NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 01 : Basic Seismological Theory, Waves on a String , Stress and Strain and seismic waves

Lecture 03: Stress tensor and its kinds

CONCEPTS COVERED

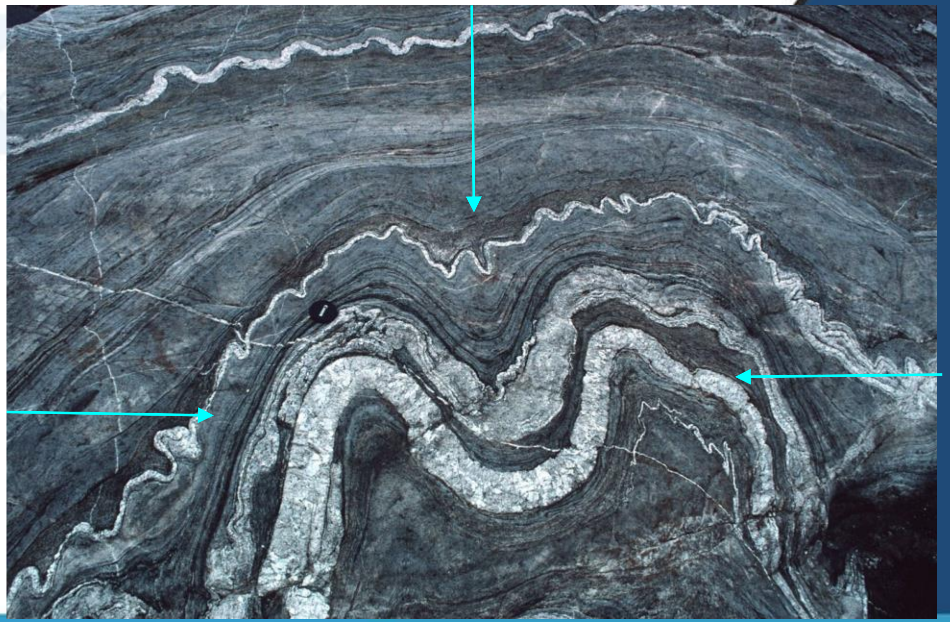
- **Stress tensor, Traction vector**
- **Normal and shear stresses**
- **Invariants of the Stress tensor**
- **Principle stresses, mean stress, deviatoric stress**

Stress

Stress plays a very crucial role in the propagation of the seismic waves.

Stress is force adjusted for the area over which it is distributed. It is the resistive or restoring force appears in the material when an external deformative force is applied

In Earth, collision of the tectonic plates and the weight of the overlying rocks exert forces on rocks at depth which results in the deformation of the rocks.



Mathematically, stress is defined as the force on the infinitesimal area. If a uniform force "F" is applied normally to the area of cross-section "A", then stress is defined as:

$$\sigma = \lim_{A \to 0} \frac{F}{A}$$

Things to be noted

- Stress is force/area that a material on the outside of the surface exerts on the material inside and is a 3x3 tensor

- The traction vector is the surface force per unit area on a plane with given normal

- The traction can be calculated using:

$$\vec{T} = \overleftrightarrow{\sigma}\hat{n}$$

- Stein and Wysession uses engineering convention that tensional (stretching) stress is positive. That is why stress in the earth is negative (compressional) values.

Things to be noted

- Stress tensor has following properties
  - Stress is a symmetric tensor
  - can be rotated it into other coordinate system.
  - Coordinate system without the shear stresses gives rise to principal stresses. These are the eigenvalues of the stress tensor.
  - The trace of the stress tensor is independent of the coordinate system used.
- The pressure is the mean of the trace of the stress tensor.
- The deviatoric stress tensor is the stress tensor minus the pressure.
- Units of stress are N/m^2 = Pascals
- 33 km increase in depth increases pressure by roughly 1 Gpa.

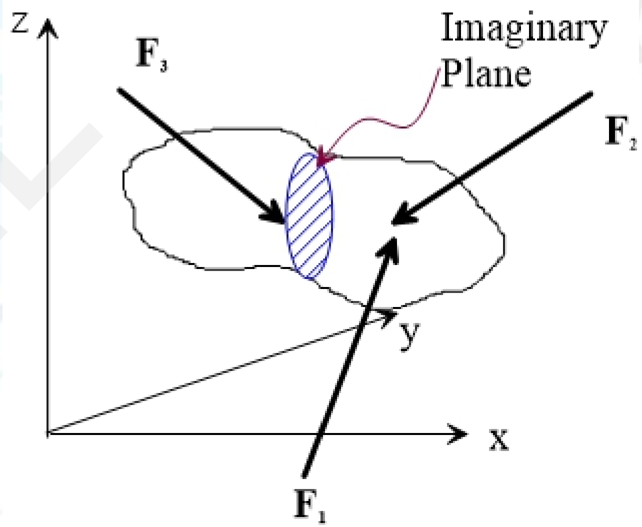
What is stress and strain ?

Forces within the Earth can be represented as stresses: force/area.

- Units = N/m^2 = Pascals.

The deformation of a body in response to stress is strain.

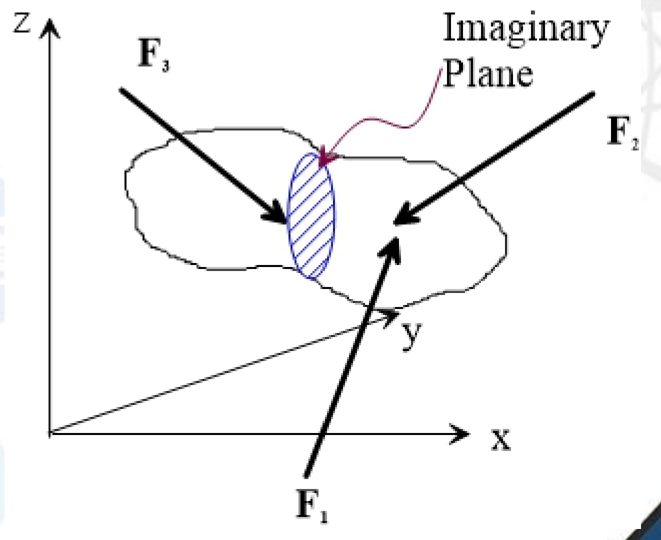
- Units = m/m, so this is unitless. Discussed in strain unit.



To make it more visible, let's consider a body acted on by different forces; F_1,F_2,F_3

Stress and strain: Body and surface forces

Consider a whole 3-D body of a material, we observe two types of forces acting on it, based on the point of their action:



- Body forces, which act on every part of the body—gravity is a good example. A body force does not require contact for its transmission.

- Surface forces, which act (are transmitted) on the boundary of this body.

Stress and strain: Considering the surface forces

Traction

Now, slicing off the right side of the body—we can represent the effects of the missing right side by an equivalent set of surface forces.

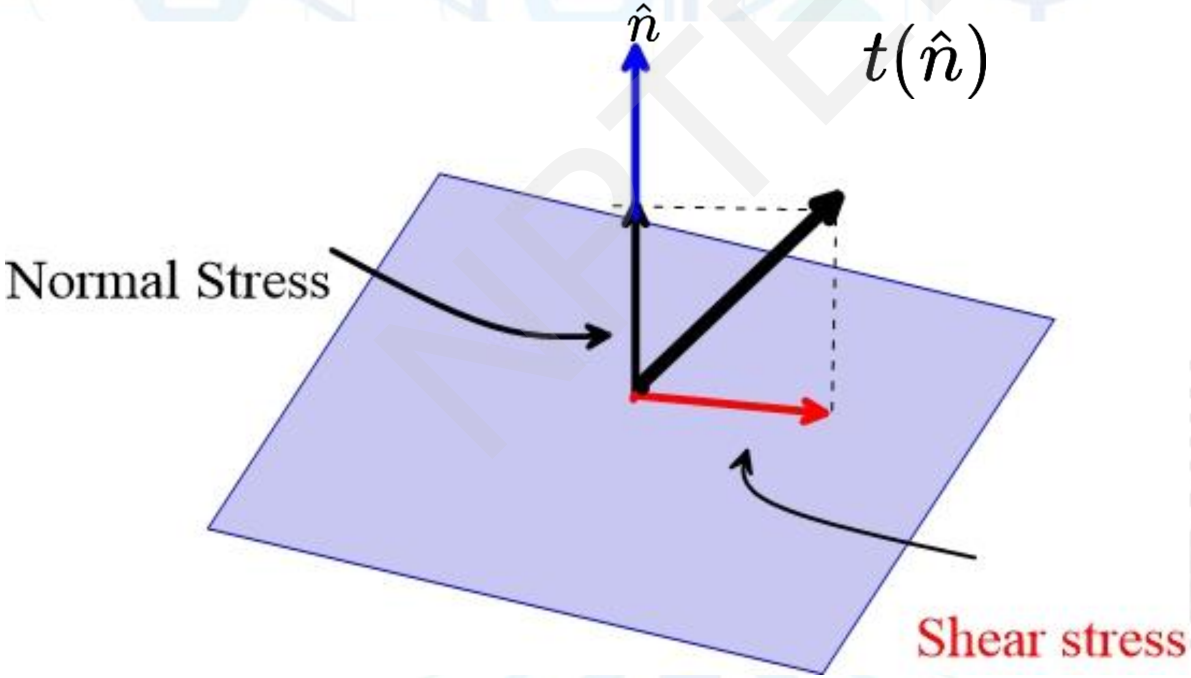
Let's consider infinitesimal elements in our plane, each with area ΔA and normal $\hat{n}$

Traction vector on an area element is expressed as : the force on the element divided by the area, in the limit as the area goes to 0. $\vec{t}(\hat{n}) = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A} = t_x \hat{x} + t_y \hat{y} + t_z \hat{z}$

Stress and Strain: Surface forces components

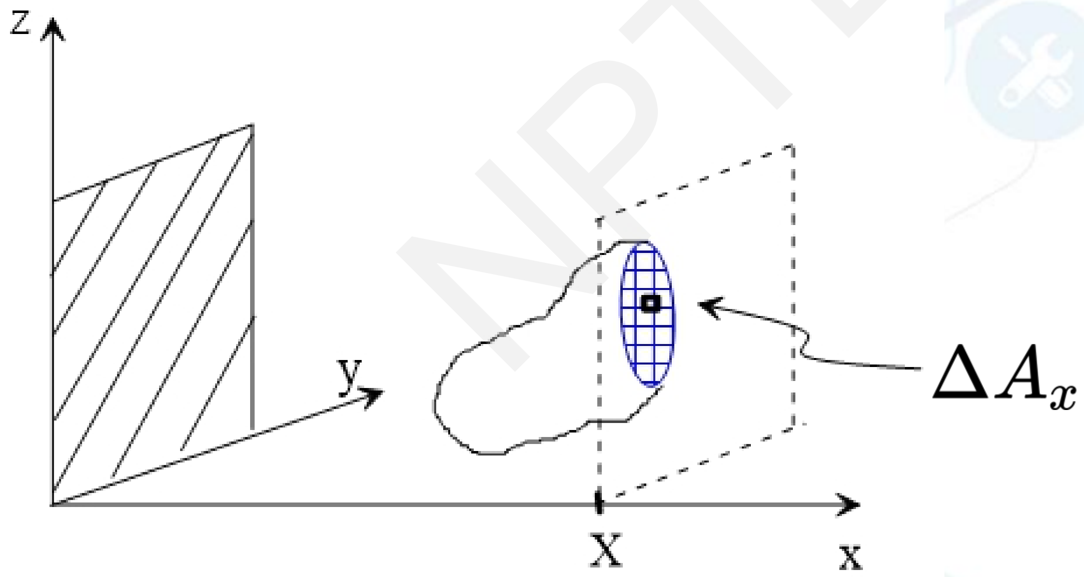
Normal and Shear Stress

This traction vector can be decomposed into components parallel to the surface—shear stress —and normal to the surface—normal stress.



Normal and Shear Stress

Direction of traction vector depends on the orientation of the plane used which in turn depends on the normal to that plane. Here, for example is the plane with a normal in the x-direction.



Normal and Shear Stress

We can define stress components on this "x" plane.

normal stress

$$\sigma_{xx} = \lim_{\Delta A_x \to 0} \frac{\vec{F}_x}{\Delta A_x}$$

shear stress

$$\sigma_{xy} = \lim_{\Delta A_x \to 0} \frac{\vec{F}_y}{\Delta A_x}$$

shear stress

$$\sigma_{xz} = \lim_{\Delta A_x \to 0} \frac{\vec{F}_z}{\Delta A_x}$$

Thus we have:

$$\sigma_{ij}$$

Index i gives direction of normal to plane acted on by a force

Index j gives direction of the force

Normal and Shear Stress

We can do this for the y-plane (plane with a normal in the y-direction), and the z-plane. With 3 possible planes to consider, and 3 possible directions for the traction vector to be resolved in each plane, this gives us 9 stress components to define the state of stress at point P.

Component of stress in x direction | Component of stress in y direction | Component of stress in z direction

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{matrix} \text{Plane x} \\ \text{Plane y} \\ \text{Plane z} \end{matrix}$$

Normal and Shear Stress

Another way of representation to write the stress tensor

Using 1,2,3, rather than x, y, z, to denote the coordinate axes…

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

The Cauchy Stress Theorem

The state of stress at a point in the body is then defined by all the stress vectors associated with all planes (infinite in number) that pass through that point.

If σ is known, the traction on any arbitrary surface can be calculated from

$$\vec{T} = \hat{n}\overleftrightarrow{\sigma}$$

In Einstein Summation Notation, this can be expressed as

$$T_i = \sigma_{ji} n_j$$

Because the stress tensor is symmetric

$$\sigma_{ji} = \sigma_{ij}$$ Symmetric Condition

This means we can rewrite the equation as:

$$T_i = \sigma_{ij} n_j \quad or \quad T = \sigma \hat{n}$$

The Cauchy Stress Theorem II

Here, According to Einstein's notation we sum over repeated indices, in this case it is j. So,

$$T_i = \sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j$$

Because i is not repeated, we do this separately, for i = 1, 2, 3. This means there are three equations here, one for each value of i.

$$T_1 = \sum_{j=1}^{3} \sigma_{1j} n_j \qquad T_2 = \sum_{j=1}^{3} \sigma_{2j} n_j \qquad T_3 = \sum_{j=1}^{3} \sigma_{3j} n_j$$

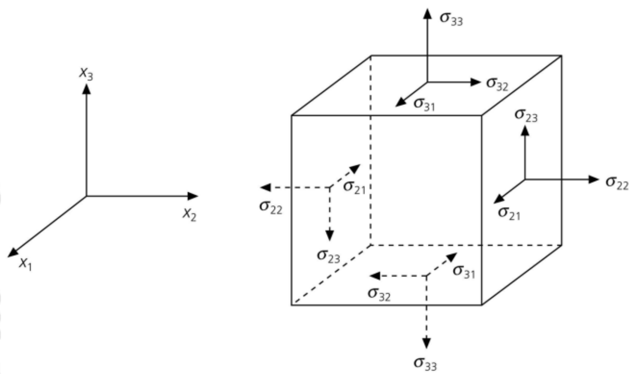
The Cauchy Stress Theorem II

Let's do an example, for the x_2-x_3 plane with a normal in the x_1 direction. This implies

$$\hat{n} = [1,0,0]$$

$$\vec{T} = \vec{\sigma}\hat{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\sigma_{11} + 0\sigma_{12} + 0\sigma_{13} \\ 1\sigma_{21} + 0\sigma_{22} + 0\sigma_{23} \\ 1\sigma_{31} + 0\sigma_{32} + 0\sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

Figure 2.3-4: Stress components on the faces of a volume element.



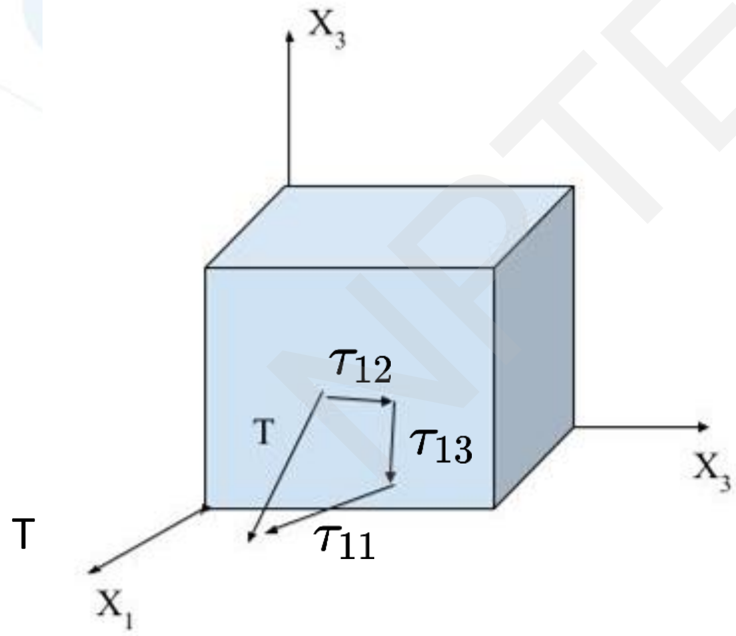
Since the stress tensor is symmetric

$$\vec{T} = [\sigma_{11}, \sigma_{21}, \sigma_{31}] = [\sigma_{11}, \sigma_{12}, \sigma_{13}]$$

One can define a normal for any arbitrary plane, so the tractions on any plane can be calculated. But don't forget the length of the normal has to be 1!

Different books use different notation. For example, the following figures, x_1, x_2, x_3 in place of x,y and z. Moreover, τ is used instead of σ.

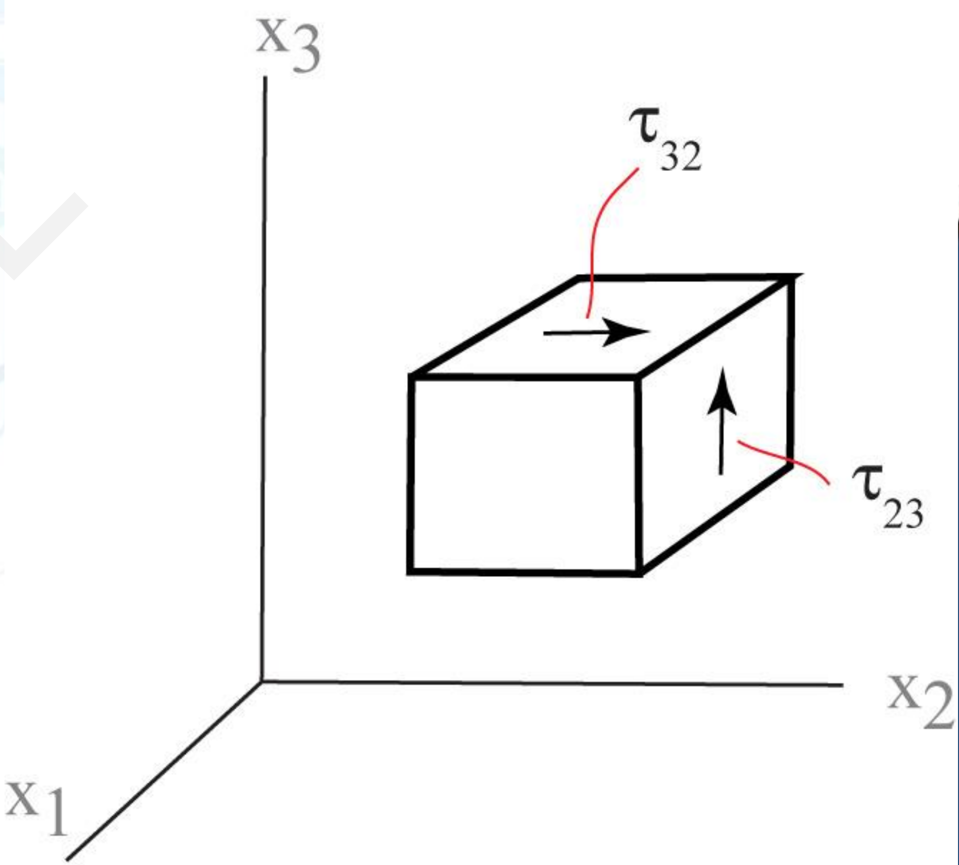
So, in this notation, τ_{ij} is the stress acting in the x_j direction.



Equilibrium Constraints

If there is no rotation, $\tau_{ij} = \tau_{ji}$. For instance, $\tau_{23} = \tau_{32}$

Otherwise, unbalanced forces would cause rotational acceleration. We can have a cube deformed such that there is a net rotation, but not a net rotational force (i.e Sum of all shear stresses is 0) , which would lead to rotational acceleration.



Equilibrium Constraints

This means that magnitude of the many terms in the stress tensor must be equal.

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

This makes the stress tensor symmetric in nature.

Equilibrium Constraints

Thus, we are left with six independent parameters, three with normal stresses (diagonal elements) and three with shears stresses (off-diagonal elements)

$$\tau = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix}$$

Shear stresses

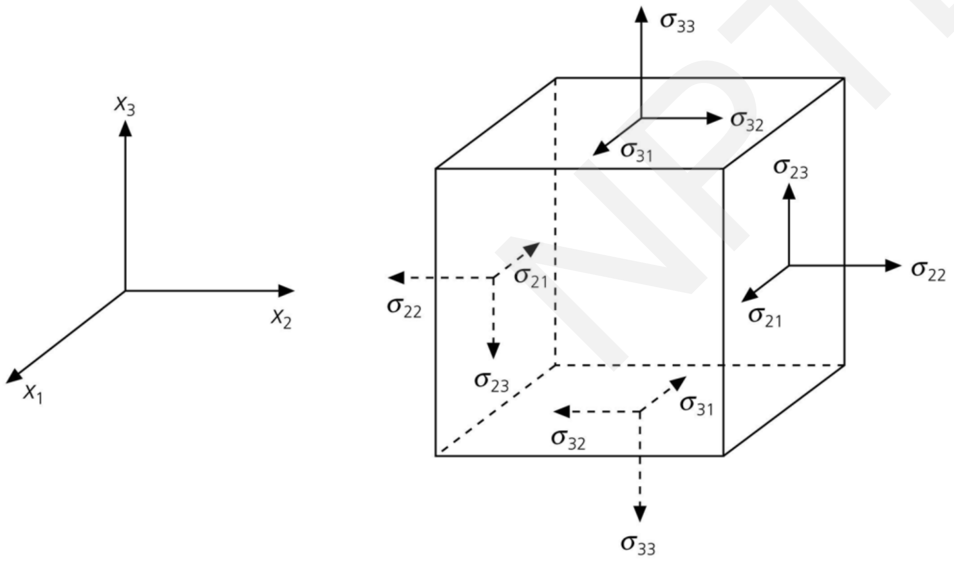
Normal stresses

Sign Conventions

- Engineering: normal stress is positive , indicate dilatational stress (volume increase).
- Geology: normal stresses is positive , indicates compressional stress (volume decrease).

Stein and Wysession use the engineering convention.

Figure 2.3-4: Stress components on the faces of a volume element.



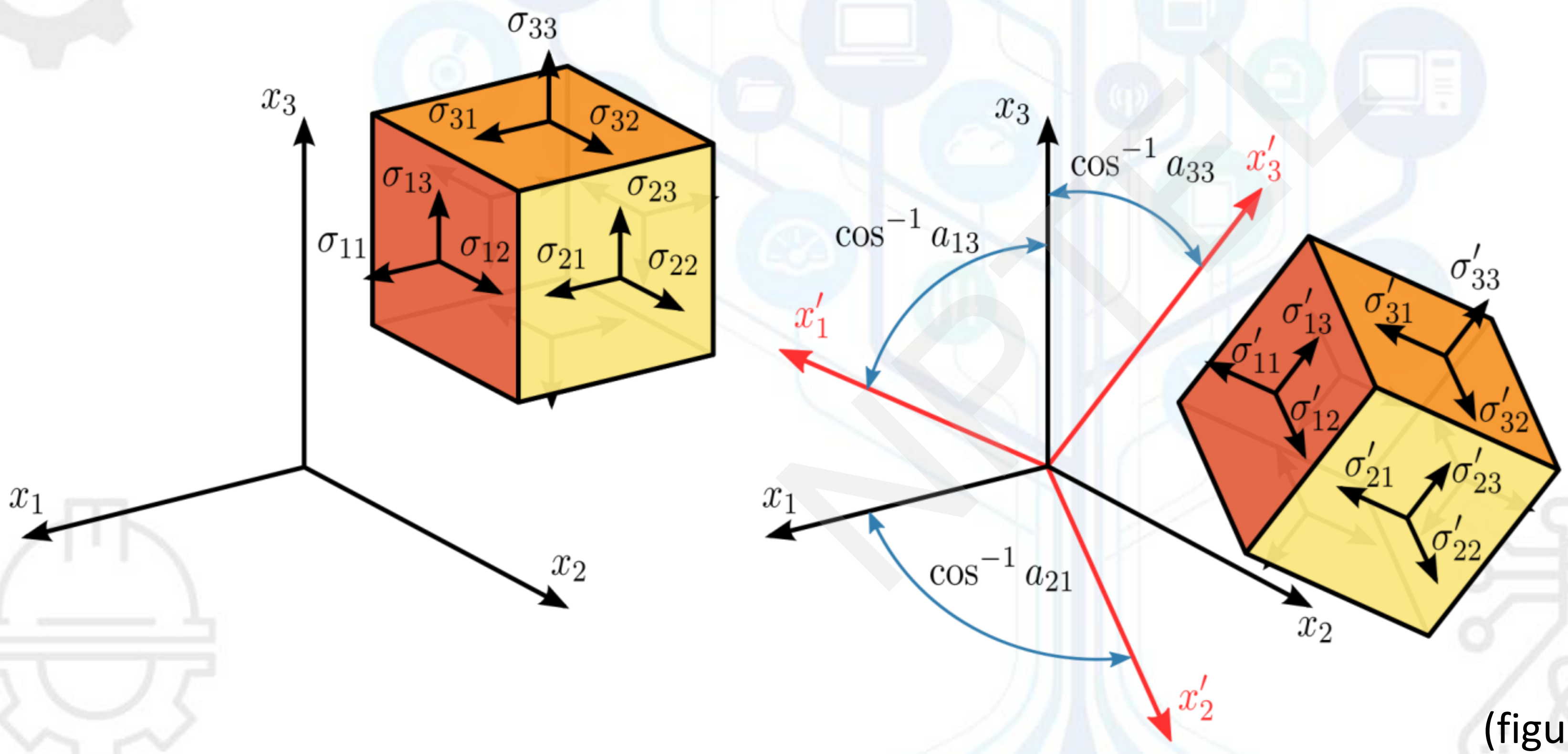
Arrows point in direction of positive stress.

Stress Units

- Units are in force/area
  - SI: 1 Pascal (Pa) = 1 N/m^2 = 10^{-5} bars
  - CGS: 1 bar = 10^6 dyn/cm^2
  - 1 atm = 1.01 bars
  - Generally, In Earth Sciences following units are used: kbar, Mpa, and Gpa. 1 Gpa = ~30 km depth = 10 kbar

Rotational Invariants of stress tensor

As we transform our coordinate system, say going from x to x', there are some aspects/quantities of the stress tensor that stay the same terms as rotational invariants.



(figure from Wikipedia)

Invariants

There are 3 invariants for the stress tensor (and indeed for any 2nd rank tensor).

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix}$$

$$= \sigma_{11}\sigma_{12} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22}$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22}$$

$$= \det(\sigma)$$

These quantities will be constant, regardless of change in coordinate system

What do the invariants mean physically?

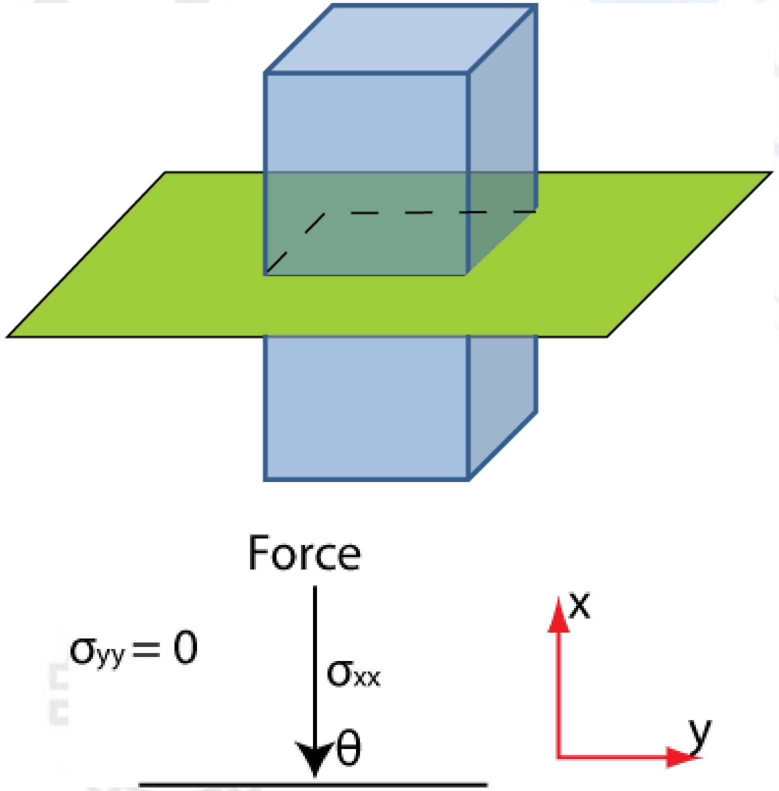
- I_1 and I_2

I_1 tells us that the sum of the normal stresses is remains unchanged. The mean of stresses in I_1 represents the physical quantity , pressure.

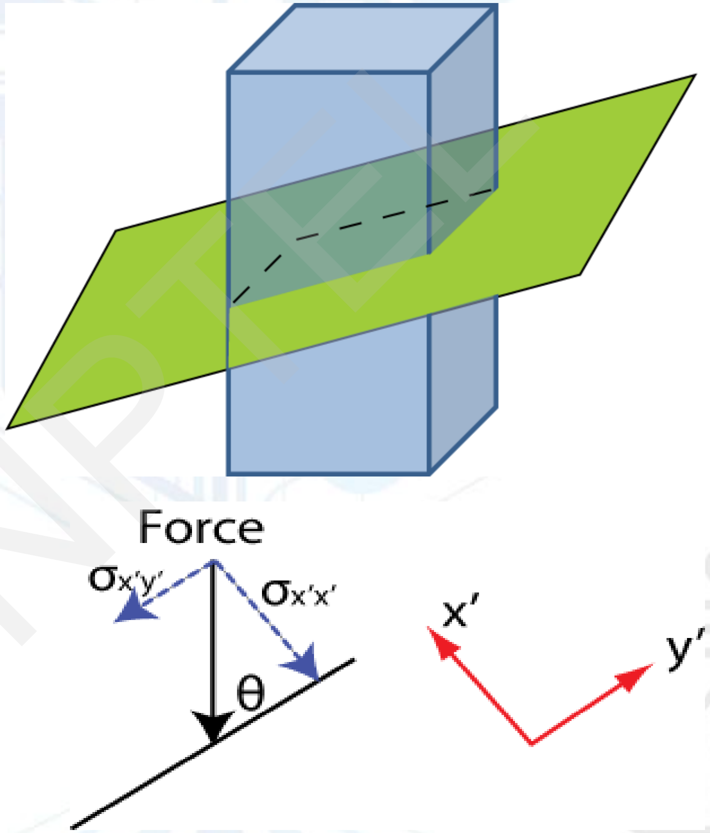
I_2 is related to the shear stress, and indicates when a ductile material will start to deform plastically (i.e. permanently). This is called the Von Mises yield criterion.

Stress and Orientation

Given a stressed solid, the magnitude of the normal and shear stresses varies on the coordinate system which we consider



No shear stress, normal stress maximized



shear stress, less normal stress

Principal Stresses

- At every point within a body, we can define a coordinate system for that there exists no shear stresses.
- This indicates all tractions are parallel to normal vectors in that coordinate system. Mathematically:

$$\vec{T}^{1} = \lambda_1 \hat{n}^{1}$$

$$\vec{T}^{2} = \lambda_2 \hat{n}^{2}$$

$$\vec{T}^{3} = \lambda_3 \hat{n}^{3}$$

- These equations represents that, in this considered coordinate system, there are three traction vectors in the direction of the three normal vectors in the coordinate system, but scaled by a factor lambda.

What this concludes?

1) There is a always set of directions given by $\hat{n}^1, \hat{n}^2, \hat{n}^3$, such that there are no shear tractions for that particular directions. These are called the principal stress directions.

2) The magnitude of the normal stresses is equal to the magnitude of the tractions, and is given by $\lambda_1, \lambda_2, \lambda_3$. These are called the principal stresses, and are commonly denoted by . $\sigma_1, \sigma_2, \sigma_3$

Calculating Principal Stresses

Cauchy's Stress Theorem tells us that the traction vector is the stress tensor times the unit normal to the plane of interest.

$$\vec{T} = \vec{\sigma}\hat{n}$$

In Einstein Summation Notation:

$$T_i = \sigma_{ij} n_j$$

This means we sum over repeated indices in multiplied quantities (in this case j). So, the above equation means:

$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$T_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$T_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Calculating Principal Stresses

We also know that there is a coordinate system such that the three traction vectors are perpendicular to three normal vectors (i.e. k is from 1 – 3 in the equation below—we are not using ESN in this equation)

$$\vec{T}^k = \lambda_k \hat{n}^k$$

For a given vector (arbitrary k), the 3 components of that vector can be described by:

$$T_i = \lambda n_i$$

Calculating Principal Stresses

Setting this equal to the CST for any one traction vector (arbitrary k), we get

$$\sigma_{ij} n_j = \lambda n_i$$

Use the Kronecker delta to observe,

$$n_i = \delta_{ij} n_j$$

Here i = free index and j = dummy index

$$For\, i = 1, \quad n_1 = \delta_{11} n_1 + \delta_{12} n_2 + \delta_{13} n_3 = \delta_{11} n_1$$

$$For\, i = 2, \quad n_2 = \delta_{21} n_1 + \delta_{22} n_2 + \delta_{23} n_3 = \delta_{22} n_2$$

$$For\, i = 3 \quad n_3 = \delta_{31} n_1 + \delta_{32} n_2 + \delta_{33} n_3 = \delta_{33} n_3$$

(The off-diagonal terms $\delta_{12} n_2$, $\delta_{13} n_3$, $\delta_{21} n_1$, $\delta_{23} n_3$, $\delta_{31} n_1$, $\delta_{32} n_2$ are struck through and marked 0.)

Calculating Principal Stress

This means that our new equation can be written as

$$\sigma_{ij} n_j = \lambda \delta_{ij} n_j$$

Putting stuff over to one side, we get:

$$\sigma_{ij} n_j - \lambda \delta_{ij} n_j = 0$$

In matrix notation, this implies:

$$\begin{bmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Calculating Principal Stresses

This is just an eigenvalue/eigenvector equation with λ being and eigen value.

$$\left|\sigma_{ij} - \lambda_i \delta_{ij}\right| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

On solving above problem and rearranging the terms, we can find the characteristic equation in terms of invariants

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

This is cubic order equation hence three eignen value will exist for three principal stresses, our principal stresses σ1, σ2, and σ3, where σ1 is the largest value, σ2 is next largest, etc.

Principal Stress Directions

After calculating the principal stresses, we can be calculate the the principal stress directions. For instance, for (the first principal stress), we can put in these values to solve for the components of each n vector

$$\begin{bmatrix} \sigma_{11}-\lambda_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda_1 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda_1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For finding the other principal stress direction we can repeat for λ_2 and λ_3.

Deviatoric Stress

Recall that the first invariant is: $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$

This is equivalent to the quantity called the trace of the stress matrix, which is the sum of the diagonal elements

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

Sum these to get the trace

Deviatoric Stress

In terms of principal stresses, $I_1 = \sigma_1 + \sigma_2 + \sigma_3$

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

From the above, mean stress can be defined as:

$M = I_1/3 = Tr(\sigma_{12}) = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$

Sum these to get the trace

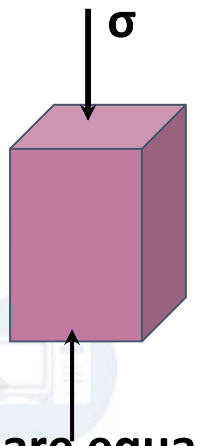
Deviatoric Stress

The deviatoric stress tensor is the stress tensor after the removal of mean stress.

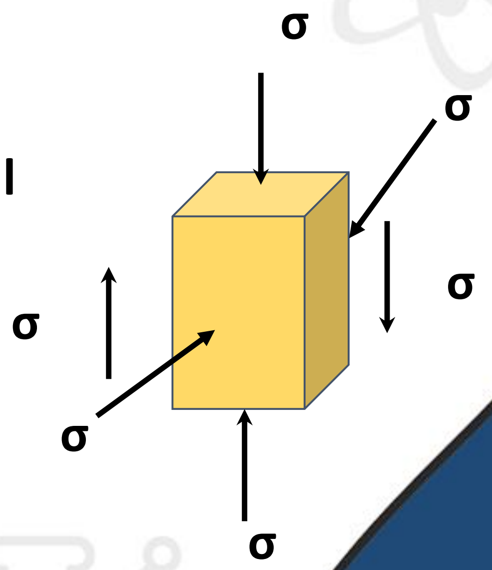
$$D = \begin{pmatrix} \sigma_{11} - M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - M \end{pmatrix}$$

Sum these to get the trace

More types of stress



- Uniaxial stress: σ_1 = something say σ and $\sigma_2 = \sigma_3 = 0$.



- Hydrostatic or lithostatic stress: when all the directional stress are equal i.e., $\sigma_1 = \sigma_2 = \sigma_3$

Differential stress: $\sigma_D = \sigma_1 - \sigma_3$

Differential stress plays an important role in the understanding of mantle rheology and anisotropy.

Pressure: $M = (\sigma_1 + \sigma_2 + \sigma_3)/3$

Sum these to get the trace

Rotation of Stress Tensor

The stress tensor can be rotated into a new coordinate frame by multiplying it through a transformation matrix, i.e., Rotation matrix

$$\sigma_{new} = R\sigma R^T$$

R is a rotation matrix rotates the cartesian coordinates in an anticlockwise direction through θ with respect to the x-axis in a 2-D system.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

It's a little complicated in 3D, as 3 angles of rotations is required to construct 3D rotation matrix. Let's ignore 3D for the time being, but you can find this on Wikipedia under "Rotation Matrix," and in the appendix of S&W.

Strain

Strain is fractional change in dimension of the body in the limit of infinitesimally small dimension. This is seen as change in shape that happens when rocks are deformed by the stress.

Strain can be categorised as :

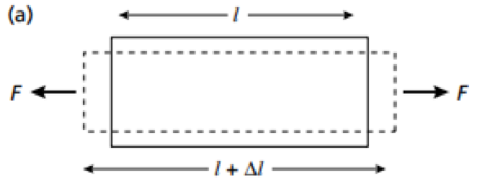
Uniaxial Longitudinal Strain

It is longitudinal strain when the material is constrained to deform uniaxially. It is developed when one direction is restricted to avoid strain and strain happens only in direction of tension.

Strain

Longitudinal Strain

The longitudinal strain or extension along an axis is fractional change in element along that axis. Mathematically, It is expressed as :



$$\text{Strain}(\epsilon) = \left(\frac{\Delta l}{l}\right)$$

Volumetric strain

It is fractional change in volume of an element when the its surface area decreases to zero.

$$\text{p}= \lim_{A\to 0}\left(\frac{\Delta v}{v}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

where u , v , and w are infinitesimal change in x, y, $and\ z$ direction.

Strain

Shear Strain

Shear strain is manifestation of angular relationship between the parts of the body. Mathematically, shear strain in a plane is half the total angular distortion.

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

where u , v , and w are infinitesimal change in x, y, and z direction.

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