NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 03 : Surface Waves and Dispersion
Lecture 04: Dispersion, Phase and group velocities

CONCEPTS COVERED

- **Dispersion**
- **Phase and group velocities**

Dispersion

- **Dispersion means that wave velocities are dependent on frequency.**
- **As an example, Snell's Law tells us that the index of refraction of light in a medium is dependant upon the speed of light in that medium.**
- **However, the speed of light in a medium (like water) is often frequency-dependent.**
- **Hence, in a raindrop, different colors will be refracted by different amounts, giving us a rainbow.**
- **More technically, dispersion implies** $v = v(\lambda)$ **.**

Another way often used to represent dispersion is to give the relationship between angular velocity ω and wave number k. We know already that (phase) velocity is a function of these two parameters:

$$v_{phase} = \frac{\omega}{k}$$

In a dispersive medium, $\omega = \omega(k)$. For instance, short period tsunami waves follow $\omega = \sqrt{gk}$. We can solve for what this means in terms of velocity.

$$v = \frac{\omega}{k} = \frac{\sqrt{gk}}{k}$$

Often dispersive relations are given in the format $f(\omega, k) = 0$, so for the tsunami example, dispersion would be represented as $\omega - \sqrt{gk} = 0$.

To explore dispersion further, we first consider the simplest example, the net effect of two harmonic waves with slightly different frequencies and wavenumbers

$$\omega_1 = \omega_0 + \delta\omega$$
$$\omega_2 = \omega_0 - \delta\omega$$

and,

$$k_1 = k_0 + \delta k$$
$$k_2 = k_0 - \delta k$$

The total displacement of the two waves is

$$\begin{aligned} u &= \cos\left(k_1 x - \omega_1 t\right) + \cos\left(k_2 x - \omega_2 t\right) \\ &= \cos\left((k + \delta k)x - (\omega + \delta\omega)t\right) + \cos\left((k - \delta k)x - (\omega - \delta\omega)t\right) \\ &= \cos\left[(kx - \omega t) + (\delta k x + \delta\omega t)\right] + \cos\left[(kx - \omega t) - (\delta k x - \delta\omega t)\right] \end{aligned}$$

Using the trig identity $\cos(\theta+\phi)+\cos(\theta-\phi)=2\cos\theta\cos\phi$

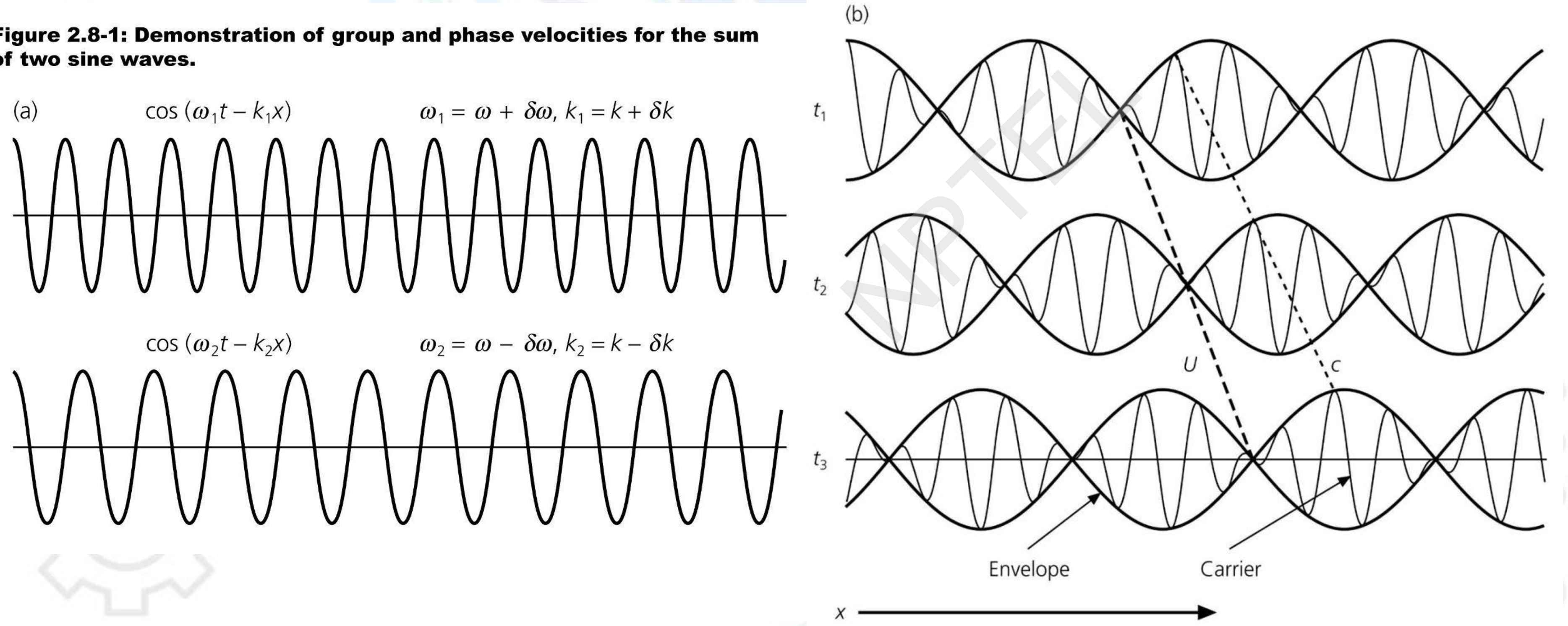
the displacement becomes $u=2\cos(kx-\omega t)\cos(\delta kx-\delta\omega t)$

This is the product of two cosine waves. The first wave has wavelength, $\lambda=\dfrac{2\pi}{k}$ **and assuming** $\delta k << k$ **that , the second will have a much larger wavelength defined by** $\lambda=\dfrac{2\pi}{\delta k}$

.

Alternatively, in the time domain, the first wave will have a higher frequency $f = \dfrac{\omega}{2\pi}$, compared to $f = \dfrac{\delta\omega}{2\pi}$.

Figure 2.8-1: Demonstration of group and phase velocities for the sum of two sine waves.



- **Hence, the first cosine term defines a wave (sometimes called the "carrier") that is modulated by a slower varying "envelope".**

- **Since, the velocity with which a wave defined by** $u = \cos(ax - bt)$ **propagates is v = b/a**

- **The velocity of the carrier wave is v = ω/k and that of the envelope is slower** $v = \dfrac{\delta w}{\delta k}$

- **We refer to the former as "phase velocity", and the latter as the "group velocity".**

- **Typically, "c" is used to denote phase velocity, and U to denote group velocity.**

- **The relationship between** $c = v_{phase} = \dfrac{\omega}{k} \; and \; U = v_{group} = \dfrac{\delta\omega}{\delta k}$ **will be defined by the dispersion relationship** $\omega = \omega(k)$**.**
  As $\delta\omega, \delta k \to 0 \qquad U \to \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$

What this means is that the most immediately recognizable wave (or a surface of constant phase) will propagate with a different velocity than the envelope. It turns out the actual energy propagation velocity is defined by the group velocity. In a medium with anisotropic velocity structure, the group and phase velocities actual may have slightly different directions.

Since, $$c = \frac{\omega(k)}{k}$$

The relationship between c and U at a given angular frequency ω can be derived using the chain rule.

$$U = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}(ck) = k\frac{\mathrm{d}c}{\mathrm{d}k} + c\frac{\mathrm{d}k}{\mathrm{d}k} = k\frac{\mathrm{d}c}{\mathrm{d}k} + c$$

In terms of wavelength, $\lambda = \frac{2\pi}{k}$ **, can be used to derive the group velocity equation.**

Using,

$$\frac{\mathrm{d}c}{\mathrm{d}k} = \frac{\mathrm{d}c}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}k}$$
$$= \frac{\mathrm{d}c}{\mathrm{d}\lambda}\left(\frac{-2\pi}{k^2}\right)$$

We can convert this to wavelength

$$k\frac{\mathrm{d}c}{\mathrm{d}k} + c = k\frac{\mathrm{d}c}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}k} + c$$
$$= k\frac{\mathrm{d}c}{\mathrm{d}\lambda}\left(\frac{-2\pi}{k^2}\right) + c$$
$$= \frac{\mathrm{d}c}{\mathrm{d}\lambda}\left(\frac{-2\pi}{k}\right) + c$$
$$= c - (2\pi)\lambda\frac{\mathrm{d}c}{\mathrm{d}\lambda}$$

In practice, the 2π is typically dropped to get

$$U = c - \lambda \frac{\mathrm{d}c}{\mathrm{d}\lambda}$$

Since at greater depths in the earth tend to have higher velocities, c increases with wavelength, so

$$\frac{\mathrm{d}c}{\mathrm{d}\lambda} > 0$$

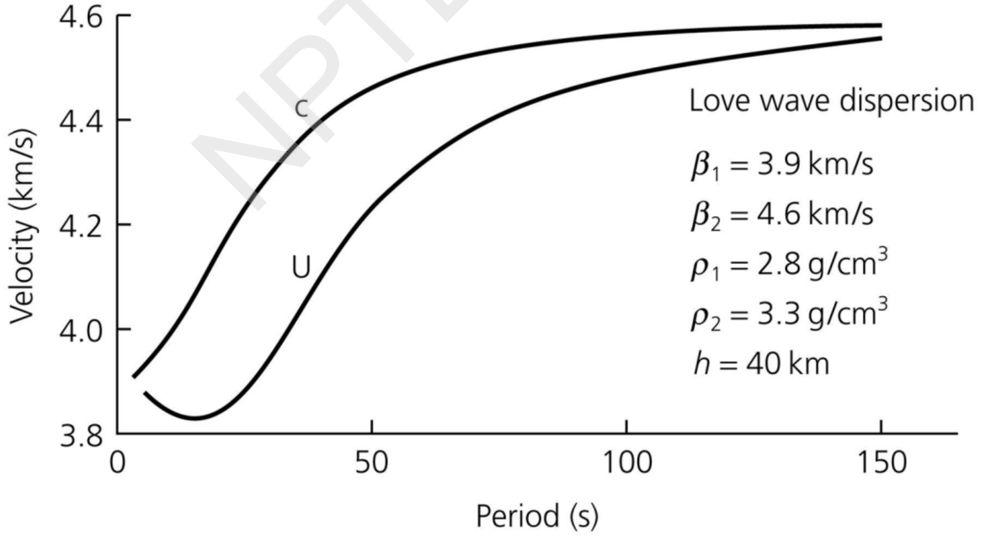
So from above relation, on average in the earth, c >U.

If the medium is non-dispersive

and the phase and group velocity will be the same.

$$\frac{\mathrm{d}c}{\mathrm{d}\lambda} = \frac{\mathrm{d}c}{\mathrm{d}k} = 0$$

Figure 2.8-2: Fundamental mode Love wave group and phase velocities.



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- **https://geologyscience.com/geology-branches/structural-geology/stress-and-strain/**
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- **Seismology course, Professor Derek Schutt, Colorado State Univ., USA.**

THANK YOU !