NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 01 : Basic Seismological Theory, Waves on a String, Stress and Strain and seismic waves

Lecture 04: Strain tensor, Constitutive equations, Hooke’s law, Elastic moduli

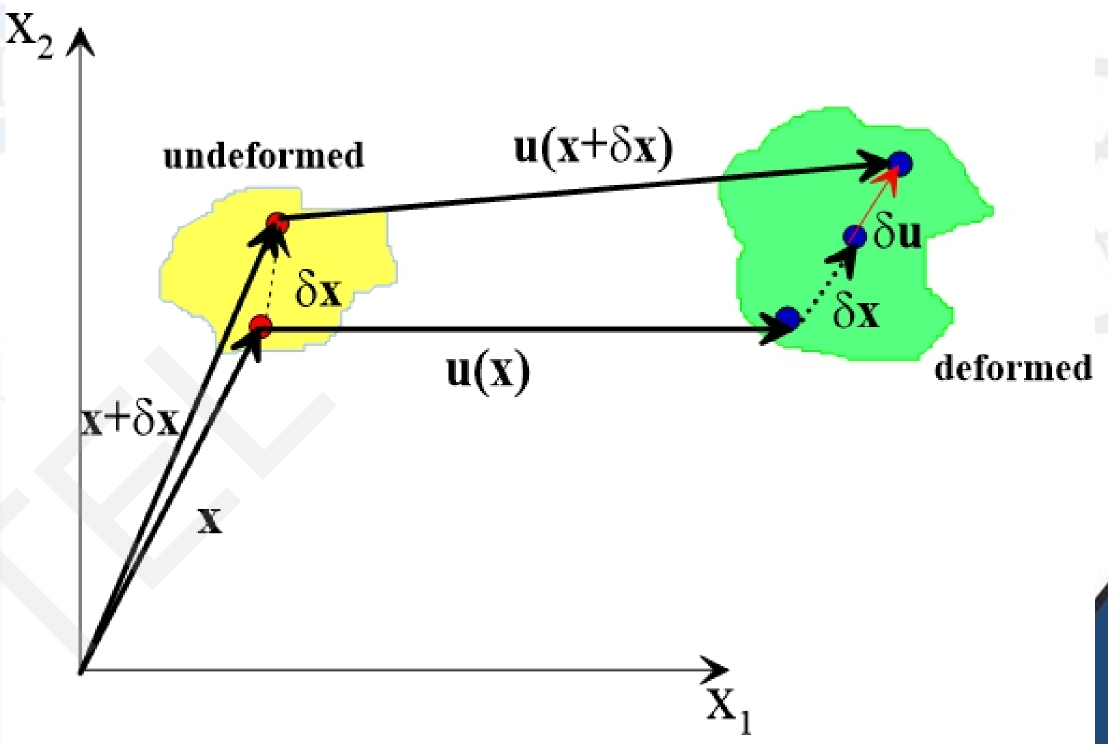
CONCEPTS COVERED

- Strain
- Constitutive equations, Hooke's law in seismology
- Hooke's law for isotropic material,
- Shear modulus, Young's modulus, Bulk modulus, Poisson's ratio

Strain

We are assuming the smooth variation of u(x) from one point to another, which is ideal case for strain in elastic body.

This means that if u(x) is the deformation at x, there is a function u(x)+du(x) that is the deformation at x + dx.



(From Stein and Wysession)

According to Taylor Series Expansion

Variation of u(x) with small change in position from x to x+δx is represented by

$$u_i\left(\mathbf{x}+\delta\mathbf{x}\right) \approx u_i\left(\mathbf{x}\right) + \nabla\mathbf{u} + \textbf{higher order terms} = u_i\left(\mathbf{x}\right) + \frac{\partial u_i\left(\mathbf{x}\right)}{\partial x_j}\delta x_j = u_i\left(\mathbf{x}\right) + \delta u_i\ ,$$

Displacement **Rotation+strain** **ignored since δx is small**

This implies (in ESN): $\delta u_i\left(\mathbf{x}\right) = \frac{\partial u_i\left(\mathbf{x}\right)}{\partial x_j}\delta x_j$

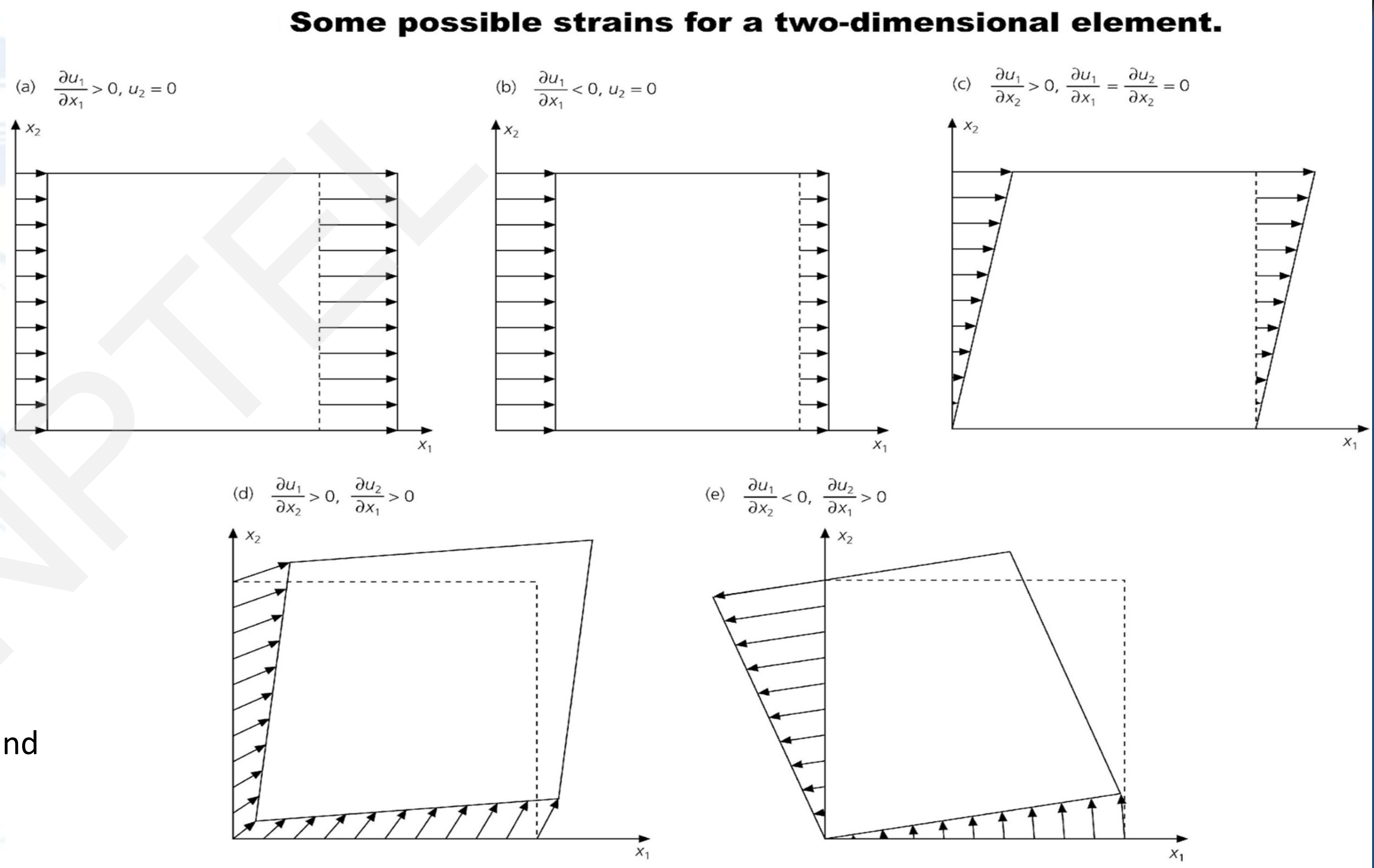
In long-hand: $\delta u_i\left(\mathbf{x}\right) = \frac{\partial u_i\left(x\right)}{\partial x_1}\delta x_1 + \frac{\partial u_i\left(x\right)}{\partial x_2}\delta x_2 + \frac{\partial u_i\left(x\right)}{\partial x_3}\delta x_3$ **for *i* = 1, 2, 3.**

δu(x) as a matrix equation

$$\delta u_i(\mathbf{x}) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$

Graphical representation of strain

- Note that mixed derivatives tend to produce shear.
- Some figures have volume change, some do not.



(From Stein and Wysession)

Using the previous equations

We have:

$$\delta u_i(\mathbf{x}) = \frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3$$

Separate into the symmetric components we can write

$$\delta u_i(\mathbf{x}) = \left(\frac{1}{2}\right).\left(\frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3\right) + \left(\frac{1}{2}\right).\left(\frac{\partial u_i(x)}{\partial x_1}\delta x_1 + \frac{\partial u_i(x)}{\partial x_2}\delta x_2 + \frac{\partial u_i(x)}{\partial x_3}\delta x_3\right)$$

According to Einstein notation

$$\delta u_i(\mathbf{x}) = \left(\frac{1}{2}\right)\left(\frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j\right) + \left(\frac{1}{2}\right)\left(\frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j\right)$$ **Here j is repeated index**

Adding $\frac{\partial u_j}{\partial x_i}$ to the first part, and subtracting from the second part

$$\delta u_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j$$

Write out summation for *j* (so you can see what it looks like in semi-non-ESN).

$$\delta u_i = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_1}\delta x_1 + \frac{\partial u_1}{\partial x_i}\delta x_1 + \frac{\partial u_i}{\partial x_2}\delta x_2 + \frac{\partial u_2}{\partial x_i}\delta x_2 + \frac{\partial u_i}{\partial x_3}\delta x_3 + \frac{\partial u_3}{\partial x_i}\delta x_3\right] +$$

$$\frac{1}{2}\left[\frac{\partial u_i}{\partial x_1}\delta x_1 - \frac{\partial u_1}{\partial x_i}\delta x_1 + \frac{\partial u_i}{\partial x_2}\delta x_2 - \frac{\partial u_2}{\partial x_i}\delta x_2 + \frac{\partial u_i}{\partial x_3}\delta x_3 - \frac{\partial u_3}{\partial x_i}\delta x_3\right]$$

Let's call the first term e_{ij} and the 2nd term ω_{ij} to get

$$\delta u_i = (e_{ij} + \omega_{ij})\delta x_j$$

e_{ij} defines the strain and ω_{ij} defines the rotation.

The e_{ij} term is called the *strain tensor.* Let's write out a few terms:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\delta x_j$$

***i*=1, *j*=1**

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$

***i*=1, *j*=2**

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Final expression for strain tensor

$$e_{ij} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

In 1, 2, 3 notation

$$e_{ij} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

In x, y, z notation

For seismology, the divergence of the displacement field is associated with change in volume which travels in the Earth as the P-wave velocity, and a rotation, the curl of the displacement field, propagates at the S-wave velocity.

Hooke's Law in Seismology

- The relationship b/w stress and strain is given by the material's constitutive equation. The simplest types of materials are linearly elastic and when Earth behaves as linearly elastic, it gives rise to seismic waves.
- Assume that material is elastic, we also assume that the displacements from an unstrained initial state are small. This assumption is generally valid for seismic waves.
- The stress and strain for a linearly elastic material are related by a constitutive equation called Hooke's law,

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

(in non-ESN)

$$= \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} e_{kl}$$

The constant C_{ijkl}, the elastic moduli, describe the properties of the material.

in the expanded form for i=1,j=1

$$\sigma_{11} = c_{1111}e_{11} + c_{1112}e_{12} + c_{1113}e_{13} + c_{1121}e_{21} + c_{1122}e_{22} + c_{1123}e_{23} + c_{1131}e_{31} + c_{1132}e_{32} + c_{1133}e_{33}$$

There are 8 more such equations for:

σ12
σ13
σ21
σ22
σ23
σ31
σ32
σ33

each with 9 C_{ijkl} terms.

Adding these up, we have 9 equations, each with 9 C_{ijkl} terms, thus 9 x 9 = 81 terms.

Symmetry of Cijkl

Recall:

- $\sigma_{ij} = \sigma_{ji}$ => 6 independent σ terms
- $e_{kl} = e_{lk}$ => 6 independent e terms.

This means:

-
- $c_{ijkl} = c_{jikl}$

  $c_{ijkl} = c_{ijlk}$

Another symmetry comes from thermodynamics:

-

$$c_{ijkl} = c_{klij}$$

These symmetries will reduce the elastic moduli from 81 terms to only 21 independent terms.

For an isotropic medium, the properties does not depend upon the direction, i.e. only two independent parameters remain.

These pairs are the Lame’s constants:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Hence, Hooke's law for an isotropic media can be written as:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$
$$= \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

where θ is the dilatation

$$\theta = e_{ii}$$
$$\equiv Tr(e_{ij})$$
$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$
$$= \nabla \cdot \mathrm{u}$$

This,

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$
$$= \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

in the expanded form it may look like as follows:

$$\sigma_{11} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{11}$$
$$\sigma_{22} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{22}$$
$$\sigma_{33} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{33}$$
$$\sigma_{12} = 2\mu e_{12} = \sigma_{21}$$
$$\sigma_{13} = 2\mu e_{13} = \sigma_{31}$$
$$\sigma_{23} = 2\mu e_{23} = \sigma_{32}$$

Elastic Parameters

"Incompressibility" or bulk modulus "K" is defined fractional change in the volume in response to a lithostatic pressure dp , such that

$$d\sigma_{ij} = dp\delta_{ij}$$

For isotropic elastic body , the resulting strains

$$-dp\delta_{ij} = \lambda d\theta\delta_{ij} + 2\mu de_{ij} \ (from\ \sigma_{ij=\lambda\theta\delta_{ij}} + 2\mu de_{ij})$$

setting i=j for normal stresses

$$-dp = \lambda d\theta + 2\mu de_{11}$$
$$-dp = \lambda\theta + 2\mu de_{22}$$
$$-dp = \lambda\theta + 2\mu de_{33}$$
$$\mathbf{-3dp = 3\lambda d\theta + 2\mu d\theta \ \ as\ \theta = e_{11} + e_{22} + e_{33}}$$

The bulk modulus is the ratio of the pressure applied to the fractional volume change

$$K = \frac{-dP}{d\theta} = \lambda + \frac{2}{3}\mu$$

Write constitutive equation in terms of the K and μ

$$\sigma_{ij} = K\theta\delta_{ij} + 2\mu\left(e_{ij} - \frac{\theta\delta_{ij}}{3}\right)$$

Elastic Parameters

Poisson's ratio gives the ratio of contraction along other two axes to the extension along the axis where tension was applied. Suppose tension is applied along x_1 axis,

$$\sigma_{11} = (\lambda + 2\mu)e_{11} + \lambda e_{22} + \lambda e_{33}$$

$$\sigma_{22} = 0 = \lambda e_{11} + (\lambda + 2\mu)e_{22} + \lambda e_{33}$$

$$\sigma_{33} = 0 = \lambda e_{11} + \lambda e_{22} + (\lambda + 2\mu)e_{33}$$

Subtracting the last two equations, will give the following,

$$e_{22} = e_{33} = \frac{-\lambda}{2(\lambda + \mu)} e_{11}$$

$$-\frac{e_{33}}{e_{11}} = \nu = \frac{\lambda}{2(\lambda + 2\mu)} \implies Poisson's\, ratio$$

and,

$$\frac{\sigma_{11}}{e_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = E$$

Here, E is the Young's modulus, the ratio of the tensional stress to the resulting extensional strain

Here is a relationships between most well-known moduli, from Stein and Wysession seismology book, Box 2.3-1 (pg. 51):

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\nu) = \frac{9K\mu}{3K+\mu} = 3K(1-2\nu)$$

$$K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\nu)}{3\nu} = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{\mu E}{3(3\mu-E)} = \frac{E}{3(1-2\nu)}$$

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(E-2\mu)}{3\mu-E} = K - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E}$$

$$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3KE}{9K-E}$$

Points to note

1. C_{ijkl} completely describe the behaviour of an elastic material.

1. λ does not have any physical meaning, but μ is called "rigidity or shear modulus".

1. A material with large μ is quite rigid and responds to a given stress with less strain & vice-versa.

1. A material in which μ is zero can not support shear stresses, & corresponds to a perfect fluid.

Many seismological problems are simplified by assuming that $\lambda=\mu$. Such a material is called a Poisson solid. In this case Poisson's ratio equals 0.25 and Young's modulus $E = (5/2)\mu$ and the Bulk modulus $K=(5/3)\mu$.

For Earth's crust, $\mu = 3 \times 10^{11}$ dyne/cm^2

whereas for steel, $\mu = 8 \times 10^{11}$ dyne/cm^2

For crust (assuming Poisson's solid), $E = 7.5 \times 10^{11}$ dyne/cm^2

For Rubber $E = 5 \times 10^9$ dyne/cm^2

What next in the que ?

Equation of motion, P-and S-waves

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THANK YOU !