NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 04 : Normal modes of the earth

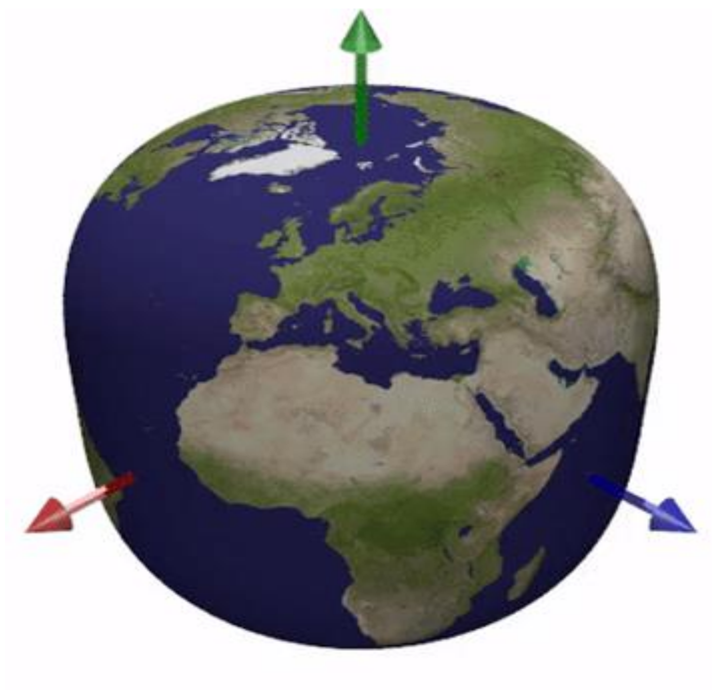
Lecture 01: Normal modes and its kinds, Normal modes on a string, Legendre polynomials, Spheroidal modes

CONCEPTS COVERED

- Normal modes
- Types of normal modes
- Normal Modes on a string
- Legendre polynomials and Legendre functions
- Spheroidal modes
- Summary

What are Normal modes?

- **Normal modes may be considered as the oscillation of a system.**
- **For a linear system, the displacement could be viewed as waves propagating along the string or as the sum of standing waves, called normal modes.**
- **Normally excited by an earthquake.**
- **Other planets and moons will also undergo free oscillations and the oscillations of the sun are studied by astrophysicists: “helioseismology”.**



Spheroidal n =0, l = 4, m=0

period ~26 min

Spheroidal n =0, l = 4, m=2

period ~26 min

From https://saviot.cnrs.fr/terre/index.en.html

Why should I care about normal modes?

a. For generations of synthetic seismograms using summation of normals modes.

a. They can be used to estimate Earth structure that may difficult to get at through other means- for example, normal modes gave the first proof for the liquid outer core.

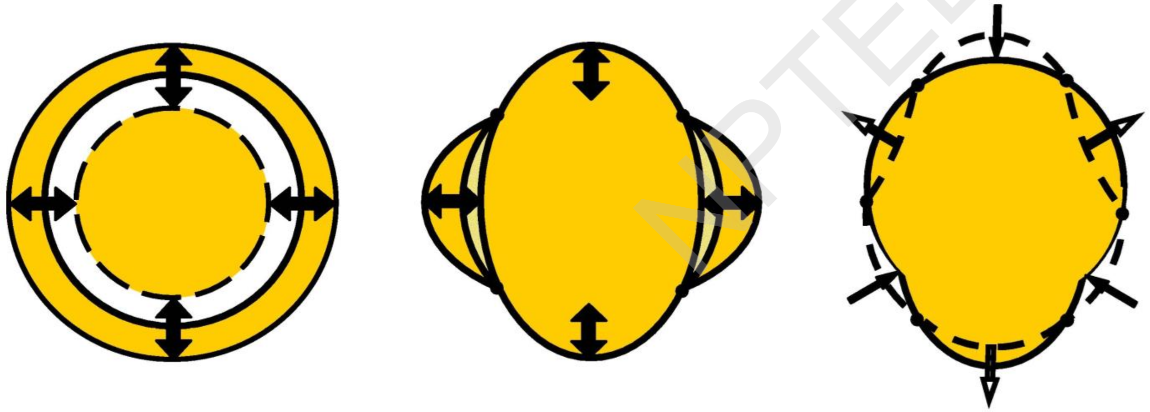
a. The normal modes deeply care about density as well as seismic velocities.

a. Earth rings like a bell after an earthquake - it feels cool.

Types of normal modes

-There are two types of normal modes.

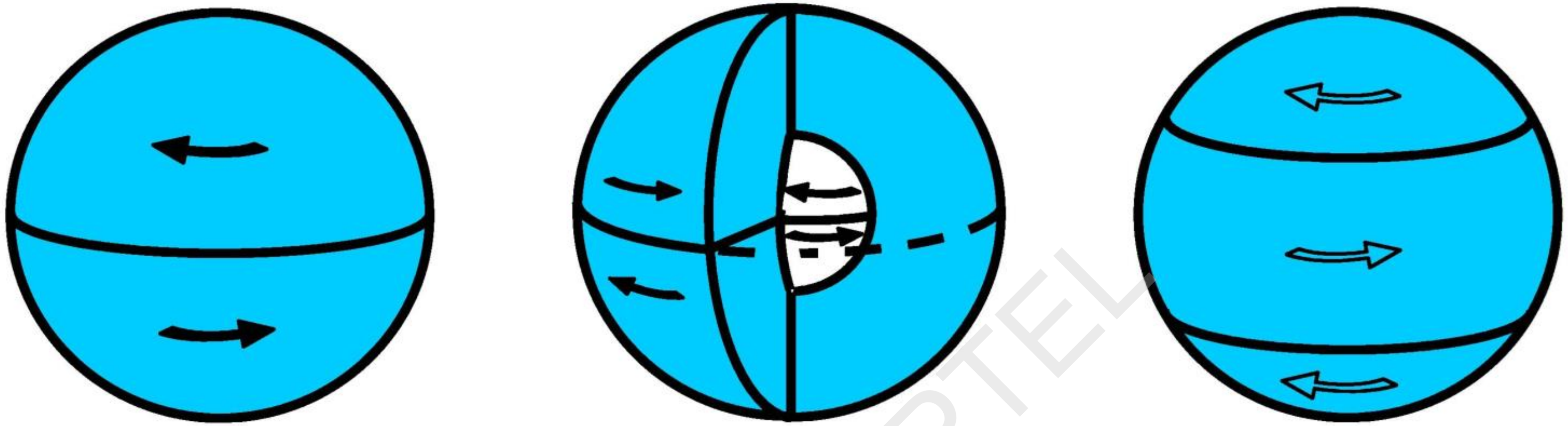
- Spheroidal modes are analogous to the P-SV motion.



Spheroidal modes ${}_0S_0$ (20.5 min), ${}_0S_2$ (53.9 min) and ${}_0S_3$ (25.7 min)

Figure from http://www.iris.iris.edu/sumatra/free_oscillations_second.htm

-Toroidal modes are analogous to Love waves or SH-motion.

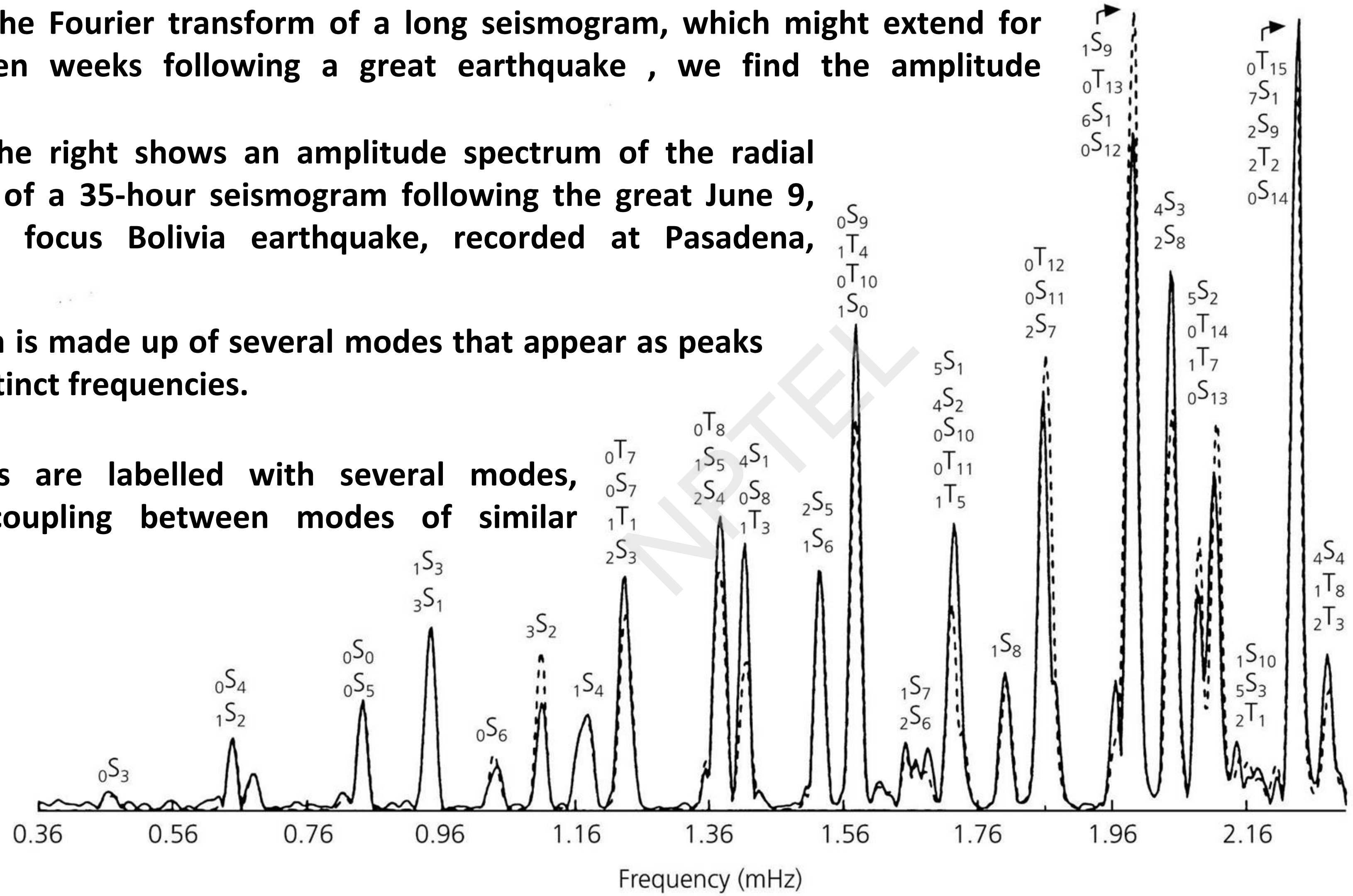


Toroidal modes ${}_0T_2$ (44.2 min), ${}_1T_2$ (12.6 min) and ${}_0T_3$ (28.4 min)

Figure from http://www.iris.iris.edu/sumatra/free_oscillations_second.htm

Figure 2.9-2: Amplitude spectrum shown mode peaks for a 35-hour record.

- If we take the Fourier transform of a long seismogram, which might extend for days or even weeks following a great earthquake , we find the amplitude spectrum.
- Figure on the right shows an amplitude spectrum of the radial component of a 35-hour seismogram following the great June 9, 1994, deep focus Bolivia earthquake, recorded at Pasadena, California.
- The spectrum is made up of several modes that appear as peaks at certain distinct frequencies.
- Many peaks are labelled with several modes, indicating coupling between modes of similar frequencies.



Normal Modes of a String

We've been looking at travelling waves as a solution to the 1-D wave equation.

A completely valid alternative is to seek solutions to the wave equation with a cos(ωt) dependence, such that:

$$y(x,t) = Y(X,\omega)\cos(\omega t)$$

For a constant property string, one solution is where the Y(x,ω) term is

$$Y(x,\omega) = \sin(\omega x/v)$$

If the string is fixed at x = 0, and x = L, then these boundary conditions imply the only frequencies that work are

$$\omega_n = n\pi v / L$$

Normal Modes on a String

Since the string can only vibrate at these discrete frequencies, these frequencies are called *eigenfrequencies*.

These eigenfrequencies correspond to the spatial terms of the solution

$$Y_n(x, \omega_n) = \sin\left(\omega_n x / v\right)$$

The complete solution is

$$Y(x,t) = \sum_{n=0}^{\infty} A_n Y_n(x, \omega_n) \cos(\omega_n t)$$

where each term (*n*) is called a normal mode.

The normal modes are orthogonal, which means if you integrate the product of two eigenfunctions over the length of the string, the result is zero.

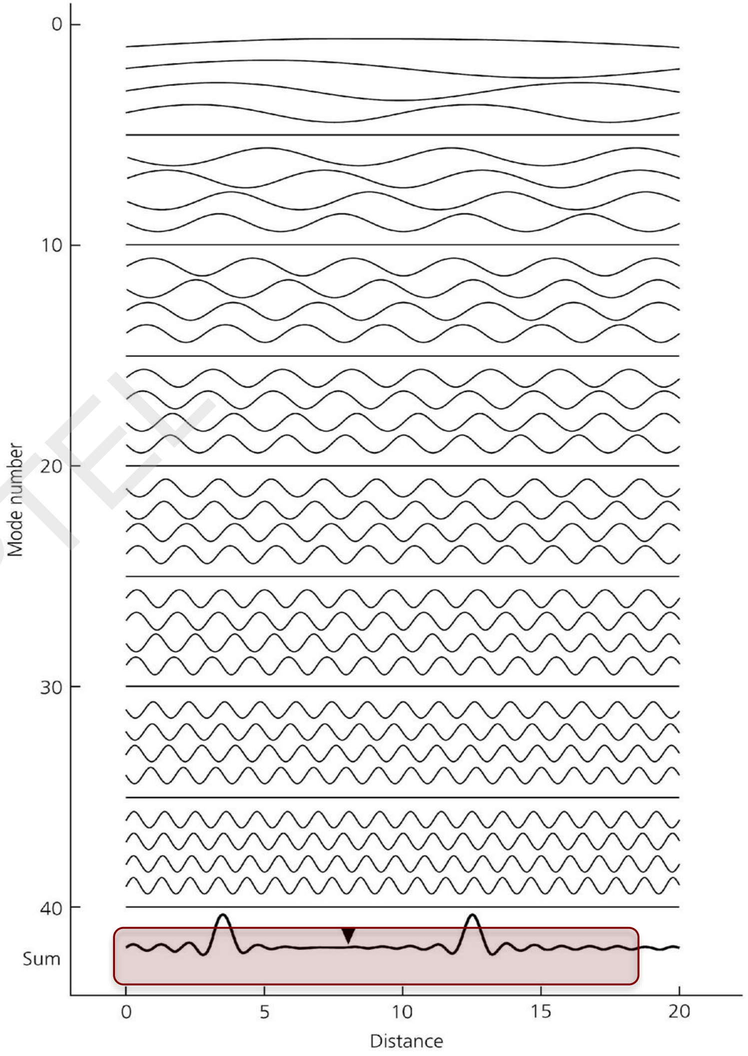
Normal Modes on a String

It may seem counterintuitive, but the normal modes are a completely valid and equivalent way to model a wave on a string, or in the earth.

Here is an example wave that is described approximately by 40 normal modes.

The amplitudes depend on the value of the eigenfunction at the point where the source excited the motion.

Figure 2.2-8: Waves on a string as a summation of modes.



We may describe elastic waves in the Earth either by propagating wave or summation of modes.

To extend string's idea into wave propagation in the three dimension spherical earth, the normal modes solution can be formulated in spherical coordinates.

Recall modes on the string:

$$u(x,t) = \sum_{n=0}^{\infty} A_n U_n(x, \omega_n) \cos(\omega_n t)$$

This is sum of standing waves or eigenfunctions, $U_n(x, \omega_n)$, each of which is weighted by amplitude A_n and vibrates at its eigenfrequency ω_n.

The eigenfunction and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

$$U_n(x, \omega_n) = \sin\left(n\pi x/L\right) = \sin\left(\omega_n x/v\right) \qquad \omega_n = n\pi v/L = 2\pi v/\lambda$$

The boundaries makes the modes! No boundaries, no discrete modes!

(an infinitely long string would have continuous modes at all frequencies)

How can we extend one and two dimensional ideas to wave propagation in the three dimensional earth?

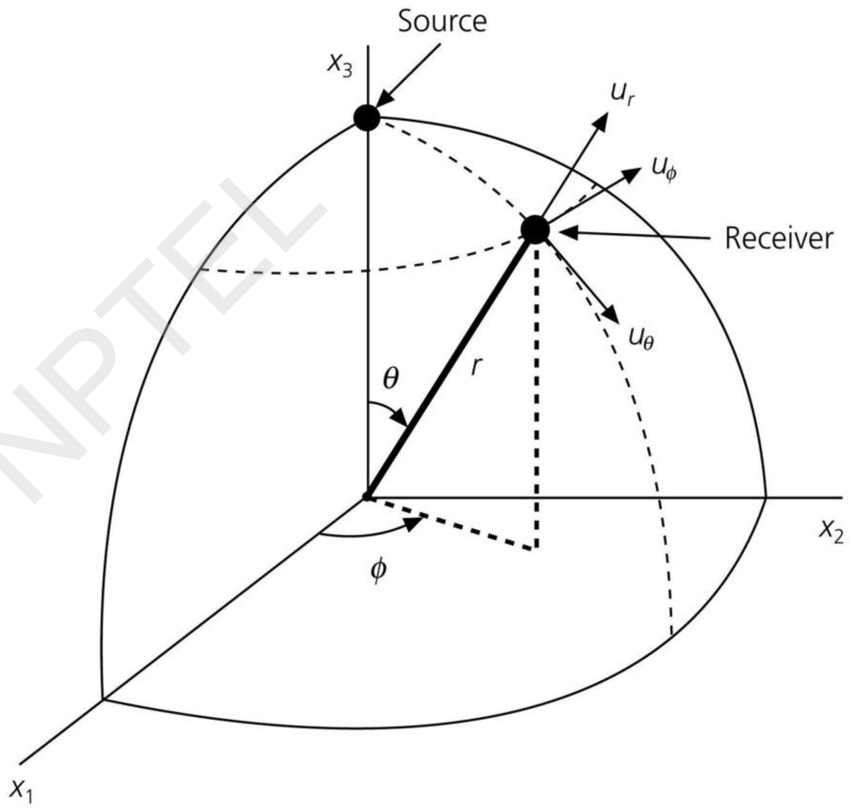
Answer is: we formulate the normal mode solution in spherical coordinates.

-Seismic source is considered at the pole as the waves propagates away from there.

-We may write the displacement vector $u(r, \theta, \phi) = (u_r, u_\theta, u_\phi)$ that satisfies the equation of motion as a function of radius r and surface position (θ, ϕ).

-In spherical coordinates, the 'radial' is the vertical and u_θ is in the direction analogous to that of plane wave propagation, and u_ϕ is transverse to it.

Figure 2.9-1: Spherical coordinate geometry for normal modes.



For displacement in 3D:

$$\mathbf{u}(r,\theta,\phi) = \sum_n \sum_l \sum_m {}_nA_l^m \; {}_ny_l(r)\mathbf{x}_l^m(\theta,\phi)e^{i_n\omega_l^m t}$$

n,l, m-radial, angular and azimuthal order

$_ny_l(r)$-scaler radial eigenfunction

$\mathbf{x}_l^m(\theta,\phi)$ - vector surface eigenfunction

$_nA^m_l$- excitation amplitudes (weights for eigenfunctions) that depend on the seismic source

How would you define a set of orthogonal functions on a sphere? (the equivalent of sines and cosines on a string)

Well, the answer is Spherical Harmonics.

For a sphere, such as the Earth, displacement on the surface can be described in an analogous way in terms of spherical harmonics.

The surface eigenfunctions are based on spherical harmonics, defined by an orthogonal set of functions called Legendre Polynomials

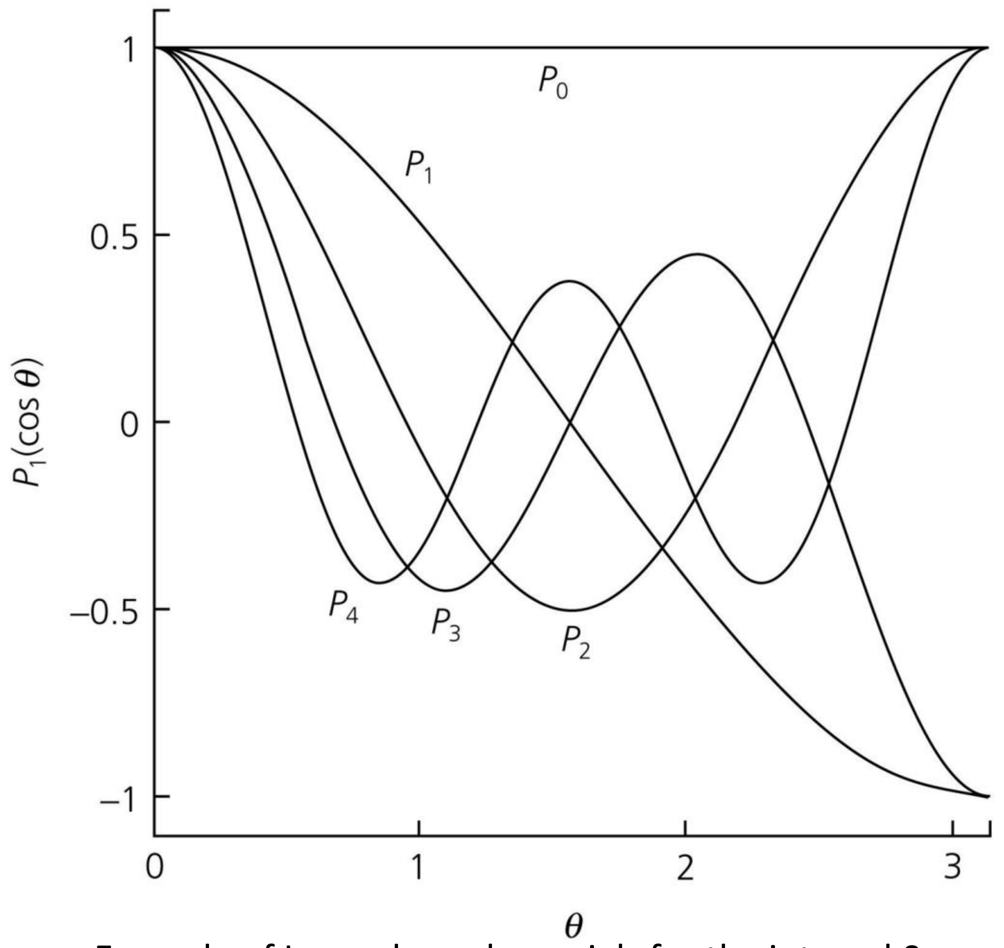
$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l$$

'l' designates the angular order of the polynomial.

$\boldsymbol{\theta}$ = angular distance from the pole (colatitude)

$\boldsymbol{\phi}$ = azimuth around the pole (longitude)

Figure 2.9-3: Examples of Legendre polynomials.



Example of Legendre polynomials for the interval 0-π used to describe the displacements associated with normal mode oscillations.

The first several polynomials are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (1/2)\left(3x^2 - 1\right)$$
$$P_3(x) = (1/2)\left(5x^3 - 3x\right)$$

On sphere, x = cos $\boldsymbol{\theta}$, so these are going to show variations with co-latitude from the source.

We want to also consider longitudinal dependence, so we expand the Legendre polynomials to a more general set of Legendre functions.

$$P_l^m(x) = \left[\frac{(1-x^2)^{m/2}}{2^l l!}\right]\left[\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l\right]$$

The azimuthal order, m, varies over $-l \le m \le l$. Note that this devolves to the Legendre Polynomial when m = 0.

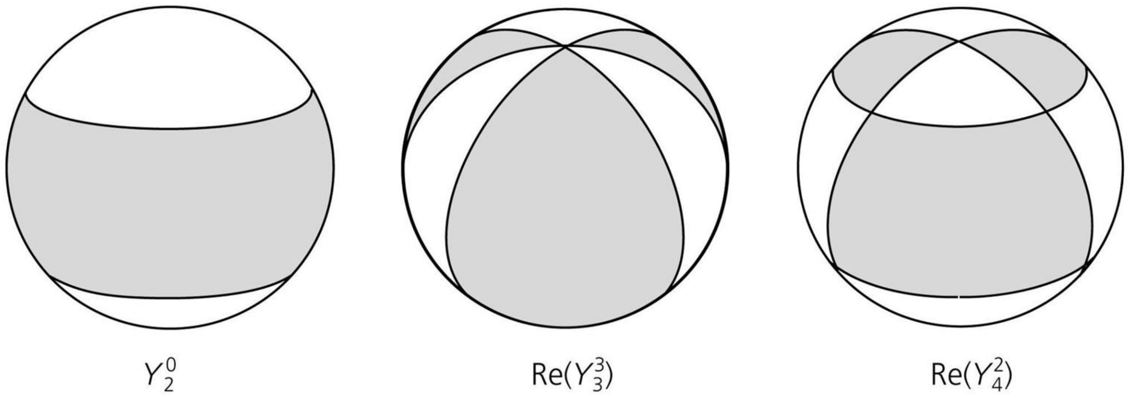
The azimuthal functions $e^{im\phi}$ and associated Legendre functions are combined to give the fully normalised spherical harmonics,

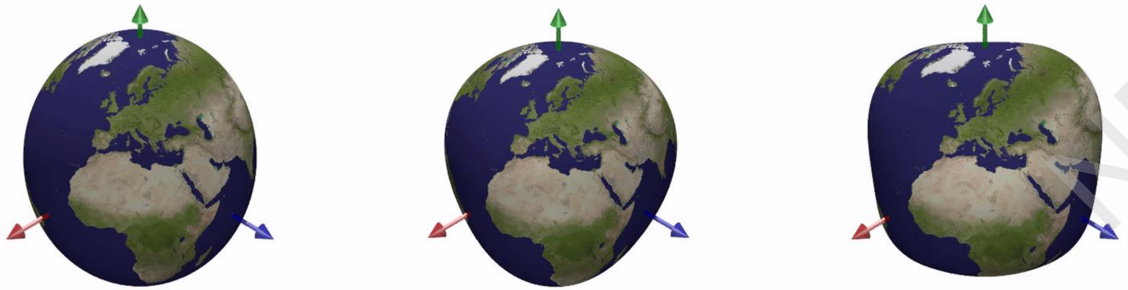
$$Y_l^m(\theta,\phi) = (-1)^m\left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

Spherical harmonics are orthogonal (i.e. the integral of the product of one with the conjugate of another over the sphere is zero.

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y_{l'}^{m'}(\theta,\phi)Y_l^m(\theta,\phi)d\theta d\phi = \delta_{l'l}\delta_{m'm}$$

Figure 2.9-4: Examples of spherical harmonics.





From https://saviot.cnrs.fr/terre/index.en.html

$$Y_l^m(\theta,\phi) = (-1)^m \left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

The angular order, *l*, gives the number of nodal lines on the surface

-If the azimuthal order m is zero, the nodal lines are small circles about the pole. these are called zonal harmonics and do not depend on ϕ.

-If m = 1, then all the surface nodal lines are great circles through the pole. These are called sectoral harmonics.

- When 0 < |m| < l, there are combined angular and azimuthal (colatitudinal and longitudinal) nodal patterns called tesseral harmonics.

Summary

- Normal modes are “Oscillation of the system” and is sum of the standing waves. Example, whole earth vibration after a large earthquake, free oscillation of sun.
- Two types of Normal modes: Spheroidal modes; analogous to P-SV motion.
  Toroidal Modes; analogous to love of SH motion.
- Boundary condition on the bounded string (at x=0 and x=L) produces normal modes, and solution is given by $Y(x,t)=\sum_{n=0}^{\infty} A_n Y_n(x,\omega_n)\cos(\omega_n t)$ $\quad Y_n(x,\omega_n) = \sin(\omega_n x/v)$ $\quad \omega_n = n\pi v/L$
- Similarly, fully normalized and orthogonal spherical harmonics for earth is given by

$$Y_l^m(\theta,\phi) = (-1)^m \left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$ Spherical harmonics

$$\mathbf{u}(r,\theta,\phi) = \sum_n \sum_l \sum_m {}_nA_l^m \, {}_ny_l(r)\mathbf{x}_l^m(\theta,\phi)e^{i_n\omega_l^m t}$$ Displacement

Where n,l and m are radial, angular and azimuthal order.

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THANK YOU !