NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 07 : Anisotropic earth structure, Attenuation and Anelasticity.
Lecture 04: Intrinsic Attenuation and Quality Factor

CONCEPTS COVERED

- **Intrinsic Attenuation**
- **Quality Factor**
- **Summary**

Recap

- Geometric spreading, scattering and multipathing reduces the amplitudes but propagating wavefield is conserved.

- Seismic velocities depend nearly linearly upon temperature, whereas attenuation depends exponentially on temperature. Thus combining velocity and attenuation studies can provide valuable information.
- Because the earth is a sphere, the ring wraps around the globe making the energy per unit wavefront of the surface waves vary as:

  $1/r = 1/(a \sin \Delta)$, where Δ is the angular distance from the source

- Amplitudes decrease as $(a \sin \Delta)^{-1/2}$, with minimum at $\Delta = 90°$, and maxima at $0°$ and $180°$.

- For Body waves, energy per unit wave front decays as $1/r^2$, and the amplitude decreases as $1/r$.

Recap

- Seismic waves refract towards low-velocity anomalies and away from high velocity anomalies.
- When multipathing occurs, the seismic waves arriving at a receiver can be viewed as having taken some ray paths in addition to the direct path, and so have sampled a larger region of the earth.
- When the heterogeneity is large compared to the wavelength, we regard the wave as following a distinct ray path that is distorted by multipathing.
- Scattering is especially important in the continental crust, which has many small layers and reflectors resulting from billions of years of continental evolution.
- The main arrival has a polarity related to the direction of propagation but the scattered energy arrives from various directions shows little or no preferred particle motion.

Intrinsic Attenuation (or Anelasticity)

➔ This is a process when seismic wave energy is converted to heat when the seismic wave produces irrecoverable deformation.

➔ To look at this process, let's examine a spring with a mass.

$$m\frac{d^2u(t)}{dt^2} + ku(t) = 0$$

u(t): displacement of mass "m"; k: spring constant

$$u(t) = Ae^{i\omega_o t} + Be^{i\omega_o t}$$

"A" and "B" are constants

(From wikipedia)

➔ Mass moves back and forth with a natural frequency $\omega_o = (k/m)^{1/2}$

➔ One example of this general solution is $u(t) = A_o \cos(\omega_o t)$

➔ This undamped oscillation continues forever, because no energy is lost.

Intrinsic Attenuation

But there is should be a loss of energy for attenuation and hence a damping term (or dashpot) is introduced. The damping force is proportional to the velocity of the mass and opposes its motion.

$$m\frac{d^2u(t)}{dt^2} + \gamma m\frac{du(t)}{dt} + ku(t) = 0$$

γ is the damping factor

To simplify this, we define the quality factor $Q = \omega_o/\gamma$

and rewrite above equation,

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_o}{Q}\frac{du(t)}{dt} + \omega_o^2 u(t) = 0$$

……equ. 1

This differential equation can be solved assuming that the displacement is the real part of a complex exponential

$$u(t) = A_o e^{ipt}$$

……equ. 2

where p is a complex number.

Intrinsic Attenuation

Substituting Eqn 2 into Eqn 1 yields

$$\left(-p^2 + ip\omega_o/Q + \omega_o^2\right) A_o e^{i(pt)} = 0$$

For this to be satisfied for all values of t,

$$-p^2 + ip\omega_o/Q + \omega_o^2 = 0 \qquad \text{……equ. 3}$$

Because p is complex, we break it into its real and imaginary parts,

$$p = a + ib, \quad p^2 = a^2 + 2aib - b^2$$

so Eqn 3 gives

$$-a^2 - 2iab + b^2 + ia\omega_o/Q - b\omega_o/Q + \omega_o^2 = 0$$

which can be split into equations for the real and imaginary parts and solved separately:

Real: $-a^2 + b^2 - b\omega_o/Q + \omega_o^2 = 0$

Imaginary: $-2iab + ia\omega_o/Q = 0$

Intrinsic Attenuation

Solving the imaginary part for “b” gives

$$b = \omega_o / 2Q$$

and putting this into the equation for the real part gives

$$a^2 = \omega_o^2 - \omega_o^2/4Q^2 = \omega_o^2\left(1 - 1/4Q^2\right)$$

Thus we define

$$\omega = a = \omega_o\left(1 - 1/4Q^2\right)^{1/2}$$

and rewrite u(t) with separate real and imaginary parts,

$$u(t) = A_o e^{i(\omega t + ibt)} = A_o e^{-bt} e^{i\omega t}$$

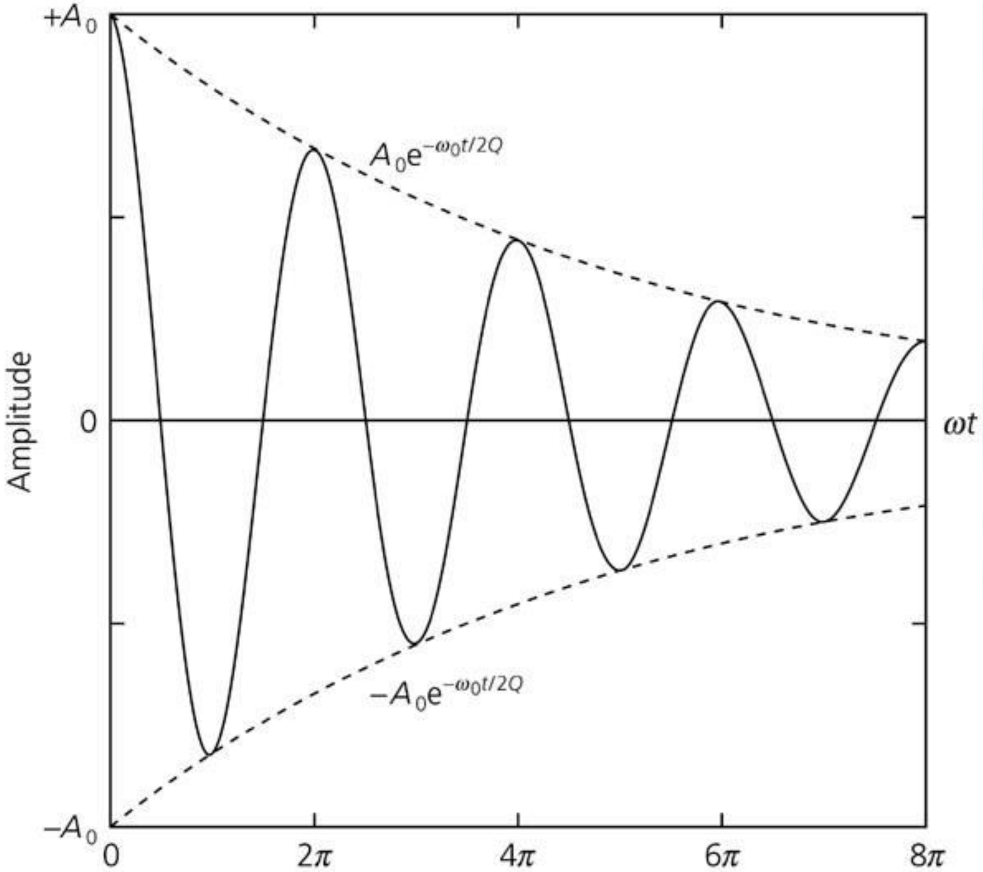
The real part is the solution for the damped harmonic displacement,

$$u(t) = A_o e^{-\omega_o t/2Q} \cos\left(\omega t\right)$$

Note that for this to be real, Q > ½. Since typically Q is much larger in the Earth, we won’t worry about this being imaginary.

Intrinsic Attenuation

Figure 3.7-11: Wave amplitude for a damped harmonic oscillator.



- It is no longer a simple harmonic oscillation. The exponential term expresses the decay of the signal's envelope, or overall amplitude, $u(t) = A_o e^{-\omega_o t/2Q}$, ……equ. 4

  which is superimposed on the harmonic oscillation given by the cosine term.

- Q is inversely proportional to the damping factor, γ, so the smaller the damping, the greater Q is.

Intrinsic Attenuation

- As the damping increases, Q decreases, so the amplitude decays faster.

- We refer to the time it takes the signal to decay to 1/e of its initial amplitude A_0 as the relaxation time.

$$t_{1/e} = 2Q/\omega_o \qquad \text{……equ. 5}$$

- Because the energy in an oscillating system is proportional to the square of the amplitude

$$E(t) = \frac{1}{2}kA(t)^2 = \frac{1}{2}kA_o^2 e^{-\omega_o t/Q} = E_o e^{-\omega_o t/Q} \qquad \text{……equ. 6}$$

- Energy decays faster than the amplitude, because the negative exponent in Eqn 6 is twice as large as in Eqn 5.

Quality factor, Q

- Attenuation for seismic waves and a variety of other physical phenomena are often discussed in terms of Q or Q^{-1}.

- Although Q has more convenient values, Q^{-1} has the advantage that it is directly, rather than inversely, proportional to the damping.

- We speak of the Q of surface waves, body waves, and crustal phases like L_g and variation within the earth of Q_α and Q_β , which controls the attenuation of P and S waves.

- The anelastic structure of the earth, given by variations in Q_α and Q_β , is analogous to the elastic velocity structure because Q can be viewed mathematically as an imaginary part of the velocity.

Quality factor, Q

We have derived the decaying oscillation which can be viewed as an oscillation with a complex frequency p

$$u(t) = A_o e^{ipt} = A_o e^{i(a+ib)t}$$

where the real and imaginary parts of the frequency are

$$a = \omega, \;\; b = \omega^* = \omega_o/2Q \approx \omega/2Q$$

assuming that attenuation is small (Q large) enough that $\omega \approx \omega_0$. Hence we write

$$Q^{-1} = 2b/a = 2\omega^*/\omega.$$

Treating the attenuation as an imaginary part of the frequency and dividing by the wavenumber lets us treat the corresponding velocity for a propagating wave as complex,

$$c + ic^* = \omega/k + i\omega^*/k = \omega/k + i\omega Q^{-1}/2k$$

So,

$$Q^{-1} = 2c^*/c$$

Thus we can express the attenuation of P- and S-waves by using the quality factors Q_α and Q_β to give imaginary parts to the velocities.

Quality factor, Q

“If there is no attenuation (Q = ∞), the frequency and the velocity have no imaginary parts.”

We pose the complex parts of the velocities in terms of the properties of the material causing attenuation by treating the elastic moduli as having imaginary parts.

For the shear velocity,

$$\beta + i\beta^{*} = \beta\left(1 + iQ_{\beta}^{-1}/2\right)$$
$$= ((\mu + i\mu^{*})/\rho)^{1/2} = \beta(1 + i\mu^{*}/\mu)^{1/2}$$
$$\approx \beta(1 + i\mu^{*}/2\mu)$$

From Taylor’s series

Since $$Q^{-1} = \frac{2w^{\cdot}}{w} = \frac{\frac{2w^{\cdot}}{k}}{\frac{w}{k}} = \frac{2\beta^{\cdot}}{\beta}$$

Comparing terms shows that $$Q_{\beta}^{-1} = \mu^{*}/\mu$$

Similarly the quality factor for P waves is given by the imaginary parts of both the bulk and shear moduli

$$Q_{\alpha}^{-1} = (K^{*} + 4/3\mu^{*})/(K + 4/3\mu)$$

Quality factor, Q

Attenuation in terms of imaginary parts of the compressibility and rigidity:

$$Q_K^{-1} = K^*/K, \qquad Q_\mu^{-1} = \mu^*/\mu = Q_\beta^{-1}$$

These quality factors are related to those for the velocities by

$$Q_\alpha^{-1} = LQ_\mu^{-1} + (1 - L)Q_K^{-1}, \qquad L = (4/3)(\beta/\alpha)^2$$

"In general little energy is lost in compression, so Q^{-1}_K is very small, and thus most of the attenuation for P waves occurs in shear, making $Q_\alpha^{-1} \approx (4/9)Q_\beta^{-1}$ **"**

Techniques for measuring Q in the earth follow from those used to measure Q for the decay of an oscillation.

Taking the natural logarithm of the envelope (in equ. 4) shows that

$$\ln A(t) = \ln A_o - \omega_o t/2Q,$$

so Q can be found from the slope of the logarithmic decay.

Quality factor, Q

Alternatively, if successive peaks one full period $T = 2\pi/\omega_0$ apart have amplitudes

$$A_1(t_1) = A_o \exp\left(-\omega_o t_1/2Q\right) \quad \text{and}$$
$$A_2(t_1+T) = A_o \exp\left(-\omega_o (t_1+T)/2Q\right),$$

their ratio is

$$A_1/A_2 = \exp\left(-\omega_o t_1/2Q - \omega_o(t_1+T)/2Q\right) = \exp\left(\pi/Q\right)$$

so,

$$Q = \pi/\ln\left(A_1/A_2\right)$$

Another way to view Q is as the number of cycles the oscillation takes to decay to a certain level. The number of cycles n, is

$$n = t/T = \omega t/2\pi \approx \omega_o t/2\pi$$

assumes that the attenuation is small enough (Q >> 1) so that $\omega \approx \omega_0$. The amplitude at time t_n, after n cycles, is

$$A(t_n) \approx A_o e^{\frac{-n\pi}{Q}}$$

A(t)

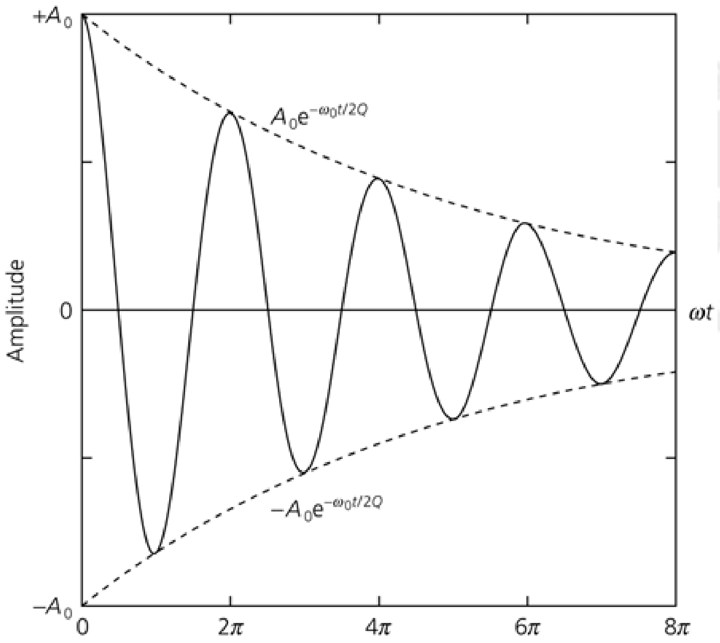
1) If we have A(t), the envelope of oscillation

$$A(t) = A_0 e^{-\omega_0 t/2Q}$$

$$\ln(A(t)) = \ln A_0 - \frac{\omega_0 t}{2Q}$$

So Q can be found from the slope of the natural log of the amplitude envelope as it changes with time.

Figure 3.7-11: Wave amplitude for a damped harmonic oscillator.



2) If you have successive peaks 1 period apart

$$A_1(t_1) = A_0 \exp(-\frac{\omega_0 t_1}{2Q})$$

$$A_2(t_2) = A_0 \exp(-\frac{\omega_0 (t_1 + T)}{2Q})$$

$$\frac{A_1}{A_2} = \exp\left[-\frac{\omega_0 t_1}{2Q} - \frac{\omega_0 (t_1 + T)}{2Q}\right]$$

Since $T = 2\pi/\omega_0$

$$\frac{A_1}{A_2} = \exp(-\frac{\pi}{Q})$$

after n cycles

$$A(t_n) \approx A_0 e^{-n\pi/Q}$$

This tells us how the amplitude of a wave varies with time at a given location. Another way to look at things is how amplitude varies in space (A(x)). Since t = x/c

$$A(x) = A_0 e^{-\omega_0 x/2cQ}$$

For ray theoretical methods, the integrated effect of attenuation along the wavepath is

$$t^* = \int \frac{dt}{Q(r)}$$

In frequency space, then

$$A(\omega) = A_0(\omega) e^{-\omega t^*/2}$$

This allows to get at variations in attenuation in the Earth. For a given earthquake observed at multiple seismometers, the relative variations in t* can be gotten at by ratio-ing the natural log of the spectrum of the event at each seismometer. The natural log of $A(\omega)$ is:

$$\ln\left(A(\omega)\right) = \frac{-\omega t^*}{2} + \ln A_0(\omega)$$

$$\frac{A_1(\omega)}{A_2(\omega)} = \underbrace{\frac{A_0(\omega)}{A_0(\omega)}}_{\text{same source}} \frac{e^{-\omega t_1^*/2}}{e^{-\omega t_2^*/2}}$$

$$ln\left(\frac{A_1(\omega)}{A_2(\omega)}\right) = \frac{-\omega}{2} t_1^* + \frac{\omega}{2} t_2^*$$

$$= \frac{1}{2}(\Delta t^*)\omega$$

Variations on this technique are regularly used in seismology.

Quality factor, Q

so, if we define n as equal to Q,

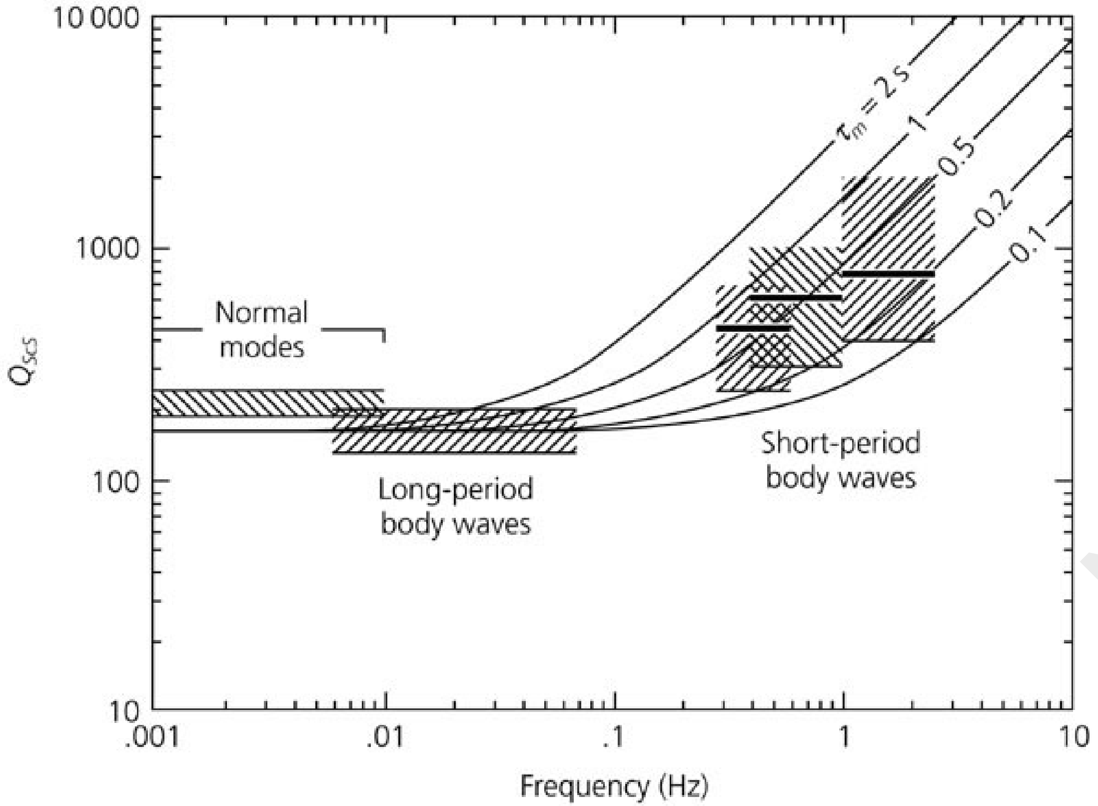
$$A(t_n) \approx A_o e^{-\pi} \approx 0.04 A_o$$

- Thus, after Q cycles, the amplitude drops to a level of $e^{-\pi}$ or 4% of the original amplitude.
- More than 95% of the amplitude is lost after Q ≈ 8 cycles.
- Q can describe the oscillation's decay in either time or space.
- For standing waves like normal mode oscillations, Q describes the decay of amplitudes with time.
- For traveling waves, we replace t with x/c, where x is the distance traveled and c is the velocity, which gives

$$A(x) = A_o e^{\frac{-\omega_0 x}{2cQ}}$$

Quality factor, Q

Figure 3.7-12: Frequency dependence of mantle attenuation.



- ➔ Frequency dependence of seismic waves in the mantle using ScS waves, a good measure of the average mantle.
- ➔ Q varies with frequency and is essentially constant at low frequencies, about 0.001 to 0.1 Hz, but then increases with frequency.
- ➔ Q values for normal modes are lower than those for higher frequency waves, such short and long period body waves.

Summary

- For the damped harmonic oscillator, displacement can be given: as: $x(t) = A_o e^{i(\omega t + ibt)} = A_o e^{-bt} e^{i\omega t}$ with $\omega = a = \omega_o\left(1 - 1/4Q^2\right)^{1/2}$ and $b = \omega_o/2Q$
- Relaxation time is given as: $t_{1/e} = 2Q/\omega_o$
- Q is inversely proportional to the damping factor, γ, so the smaller the damping, the greater Q is.
- Energy decay is given as: $E(t) = \frac{1}{2}kA(t)^2 = \frac{1}{2}kA_o^2 e^{-\omega_o t/Q} = E_o e^{-\omega_o t/Q}$
- Quality factor can be formulated as: $Q^{-1} = 2b/a = 2\omega^*/\omega.$ and $Q^{-1} = 2c^*/c$

$$Q_\beta^{-1} = \mu^*/\mu \qquad Q_\alpha^{-1} = (K^* + 4/3\mu^*)/(K + 4/3\mu)$$

Summary

- $Q = \pi / \ln{(A_1/A_2)}$ **A_1 and A_2 are the amplitudes of two consecutive cycles**

- **For traveling waves, we replace t with x/c, where x is the distance traveled and c is the velocity, which gives**

$$A(x) = A_o e^{\frac{-\omega_O x}{2cQ}}$$

- **Q values for normal modes are lower than those for higher frequency waves, such short and long period body waves.**

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THANK YOU !