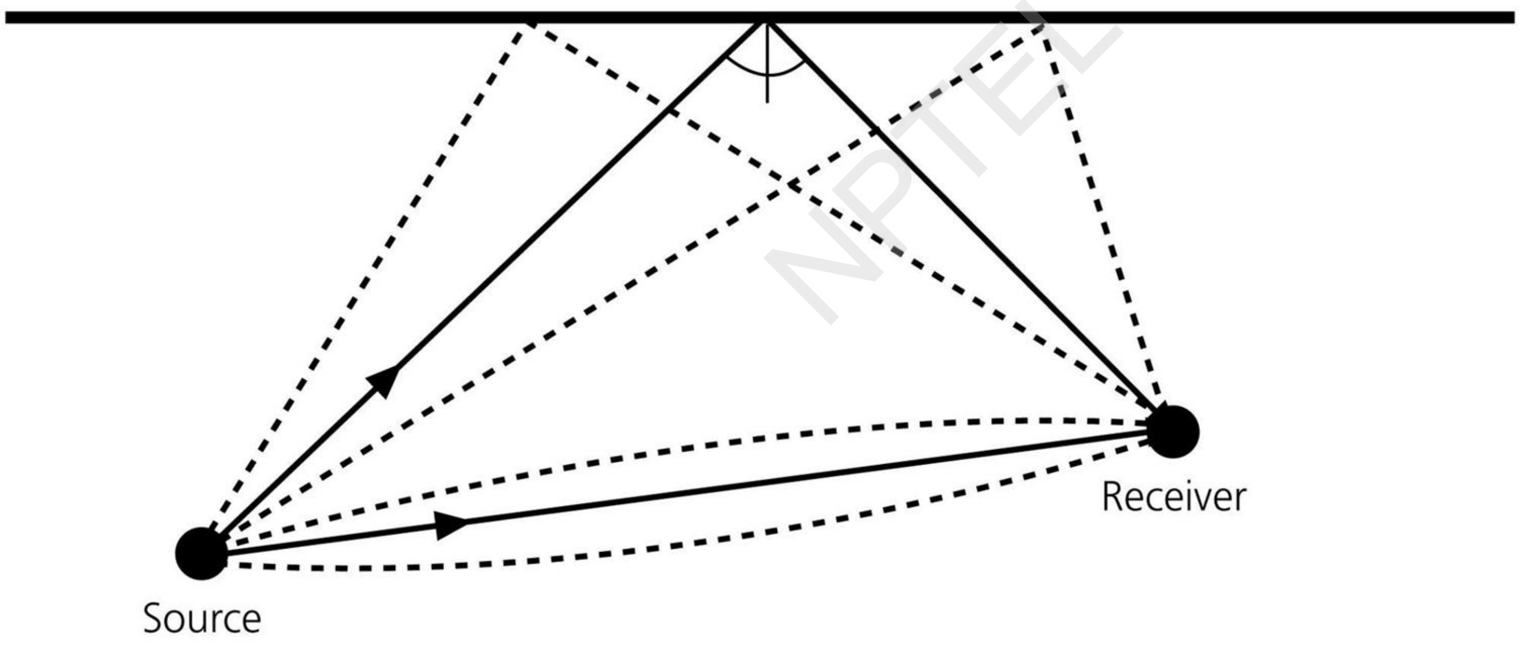
$$\begin{bmatrix} R_P \\ R_S \\ T_P \\ T_S \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 & \frac{V_{P_1}}{V_{S_1}}\cos\phi_1 & \frac{V_{P_1}}{V_{P_2}}\sin\theta_2 & \frac{V_{P_1}}{V_{S_1}}\cos\phi_2 \\ \cos\theta_1 & \frac{V_{P_1}}{V_{S_1}}\sin\phi_1 & \frac{V_{P_1}}{V_{P_2}}\cos\theta_2 & -\frac{V_{P_1}}{V_{S_1}}\sin\phi_2 \\ \sin 2\theta_1 & -\left(\frac{V_{P_1}}{V_{S_1}}\right)^2\cos 2\phi_1 & \frac{\rho_2 V_{S_2}^2 V_{P_1}^2}{\rho_1 V_{S_1}^2 V_{P_2}^2}\sin 2\theta_1 & \left(\frac{\rho_2}{\rho_1}\cdot\frac{V_{P_1}^2}{V_{S_1}^2}\right)\cos 2\phi_2 \\ -\cos 2\phi_1 & -\sin 2\phi_1 & \frac{\rho_2}{\rho_1}\cos 2\phi_2 & \frac{\rho_2}{\rho_1}\sin 2\phi_2 \end{bmatrix} \begin{bmatrix} \sin\theta_1 \\ \cos\theta_1 \\ \sin 2\theta_1 \\ \cos 2\phi_1 \end{bmatrix}$$

In geophysics and reflection seismology, the **Zoeppritz equations** are a set of equations that describe the partitioning of seismic wave energy at an interface, due to mode conversion. They are named after their author, the German geophysicist Karl Bernhard Zoeppritz, who died before they were published in 1919.

The equations are important in geophysics because they relate the amplitude of P-wave, incident upon a plane interface, and the amplitude of reflected and refracted P- and S-waves to the angle of incidence. They are the basis for investigating the factors affecting the amplitude of a returning seismic wave when the angle of incidence is altered — also known as amplitude versus offset analysis — which is a helpful technique in the detection of petroleum reservoirs. (Wikipedia)

Fermat’s Principle

It states that the ray paths between two points are those for which travel time is an extremum, which is a minimum or maximum travel time.

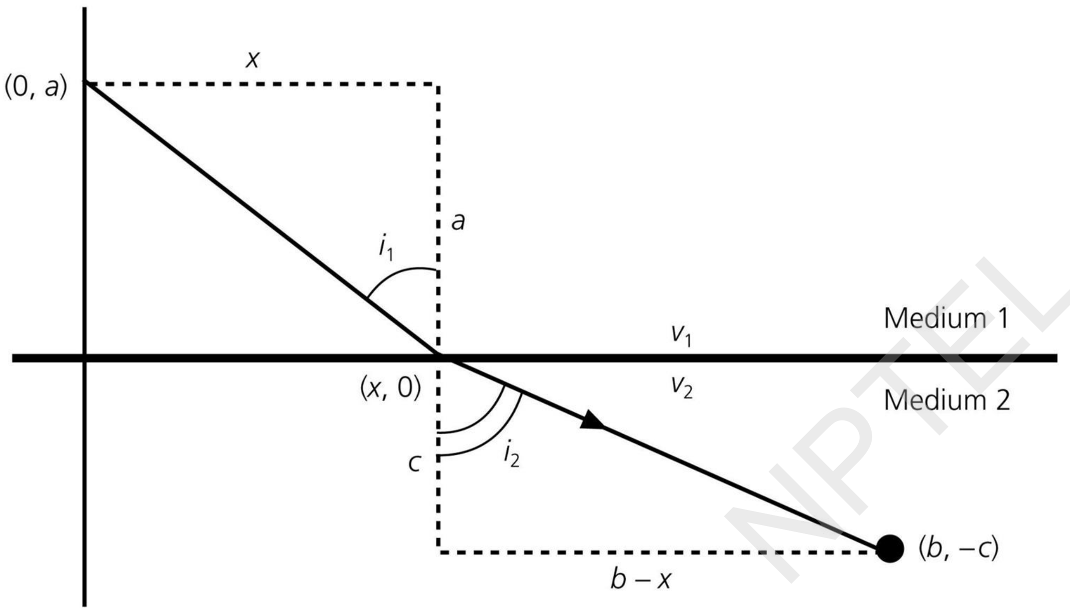


- The simplest case is two points in homogeneous half space; the time needed to traverse the straight line connecting the points is less than for adjacent paths.

- A second ray path for which the time is a minimum compared to adjacent path is that of the reflected ray satisfying Snell's Law.

- The direct ray path corresponds to an absolute minimum of the travel time, whereas the reflected ray corresponds to a local minimum.

Derivation of Snell's Law from Fermat's Principle



Consider the possible ray paths between the point (0,a) in medium 1, with velocity V_1 and the point (b,-c) in medium 2, with velocity V_2.

The travel time as a function of x is

$$T(x) = \frac{\sqrt{(a^2 + x^2)}}{v_1} + \frac{\sqrt{\left((b-x)^2 + c^2\right)}}{v_2}$$

To find the path for which the travel time is an extremum, we differentiate w.r.t x and set the result equal to zero.

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = \frac{x}{v_1 \cdot \sqrt{(a^2 + x^2)}} - \frac{(b-x)}{v_2 \cdot \sqrt{\left((b-x)^2 + c^2\right)}}$$

$$= \frac{\sin i_1}{v_1} - \frac{\sin i_2}{v_2} = 0$$

Which yields Snell's Law

$$\frac{\sin i_1}{v_1} = \frac{\sin i_2}{v_2}$$

Huygen's Principle

- In some applications, Geometric Ray Theory fails to explain all observation.

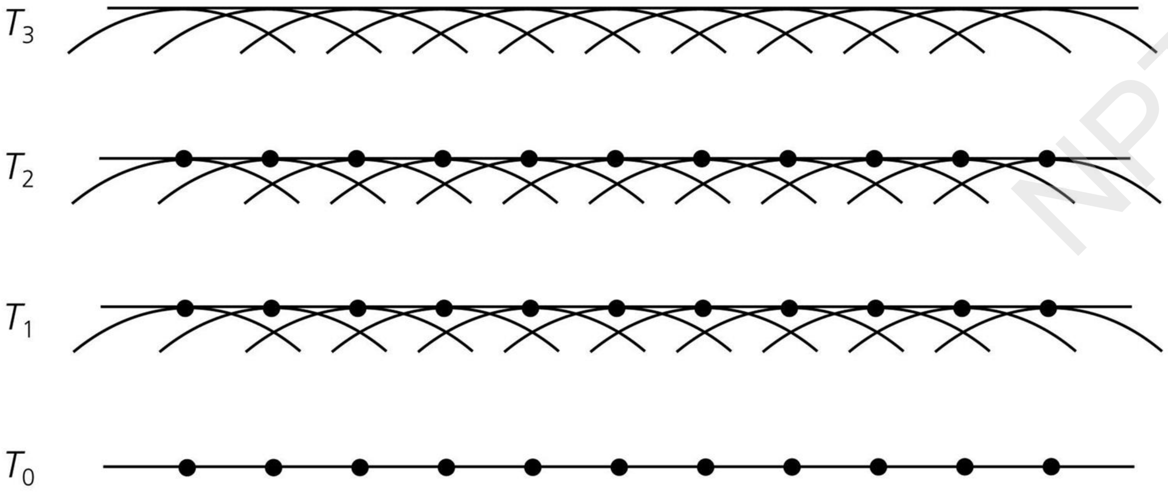
For example,

-Waves bend or diffract around the Earth's core and so reach places to which Snell's Law predicts no ray path.

-Although, similarly ray theory says that no energy is transmitted when a wave is incident on an interface at an angle greater than the critical angle, some energy in fact transmitted

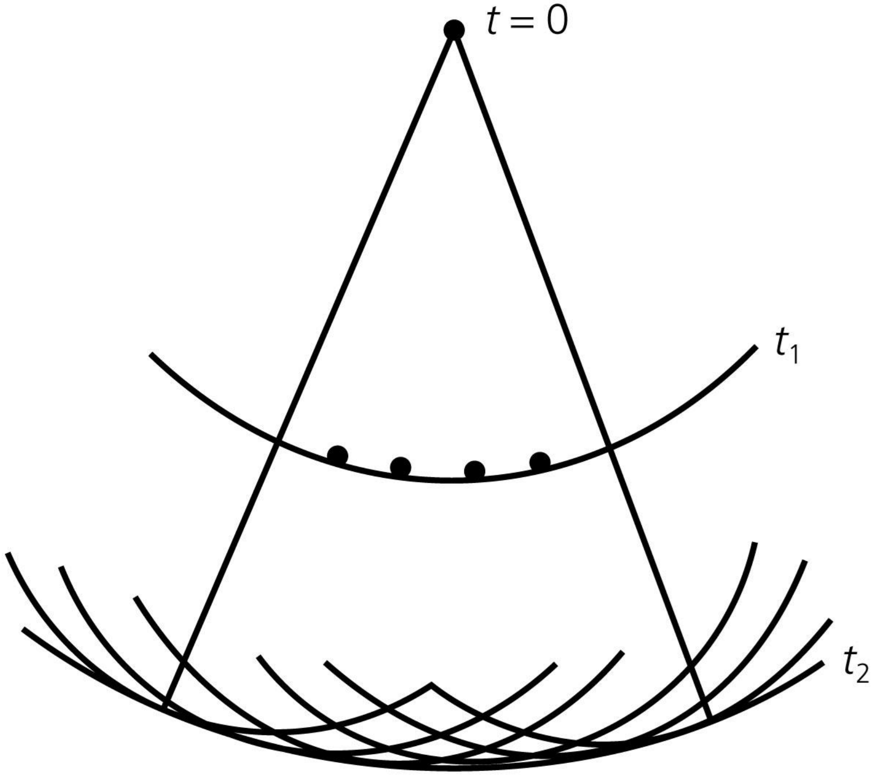
Huygen’s Principle

- **Huygen’s principle can explain some of these observations by considering each point on a wavefront is considered to be Huygen’s source that give rise to another circular wavefront. These wavefronts interfere constructively to give a circular wavefront, and interfere destructively everywhere else. In three dimensions, the wavefront are spherical.**



Demonstration of Huygen’s principle for the propagation of a straight wavefront. Successive wave fronts are generated by drawing a circular wave from each point on the previous wavefront and then drawing a line tangent to the circles. The circular wavefronts are assumed to interfere destructively everywhere else.

Huygen's Principle



How circular wave fronts can be generated by treating each point on the initial wavefront as a point source of wave energy.

Summary

-Snell's law, for a wave moving from one medium to another $\dfrac{\sin\theta}{v_1} = \text{constant}$

-Take off angle is the angle at which ray leaves from the source measured in the plane containing source and receiver

-In order to derive the reflection and transmission coefficients, two boundary conditions has been applied: 1. The wave is continuous; 2. Tractions are equal

$$R = \frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

$$T = \frac{2\rho_1\beta_1 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}$$

-Fermat's Principle, it states that the ray paths between two points are those for which travel time is an extremum, which is a minimum or maximum travel time.

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THANK YOU !