NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 03 : Surface Waves and Dispersion
Lecture 02: Rayleigh Waves and its particle motion

CONCEPTS COVERED

- Rayleigh waves
- Particle motion of Rayleigh waves
- Summary

Rayleigh Waves

Rayleigh waves are a combination of trapped or evanescent P and SV waves that exist at the top of a halfspace (where there is a 0 traction surface condition).

Let’s consider the P and SV solutions to the wave equation. These are potentials.

$$\textbf{P-wave}: \ \phi = Ae^{-i(\omega t - k_x x - k_x r_\alpha z)}$$

$$\textbf{SV} - \textbf{wave}: \ \Psi = Be^{-i(\omega t - k_x x - k_x r_\beta z)}$$

Recall r is the ratio of the vertical to horizontal wave numbers, and c_x is the apparent horizontal velocity of a wave.

$$\begin{aligned} r_\alpha &= \frac{k_{z_\alpha}}{k_x} = c_x \eta_\alpha \\ &= c_x \left(\frac{1}{\alpha^2} - p^2 \right)^{1/2} \\ &= \left(\frac{c_x^2}{\alpha^2} - c_x^2 p^2 \right)^{1/2} \end{aligned}$$

As we know,

$$p = \frac{k_x}{\omega} = \frac{1}{c_x}$$

$$\implies r_\alpha = \left(\frac{c_x^2}{\alpha^2} - 1\right)^{1/2}, \text{ likewise for } r_\beta$$

$$r_\beta = \left(\frac{c_x^2}{\beta^2} - 1\right)^{1/2}$$

Let's assume r_α and r_β are imaginary, analogous to how we dealt with evanescent waves in the last lecture. For this condition to hold: $c_x < \beta < \alpha$

Thus,

$$r_\alpha = -i\left(1 - \frac{c_x^2}{\alpha^2}\right)^{1/2}, \text{ and } r_\beta = -i\left(1 - \frac{c_x^2}{\beta^2}\right)^{1/2}$$

Another condition that must hold at a free surface (like the surface of the Earth) is that shear tractions must be 0.

So,

$$\sigma_{xz} = \sigma_{zz} = 0$$

We can calculate these from

$$\sigma_{ij=\lambda\theta\delta_{ij}+2\mu e_{ij}}$$

Now, relate the stress to deformation

$$\sigma_{xz}(x,0,t) = 0 = 2\mu e_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\sigma_{zz}(x,0,t) = \lambda\theta + 2\mu e_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z}$$

For the next step, we will write “u” in terms of the potentials, for a P-SV system. To do this, we’ll have to use eq. 2.5.5 in the book, which was derived as part of the potential field treatment of Snell’s Law, which we skipped. In this, for a horizontal boundary,

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \Psi}{\partial z},\ u_z = \frac{\partial \phi}{\partial z} - \frac{\partial \Psi}{\partial x}$$

This can be plugged into the equations for stress to get

$$\sigma_{xz}(x,0,t) = 0 = 2\mu e_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \mu\left(\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2}\right)$$

$$\sigma_{zz}(x,0,t) = \lambda\theta + 2\mu e_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2\mu\left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial x \partial z}\right)$$

Now add in the equations for the two potentials

$$\textbf{P-wave :}\ \phi = Ae^{-i(\omega t - k_x x - k_x r_\alpha z)}$$

$$\textbf{SV} - \textbf{wave :}\ \Psi = Be^{-i(\omega t - k_x x - k_x r_\beta z)}$$

and take the respective derivatives.

Here's I'll skip the details:

$$\sigma_{xz}(x,0,t) = 0 = 2r_\alpha A = \left(1 - r_\beta^2\right)\beta$$

$$\sigma_{zz}(x,0,t) = 0 = \left[\lambda\left(1 + r_\alpha^2\right) + 2\mu r_\alpha^2\right]A + 2\mu r_\beta B$$

Since, $\left(1 + r_\alpha^2\right) = \dfrac{c_x^2}{\alpha^2}$ and $\rho\alpha^2 = \lambda + 2\mu$ **and** $\mu = \rho\beta^2$, we can substitute in to get

$$2\left(\frac{c_x^2}{\alpha^2} - 1\right)^{1/2} A + \left(2 - \frac{c_x^2}{\beta^2}\right)B = 0$$

$$\left(\frac{c_x^2}{\beta^2} - 2\right)A + 2\left(\frac{c_x^2}{\beta^2} - 1\right)^{1/2} = 0$$

These equations have a non-trivial solution if the determinant is 0, the resulting equation is called the Rayleigh function:

$$\left(2 - \frac{c_x^2}{\beta^2}\right)^2 + 4\left(\frac{c_x^2}{\beta^2} - 1\right)^{1/2}\left(\frac{c_x^2}{\alpha^2} - 1\right)^{1/2} = 0$$

Another way you might see this equation is like this:

$$\left(2p^2 - \frac{1}{\beta}\right)^2 + 4p^2\left(p^2 - \frac{1}{\alpha^2}\right)^{\frac{1}{2}}\left(p^2 - \frac{1}{\beta^2}\right)^{\frac{1}{2}} = 0$$

To simplify this, let’s assume a poisson solid (υ=0.25): then $\frac{\alpha^2}{\beta^2} = 3$. The determinant becomes:

$$\left(\frac{c_x^2}{\beta^2}\right)\left[\frac{c_x^6}{\beta^6} - 8\frac{c_x^4}{\beta^4} + \left(\frac{56}{3}\right)\frac{c_x^2}{\beta^2} - \frac{32}{3}\right] = 0$$

There are three roots to this equation, and only one satisfies the requirement that $c_x < \beta < \alpha$

$$c_x = \left(2 - \frac{2}{\sqrt{3}}\right)\beta = 0.92\beta$$

Now that we know c_x , we can solve for B in terms of A (there are the amplitudes from the potentials, back at the beginning of this derivation):

$$B = A \frac{\left(2 - \frac{c_x^2}{\beta^2}\right)}{2r_b}$$

This can be plugged back into the potentials, and then solved for displacement,

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}$$
$$u_x = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}$$

And take the real component of u_x and u_z to get:

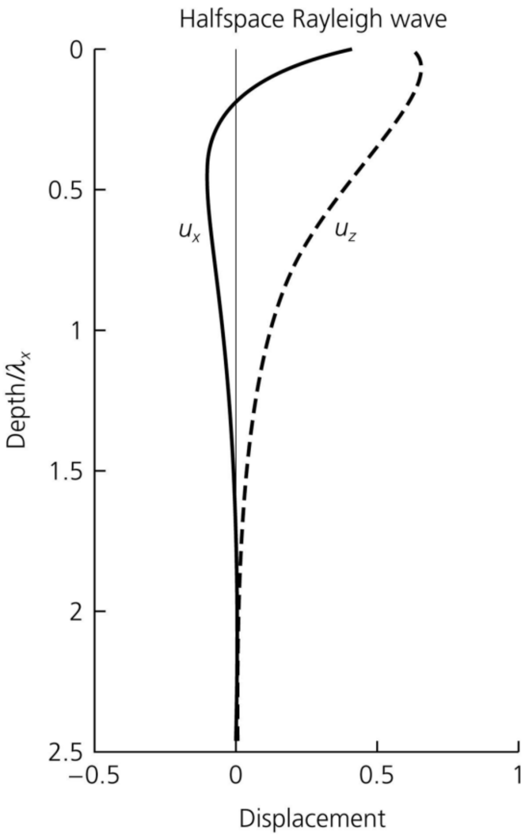
$$u_x = Ak_x \sin(\omega t - k_x x)\left[e^{-0.85k_x z} - 0.58e^{-0.39k_x z}\right]$$
$$u_z = Ak_x \cos(\omega t - k_x x)\left[-0.85e^{-0.85k_x z} - 1.47e^{-0.39k_x z}\right]$$

Here’s what this looks like:

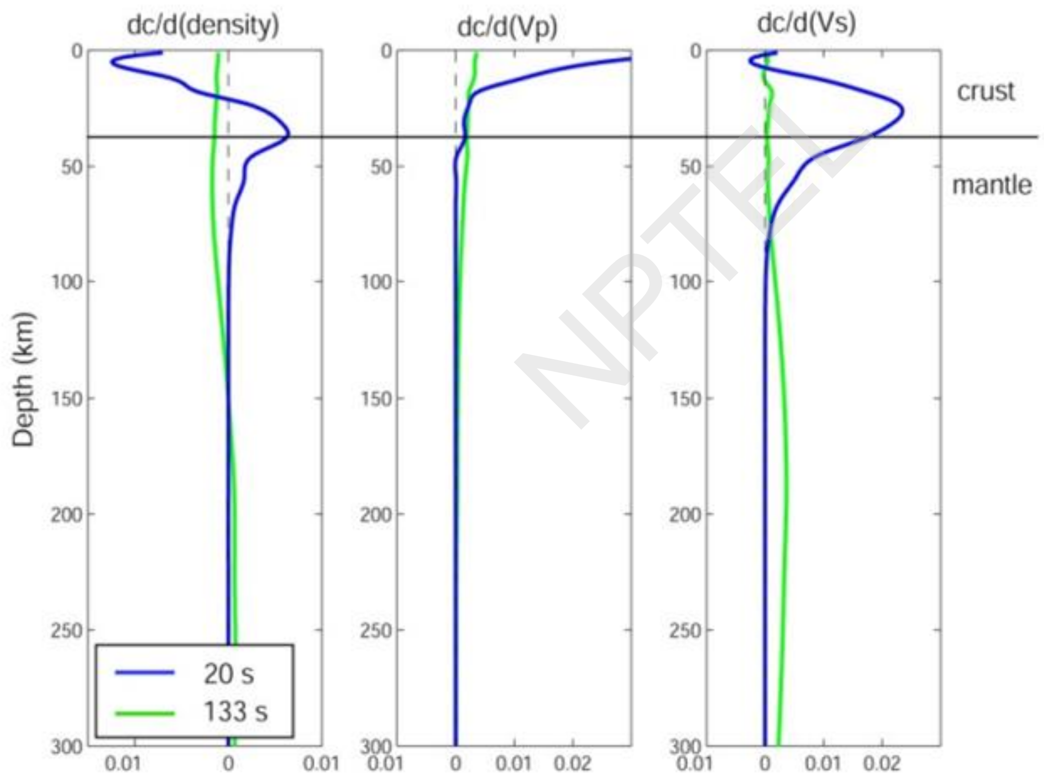
- **The amplitude of displacement diminishes with depth.**

- **The longer wavelengths will have displacement at greater depths, since the y-axis on the figure is a function of depth/λ_x**

Another way of putting this is that the sensitivity of the Rayleigh waves to structure within the earth varies with the wavelength of the wave.



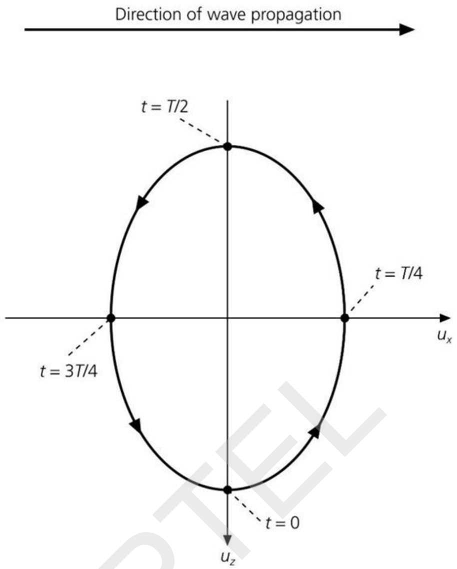
Example of how Rayleigh wave velocities (c, which corresponds to c_x in our equations) are affected by variations in density, P-wave velocity (V_p), and S-wave velocity(V_s), for two different periods waves: one with the period of 20s and one with a period of 133s. This roughly corresponds to range of wave periods over which we see Rayleigh waves from teleseismic earthquakes.

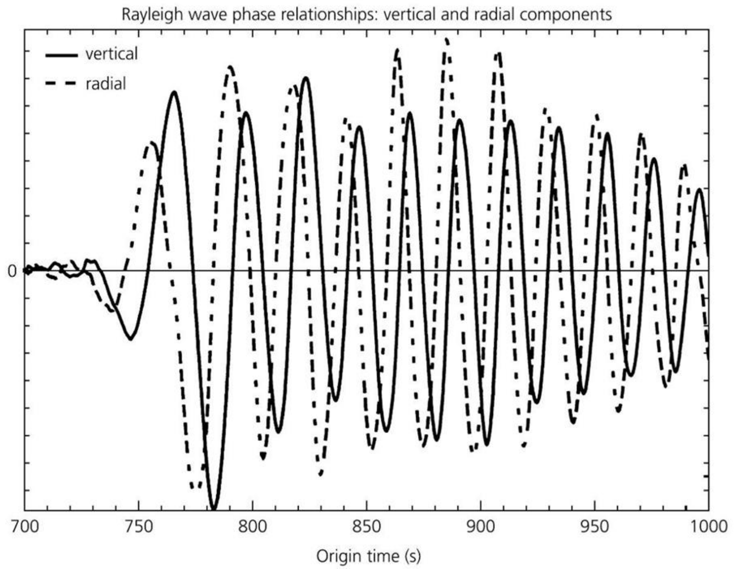


Particle motion of Rayleigh Waves

Let's look at particle motion due to Rayleigh waves. By particle motion, I mean the displacement of a specific point in the ground.

$$\text{At } z = 0$$
$$u_x = 0.42Ak_x \sin(\omega t - k_x x)$$
$$u_z = 0.62Ak_x \cos(\omega t - k_x x)$$





Since cos and sin are 90° out of phase, this means a particle on the surface is going through elliptical motion. Moving to greater depths, the sense of rotation changes, from counter-clockwise (retrograde), to clockwise (prograde). You can see this in the figure above, where u_x becomes negative.

Particle motion of Rayleigh Waves

- Note that the exponential decay of the displacement is a function of the wavenumber, $k_x = \frac{2\pi}{\lambda}$

- Thus, the degree of exponential decay diminishes with increasing wavelength.

- In other words, a longer wavelength Rayleigh wave will see deeper structure in the Earth (see figure to right).

- Since, in general, velocity increases with depth in the Earth, this makes Rayleigh waves dispersive (velocity changes with wavelength).

- This dispersion can be measured to infer the depth-variation of velocity in the Earth.

Summary

Surface waves are the dominant type of wave in a typical seismogram; they can travel multiple times around the world, and cause much of the damage in earthquakes.

Surface waves are useful in Earth Science because different wavelength waves are sensitive to different depths of Earth structure. Thus, if we can measure surface wave velocities as a function of wavelength, we can infer the structure of the Earth. This makes us seismologists happy.

Some features of Rayleigh waves:

1) Rayleigh waves have P-SV motion. We see them on the vertical and radial channels of seismograms.
2) They need no special conditions to exist, unlike Love waves.
3) They are not inherently dispersive.
4) In the Earth, they are dispersive.

REFERENCES

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THANK YOU !