FEM and Constitutive Modelling in Geomechanics
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Lecture: 3
Development of Equilibrium Equations for 1-D Systems

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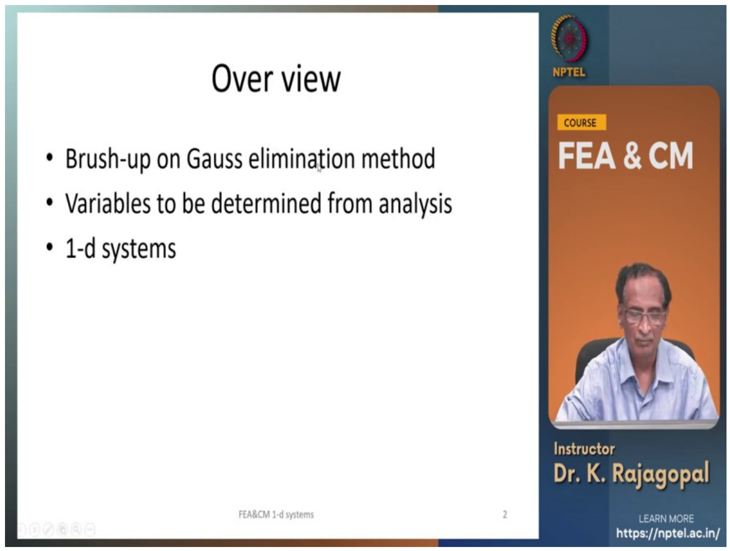

So, hello students let us continue from our previous lecture. In the previous lecture we had seen some basics of the Matrix algebra. We require the Matrix operations for understanding all our subsequent calculations and in today's lecture we will look at the axially loaded one-dimensional systems we will see how to analyze them and just a brief brush up on the Gauss elimination method; before we go with the today's topic.

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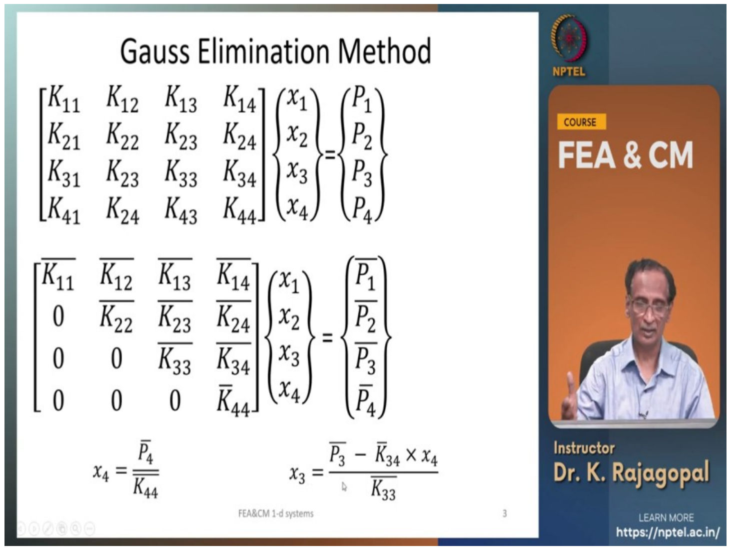

So, we had seen that if you have system of simultaneous equations with variables x1, x2, x3, and x4 we can solve it by either Matrix inversion or by Gauss elimination method.

And between these 2 are the Gauss elimination method we had discussed is more suitable for finite element calculations because we have a bandwidth Matrix where significant number of stiffness coefficients away from the diagonal are zero.

Gauss Elimination Method

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{23} & K_{33} & K_{34} \\ K_{41} & K_{24} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

$$\begin{bmatrix} \overline{K_{11}} & \overline{K_{12}} & \overline{K_{13}} & \overline{K_{14}} \\ 0 & \overline{K_{22}} & \overline{K_{23}} & \overline{K_{24}} \\ 0 & 0 & \overline{K_{33}} & \overline{K_{34}} \\ 0 & 0 & 0 & \overline{K}_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \overline{P_1} \\ \overline{P_2} \\ \overline{P_3} \\ \overline{P_4} \end{Bmatrix}$$

$$x_4 = \frac{\overline{P}_4}{\overline{K_{44}}} \qquad x_3 = \frac{\overline{P_3} - \overline{K}_{34} \times x_4}{\overline{K_{33}}}$$

So, we first convert our system of equations into an upper diagonal matrix like this. So, that in the last equation we have only one coefficient and then we can get the last variable x 4 as P 4 bar by by K 44 bar and once we determine the x 4 we can determine x 3 from the previous equation because we have 2 unknown 2 quantities here x 3 and x 4 and x 4 was already determined and now we can determine x 3 like this and continue the process x 2 x 1 and so on.

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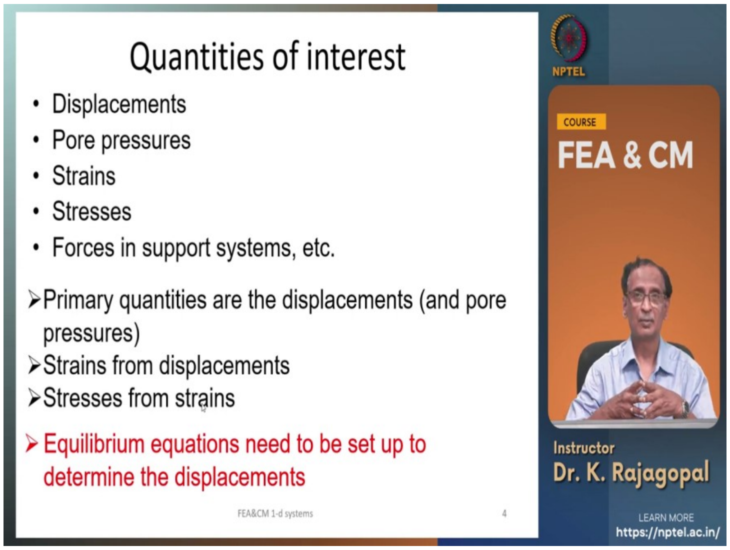

So, now let us see what are all the quantities that we solve for in any geotechnical problem? The first quantity is a displacements. Displacements could be only one when we are dealing with one dimensional problems are 2 at each node when we are dealing with the 2 dimensional and three when we are dealing with the three dimensional problems. And then we are we have the pore pressures and then we have the strains and then stresses and then the forces in the support systems are in the soil and so on.

And the primary quantities that we determine are the displacements than in the problems with water we also have the pore pressure and we determine the strains from the displacements using the compatibility equations. And then we can determine the stresses from the strains using the constituted equations that we will see later in some other lectures. And we need to set up some equilibrium equations.

So, that we can determine the displacements that is some stiffness multiplied by displacement is equal to force and what we mean by equilibrium equation is the applied force should be exactly equal to the reaction force and that is the basis on which we will develop all our equations and initially we will discuss only the problems with the only displacements that is the elasticity problems and later we will introduce water and we discuss the poor elastic problems later.

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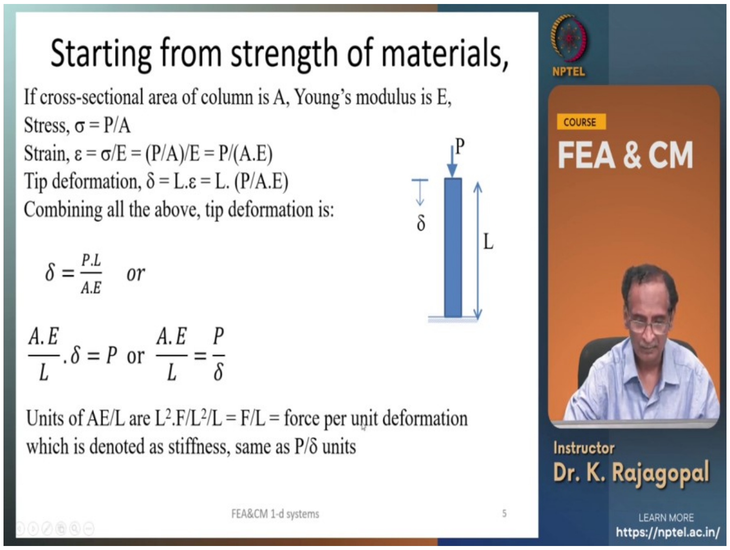
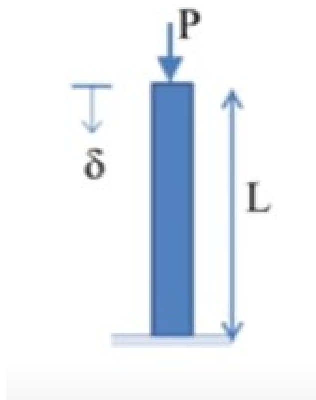


So, let us start from the most fundamental strength of materials because all of us are familiar with this famous equation Delta is equal to PL by AE that is the the equation that we study the first thing in any of the strength of materials courses.



Let us look at columnar element of length L and area A and let its Young's modulus be E and let us say that we applied a load of P and we would like to know what is the displacement at the top.

And for doing that we can do the following we can determine the stress as the load divided by the area P by A

$$\sigma = P / A$$

The strain is the stress divided by the Young's modulus and the stress is P by A and E is the Young's modulus.

$$\acute{\varepsilon} = \sigma / E = P / (A.E)$$

So, P by AE and once we get the strain the tip deformation Delta is L times Epsilon where L is the length of the element. So, it is L times Del Epsilon is PL by AE.

$$\delta = L.\acute{\varepsilon} = L.\ (P / A.E)$$

And so, our Delta the tip displacement is PL by A E or we can also write this in a slightly different form as A E by L times Delta is P or A E by L is P by Delta.

$$\delta = \frac{P.L}{A.E} \quad or$$

$$\frac{A.E}{L}.\delta = P \text{ or } \frac{A.E}{L} = \frac{P}{\delta}$$

And let us look at the units for these quantities. Say the units of this product A E by L, A has the units of area L square P is the force per unit area of the stress F by L square and the whole thing divided by L. So, that is in the units of F by L that is the force per unit deformation.

And this A E by L is a quantity that we call as the stiffness and that has the units of force per unit deformation. And on the right hand side also we have the P by Delta, P has the units of force and Delta has the units of displacement or the length. So, it is P by Delta. So, both the left hand side and the right hand side they have the same units. And so, A E by L is our stiffness and that multiplied by some displacement is equal to the force.

And actually we can think of it in some other manner on the right hand side we have the applied force that is the that is what we are applying on the on the system and A E by L times Delta is the reaction force. So, at some point the reaction force should be exactly equal to the applied force and that is what we call as equilibrium.

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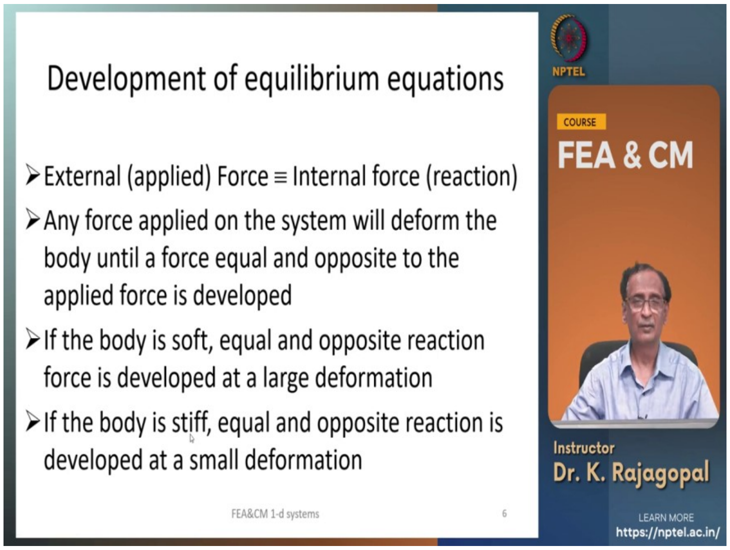

And so, what we mean by the equilibrium is externally applied force should be exactly equal to the internal force of the reaction that is a reaction is developed by the body to resist the applied forces. And so, any force that we apply on the system will produce some deformations that we may be able to measure with a ruler or with a micrometer or something depending on the magnitude of these deformations.

And and this body will go on deforming until an equal and opposite force is applied a force is developed to oppose the applied forces right. And if the body is very soft equal and the opposite reaction force is developed at a large deformation and if the body is stiff equal and opposite reaction is developed at a small deformation. So, that explains why if you have a very stiff spring and you apply some force it will deform very little but if you have a soft spring it will deform a large deformed by a larger amount.

And the governing principle is they will deform until an equal and opposite force is developed to oppose the opposed applied force.

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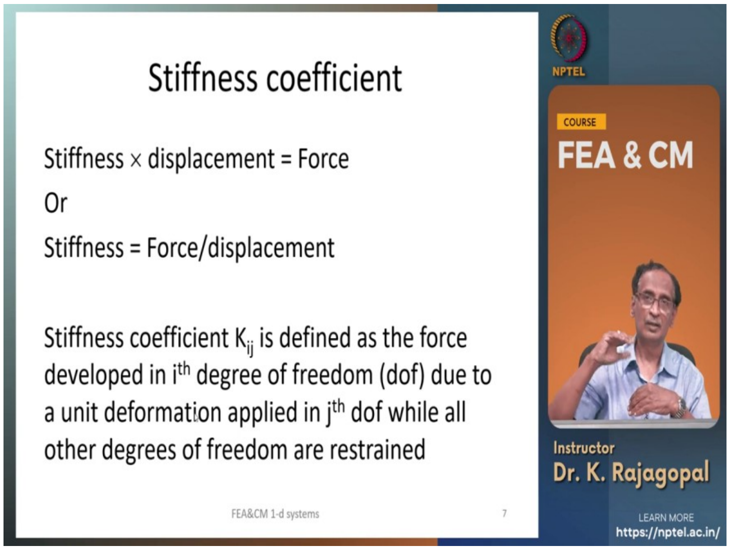

So, this stiffness coefficient is required because we require to calculate the reaction force that is calculated as stiffness times displacement and that is equal to the applied force on the right hand side. The stiffness is a force divided by the displacement and in general we Define stiffness coefficient K ij is actually we are going to deal with the multi degree of freedom system.

So, we have to develop these systems for large number of displacements or the degrees of freedom and our stiffness coefficient K ij is defined as the force developed in the ith degree of freedom dof due to a unit deformation applied in the jth degree of freedom. So, we apply some unit displacement in one direction and then what is the reaction force developed in the i direction that is called as the as the stiffness coefficient and while we are doing this we constrain all the other degrees of freedom that is the basic definition for K ij.

And let us try to develop these coefficients for simple systems before we go on to the continuum that is our soil.

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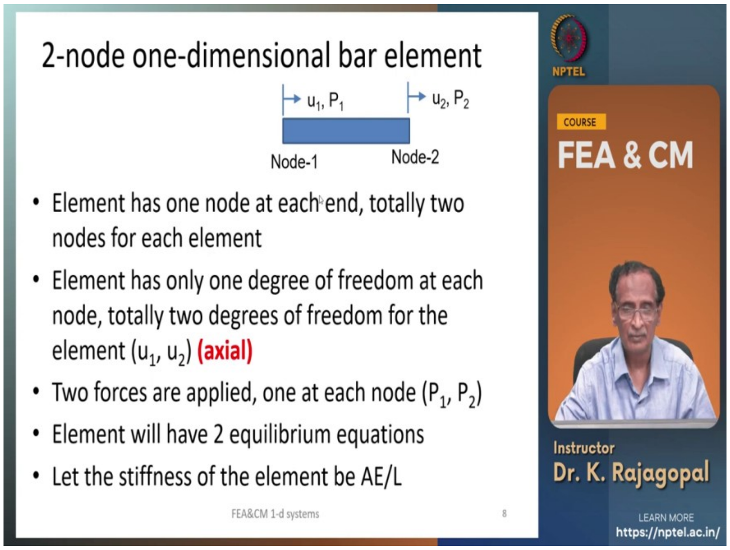
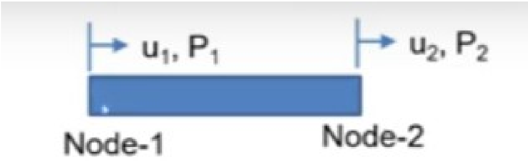


And let us start with one dimensional bar element that is purely axial all the deformations are only in the axial Direction and also the forces are in the axial Direction either tension or compression.



Now let us consider a simple 2 dimensional sorry one dimensional 2 node bar element with the 2 nodes node one and node 2 and 2 deformations u 1 and u 2 and 2 applied forces P 1 and P 2 and this element has nodes at 1 at each end.

So, Node 1 on the Left End node 2 at the right hand and this element has 2 axial degrees of freedom u 1 and u 2 and then 2 axial forces P 1 and P 2 and since we have 2 degrees of freedom this element will have 2 equilibrium equations that is a reaction at Node 1 R 1 should be exactly equal to P 1 and the reaction at node 2 should be exactly equal to the applied force P 2 and let us say that our stiffness is AE by L where A is the cross-sectional area of the element E is the young's modulus and L is the length of the of the element.

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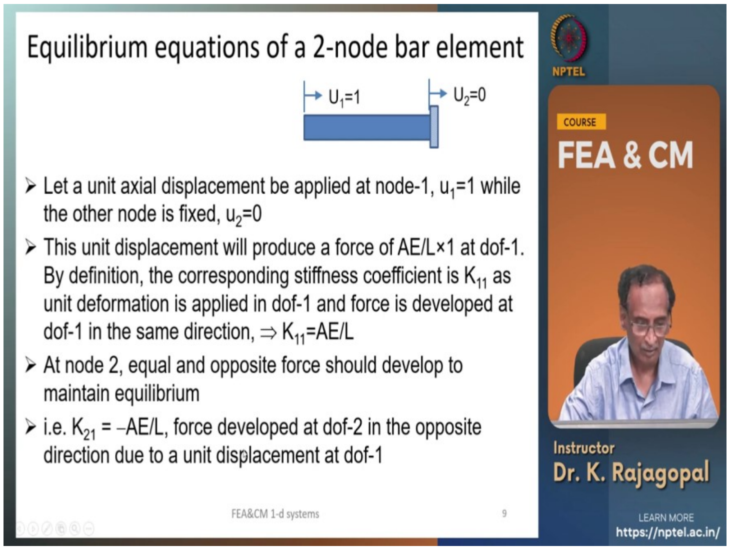

So, let us apply unit deformations in different directions and then measure the forces. So, that we can define our stiffness coefficients K ij and to start with let us apply a unit deformation a degree of freedom 1 that is u 1 is equal to 1. And while we are doing that we constrain this node or this degrees degree of freedom u 2 to 0. And so, the unit deformation at 1 will produce a force equal to A E by L times 1.

Because A E by L is our stiffness that is relate that is related to the force through the displacement. So, at degree of freedom 1 the force developer is AE by L and by definition the corresponding stiffness coefficient is K 11 that is the force developed in degree of freedom 1 when a unit displacement is applied in the same degree of freedom one and the force developed is K 11 that is AE by L.

$$K_{11} = A.E / L$$

And at node 2 equal and the opposite Force should develop to maintain the equilibrium and at node 2 at the degree of freedom 2 now the stiffness coefficient will be K 21. The force developed at the degree of freedom 2 because of a unit displacement applied a degree of freedom one. So, that K 21 will be our needs to be equal to - a by L. So, that we have equal and opposite term reaction force at the other end.

$$K_{11} = - A.E / L$$

So, the force developed at degree of freedom 2 is - A E by L due to a unit displacement a degree of freedom 1.

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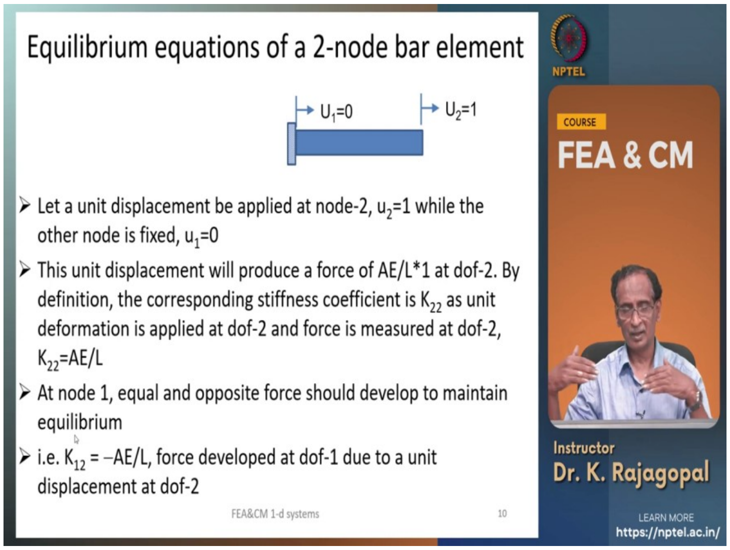

And similarly we can apply a unit displacement at a degree of freedom 2 while we constrain this degree of freedom from moving. So, by doing this we can write that K 22 is a by L that is the force developed in degree of freedom 2 because of a unit displacement a degree of freedom 2. And the force developed at degree of freedom 1 because of a unit displacement a degree of freedom 2 will be K 12 and that should be equal to - AE by L to maintain our equilibrium.

And otherwise the system will undergo some undue deformations that we call as rigid body deformations. So, here our K 12 is - AE by L that is the force developed at a degree of freedom one because of a unit displacement at degree of freedom 2.

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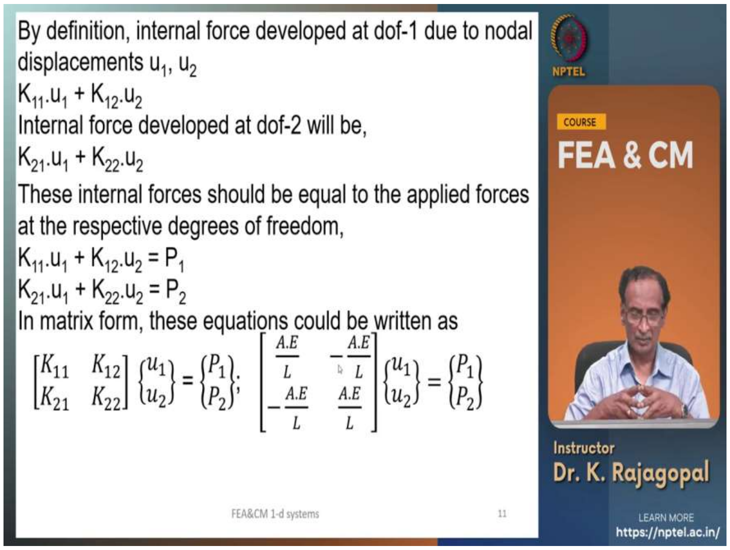

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}; \quad \begin{bmatrix} \frac{A.E}{L} & -\frac{A.E}{L} \\ -\frac{A.E}{L} & \frac{A.E}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

So, by combining these the internal forces developed can be written like this the internal Force developed a degree of freedom 1 because of displacements u 1 and u 2 will be

$$K_{11}.U_1 + K_{12} . U_2$$

and as we have seen K 11 is the force developed because of a unit displacement. So, that is the stiffness coefficient and if you apply a deformation of u 1 the force developed is K 11 times u 1.

And similarly the force developed in degree of freedom 1 because of a unit displacement to degree of freedom 2 is K 12. So, K 12 times u 2 is the reaction force development in degree of freedom 1. And similarly the internal Force developed in a degree of freedom 2 will be K 21 u 1 + K 22 u 2 and these are the internal forces or the reaction forces and these should be exactly equal to the applied forces P 1 and P 2.

$$K_{11}.u_1 + K_{12}.u_2 = P_1$$
$$K_{21}.u_1 + K_{22}.u_2 = P_2$$

So, K 11 u 1 + K 12 u 2 is P 1 and K 21 u 1 + K 22 u 2 is P 2 and we can write this in a matrix form like this

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}; \quad \begin{bmatrix} \frac{A.E}{L} & -\frac{A.E}{L} \\ -\frac{A.E}{L} & \frac{A.E}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

K 11 K 12 K 21 K 22 u 1 u 2 is equal to P 1 P 2, P 1 P 2 are the applied forces and K 11 K 12 and all these things are numerical equal to AE by L and K 11 and K 22 they are positive quantities because they refer to their own degrees of freedom that is K 12 K 21 are the cross terms the force developed at some other degree of freedom because of unit displacement at this degree of freedom and by definition that should be -.

So, there you go this is the set of equilibrium equations and then if you have the material properties the cross-sectional area Young's modulus and then the length of the element we can determine the stiffness coefficients and then get our stiffness Matrix and P 1 P 2 are the applied forces these are known and u 1 u 2 are the displacements. So, now the question is say if I give you the material properties and then the applied load can you determine u 1 and u 2.

That is can we invert this Matrix basically that is what we mean if you are able to invert this Matrix we can determine u 1 and u 2. So, we can easily find out if we can invert this or not by doing what by calculating the determinant of this Matrix. So, if you calculate the determinant A E by L multiplied by A E by L - of these 2 quantities and we see that the determinant is 0. So, that means that although we have the system of equations that we need we cannot solve. Before we solve we need to do something else that we will see later.

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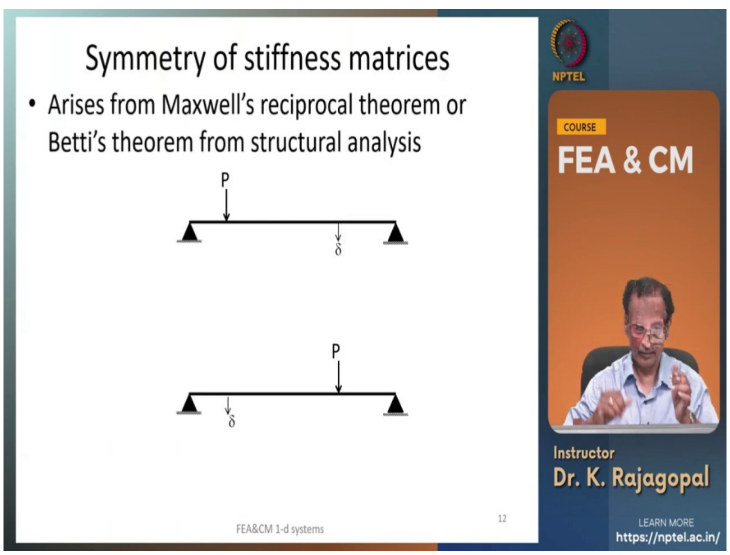
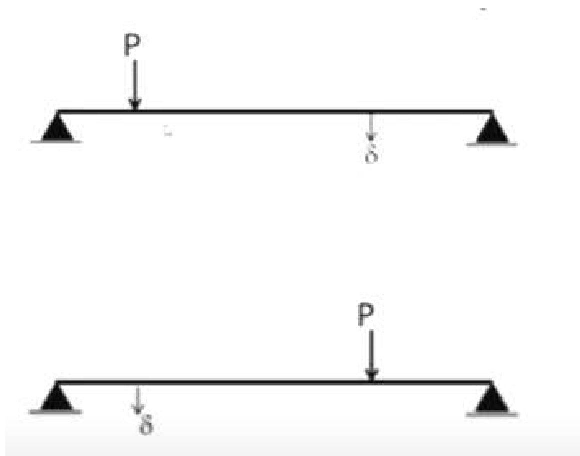


And now see we see that our system of equations gives us symmetric stiffness Matrix. See K 12 and K 21 are the same - AE by L and this symmetry we can imagine by looking at the reciprocal theorem Maxwell reciprocal theorem.



That is so, if we have a structure with a load applied at one point and you measure the displacement at some other point or apply the load at the other point and measure the displacement at this point the Delta should be the same.

Whether you apply the load here and measure the displacement here or apply the load here and measure the displacement here that is the Maxwell's reciprocal theorem and because of that we see that all our stiffness matrices they are symmetric.

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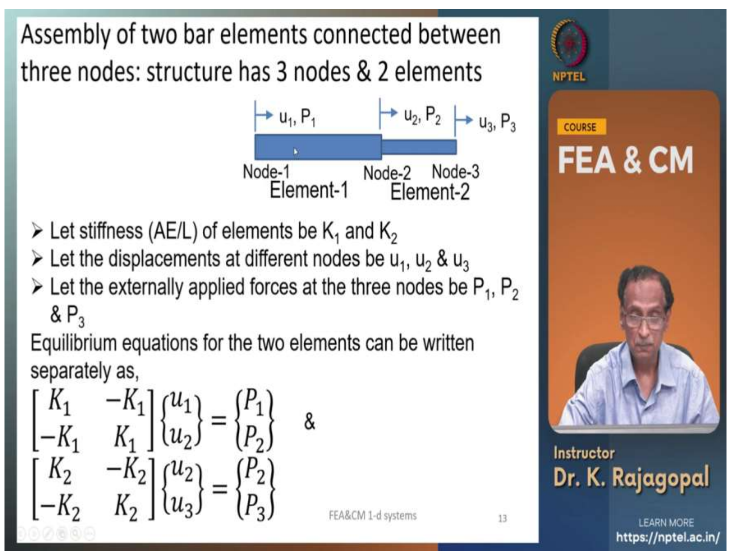
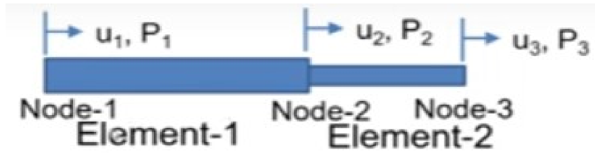


So, now let us continue for a larger body of of our structure let us consider now 2 elements element one and element two and it is not necessary that they have the same cross-sectional area under same properties and other things.



Let it let each one have their own properties and now we have three nodes node one node 2 node three and three displacements u 1 u 2 and u 3 and the three applied forces P 1 P 2 and P 3.

And let the stiffness of these 2 elements be K 1 and K 2 because theoretically each element can be made of different material they may have different length and different cross sectional areas and so on. And let the displacements be u 1 u 2 and u 3 and the applied forces are P 1 P 2 and P 3. And the equilibrium equations for the 2 elements can be written separately for element 1 and element 2 separately like this.

$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

$$\begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}$$

See for element 1 that is connected to degrees of freedom u 1 and u 2 we can write like this K 1 - K 1 - K 1 K 1 u 1 u 2 is equal to P 1 P 2. Similarly for element 2 it is connected to degrees of freedom u 2 and u3 and the applied forces are P 2 and P 3. So, this is our system of equations for element 2. And but then our structure has three degrees of freedom.

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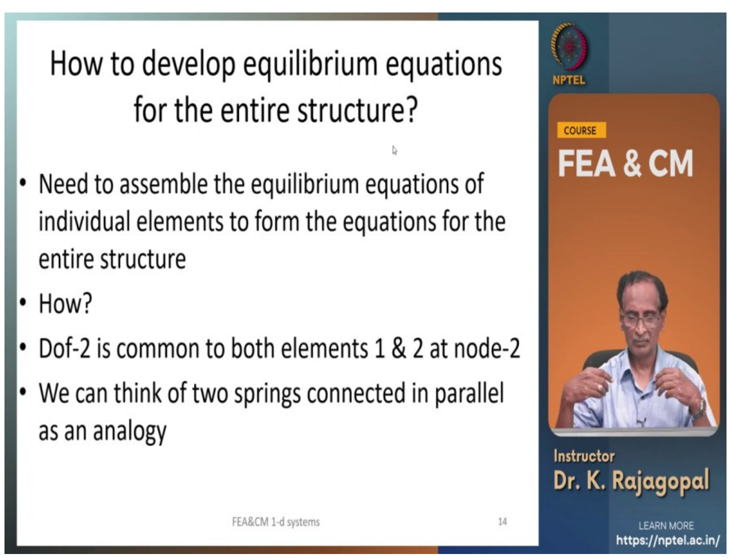

And now we have to see how we can develop the equilibrium equations for the entire structure. See now we have determined the equilibrium equations at the element level at for element 1 and element 2 separately. And we need to form the equilibrium equations for the entire structure and that we can do by assembling assembly of the contributions from different elements. And so, if we assemble we get the equations for the entire structure but how.

So, if you look at the degree of freedom 2 is common for element 1 and element 2. So, the displacement is the same at these 2 nodes for both elements one and element 2 right. And so, it is actually we can think of 2 Springs connected in parallel because these axial elements you can imagine them as the spring elements just string because that is what they are.

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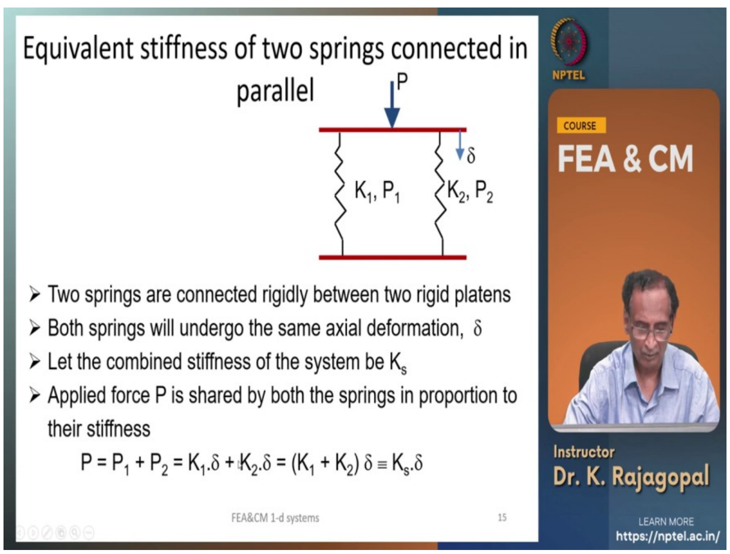
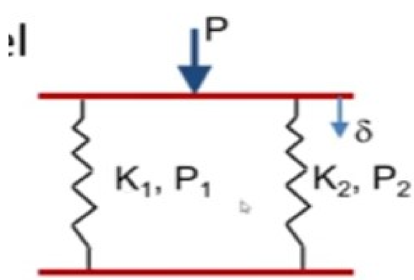


And let us consider that 2 elements K 1 and sorry element one and element 2 are connected rigidly between 2 patterns and the stiffness of element one is K 1 and stiffness of element 2 is K 2.



And when we apply any Force both will undergo the same deformation because they are connected with the rigid pattern the same displacement Delta in both the elements. And let us say that the combined stiffness is KS that is the contribution from element 1 and contribution from element 2.

And so, the applied force P is shared by both the Springs in proportion to their stiffness. So, the force developed in Spring 1 is k1 times Delta and the force developed in Spring 2 is K 2 times Delta and the total Force P is P 1 + P 2 that is K 1 times Delta + K 2 times Delta. And let us say that our combined stiffness is KS and if you look at this the P is KS times Delta the KS is the combined the stiffness of the system and P 1 P 2 are K 1 Delta and K2 Delta.

$$P = P_1 + P_2 = K_1.\delta + K_2.\delta = (K_1 + K_2)\,\delta \equiv K_s.\delta$$

So, if you look at this the combined stiffness is K 1 + K 2 that should give us the clue on how to assemble the contribution from these two from these 2 elements.

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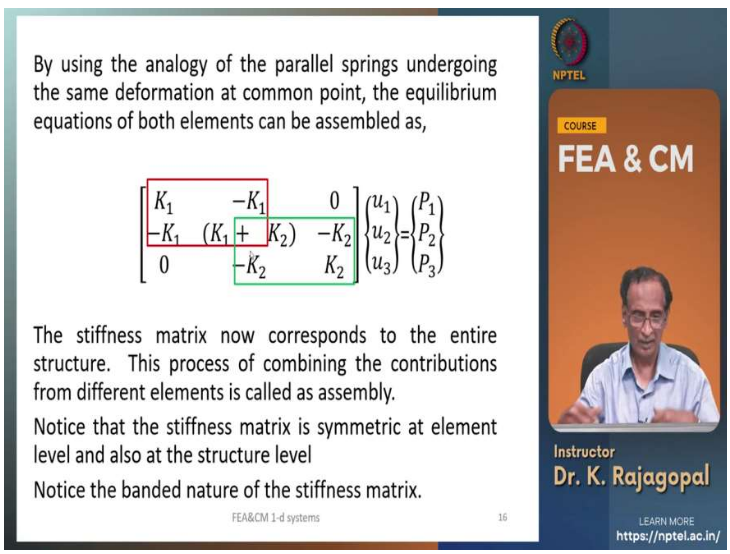

So, if you assemble a degree of freedom to the contribution is added up. So, we have K 1 and K 2, K 1 + K 2.

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

And so, this is how we do. And so, this K 1 - K1 - K 1 K 1 is the contribution from element 1 and from element 2 we have K 2 - K 2 - K 2 and K 2 and at this degree of freedom 2 the contributions of both the elements are added together. So, we have this K 1 + K 2. So, now the stiffness Matrix and the equations they correspond to the entire structure.

And we if we are able to solve this system of equations we can get our unknowns u 1 u 2 u 3. And so once again we see that our stiffness Matrix is banded and unsymmetric - K 1 - K 1 - K 2 K 2 and you see a 0 here. And it is not difficult to imagine why we got 0 here. If you look at if you look at the structure see this degree of freedom 3 is not connected to degree of freedom 1.

So, if I apply some unit displacement or in some force here only this the u 2 might react because it is connected to u 3 but u 1 is far away and it is not directly connected to u 3 server stiffness Matrix will look like this K1 - K 1 0 this 0 just simply means that if we apply anything at degree of freedom 1 the degree of freedom 3 is not affected.

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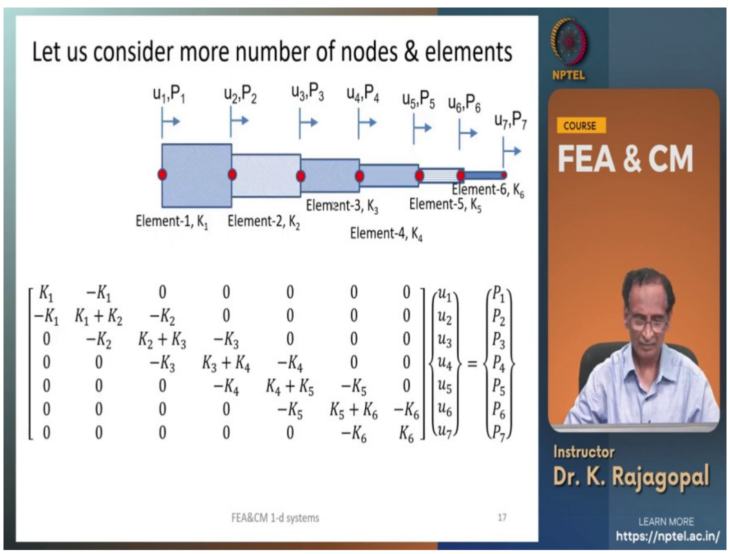
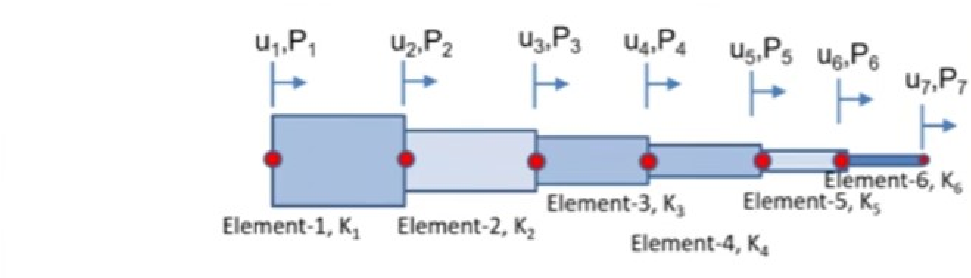


And now let us consider a bigger element bigger structure. So, now with 6 elements and 7 degrees of freedom u 1 u 2 u 3 u 4 u 5 u 6 u 7 and now you see the banded nature of this stiffness Matrix K 1 - K 1 - K 1 K 1 + K 2 and so on.



$$\begin{bmatrix} K_1 & -K_1 & 0 & 0 & 0 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 & 0 & 0 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 & 0 & 0 & 0 \\ 0 & 0 & -K_3 & K_3+K_4 & -K_4 & 0 & 0 \\ 0 & 0 & 0 & -K_4 & K_4+K_5 & -K_5 & 0 \\ 0 & 0 & 0 & 0 & -K_5 & K_5+K_6 & -K_6 \\ 0 & 0 & 0 & 0 & 0 & -K_6 & K_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{Bmatrix}$$

And these are the the displacements u 1 u 2 u3 u4 u for u 6 u 7 and then these are the applied forces and um the determinant of this bigger Matrix is also zero that we do not need to doubt.

Because that one element level we have seen then we can extrapolate the same thing and for even for 2 element Case where we have a three by three Matrix we can theoretically calculate and see that the determinant is zero. And if you take lot of effort and find the determinant of the center Matrix we will see that it is zero. We do not need to actually calculate.

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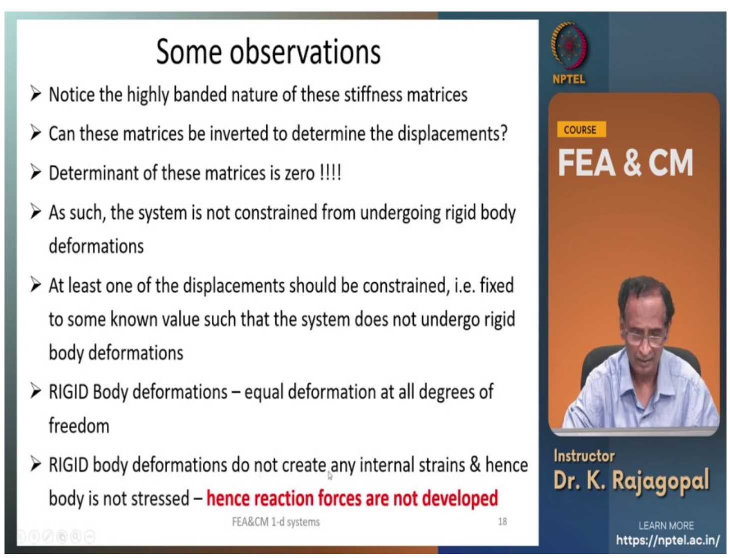

So, the first thing that we need to notice is that the stiffness Matrix is highly banded and most of the elements are zero meets column. So, if you apply a Gauss elimination procedure for this system we just need to eliminate one value below the diagonal in each of these columns.

So, it becomes very simple but then if you have to invert this Matrix you need to spend a lot of time and the determinant of these matrices is also zero.

So, we cannot really do anything we cannot invert these matrices and find our displacements and as such these systems are not constrained and they may undergo rigid body deformations a rigid body deformation is something very simple. So, I take this pen and move it in the air somewhere and all the points on this and this pen are moving by the same amount. So, that means that there is no relative deformation between the 2 points within the pen and there is no strain within this within this pen.

And if there are no strains they will not be any internal forces or internal strains. So, to make this system stable we need to apply some constraint at least one constraint because in a one-dimensional system we just require one constraint and one of the displacements we can fix to some known value it could be zero or it could be something else. So, that we can prevent the system from undergoing rigid body deformations.

And the rigid body deformation is a deformation such that all the degrees of freedom undergo the same deformation equal deformation. And the rigid body deformations as such they do not create an internal strains and hence the body is not stressed our body is not stressed. So, if I keep the pen here or somewhere else the forces that are developed within the body are zero and if the body is not stressed it will not be able to produce any reaction force.

So, we can say that there is it body a deformation does not satisfy our equilibrium equations because we are applying some external Force but the internal reaction is zero. So, that means that any body that is undergoing rigid body deformations will undergoing infinite displacements because there is no limit on the on the on the displacements because the reaction force is not developed.

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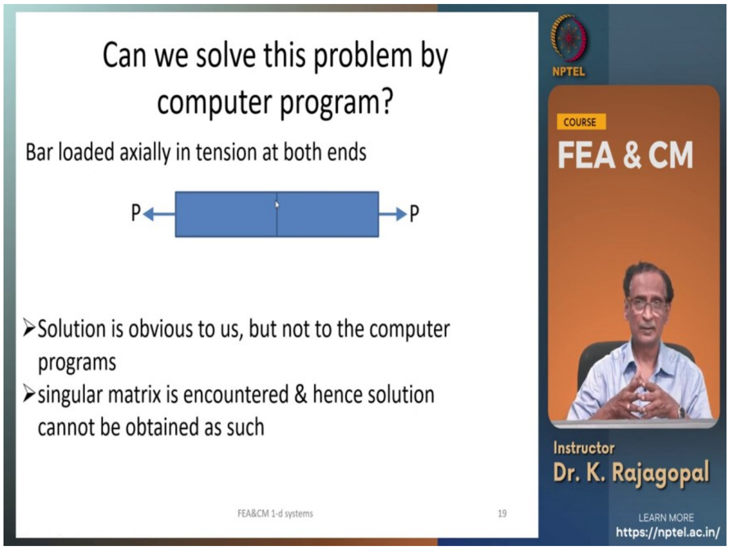

So, let us look at conceptually very simple problem. Let us take one axial element and then go on pulling at the 2 sides by an equal amount. And we know that this system is in equilibrium because we have applied equal and opposite forces at the 2 ends. And the solution is obvious to us the axial force is equal to P that is the tensile force and the stress is P by Del P by A.

But then if you pose this problem to any computer it will say that no it is this system is unstable because there is no constraint because you cannot really invert the matrices and these matrices are called a singular matrices and as such we cannot solve this. And we can do a small trick to make this table see at this neutral point at the exactly at the mid length of this element what will be the displacement.

See here at this end let us say that there is a positive displacement and at this end it will have an equal displacement but in the negative direction and exactly at the midpoint the displacement should be zero because that is the Symmetry point and in between we will have a smaller displacement of either positive or negative magnitudes.

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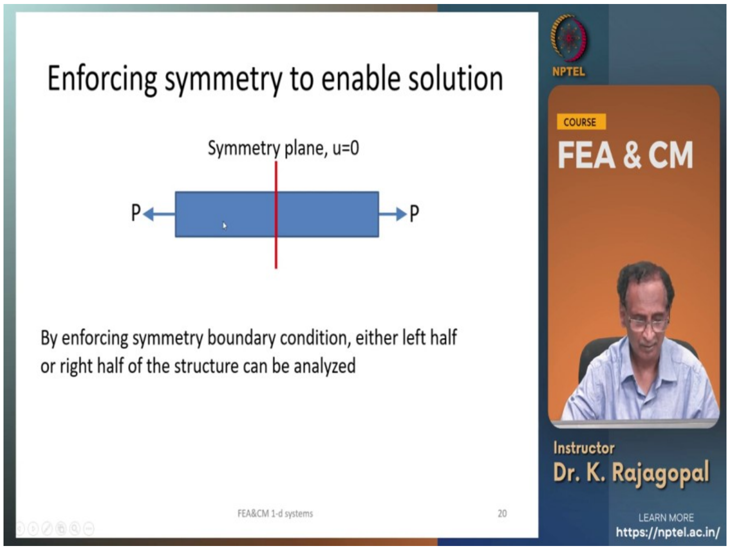

So, the Symmetry plane our displacement is zero. So, this is what we can enforce on the system that at this point my displacement is 0 and then you apply the force and then find the deformations. And once we apply this you know deformation constraint on the Symmetry plane then we can either solve the right hand side the half of the problem or the left hand side it does not matter because both will give the same result.

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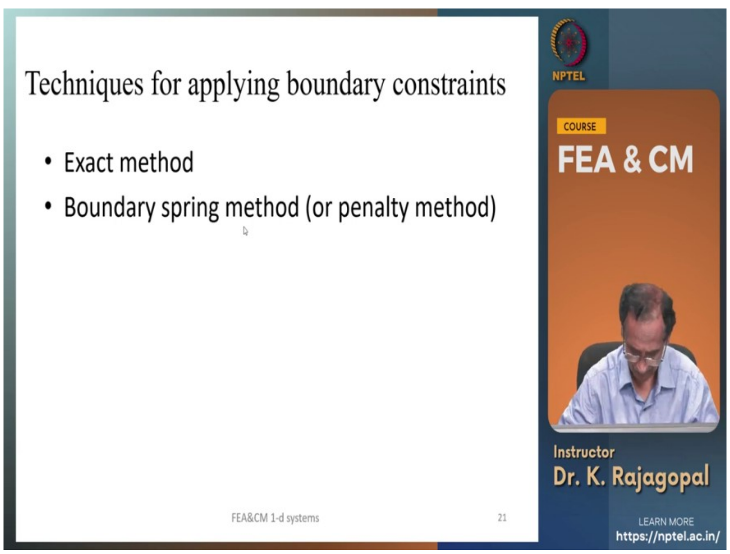

And but then how do we apply some constraints that we call as the boundary constraints and there are 2 methods one is the exact method and the other is the boundary spring method at the penalty method. And I will explain both of them very simple.

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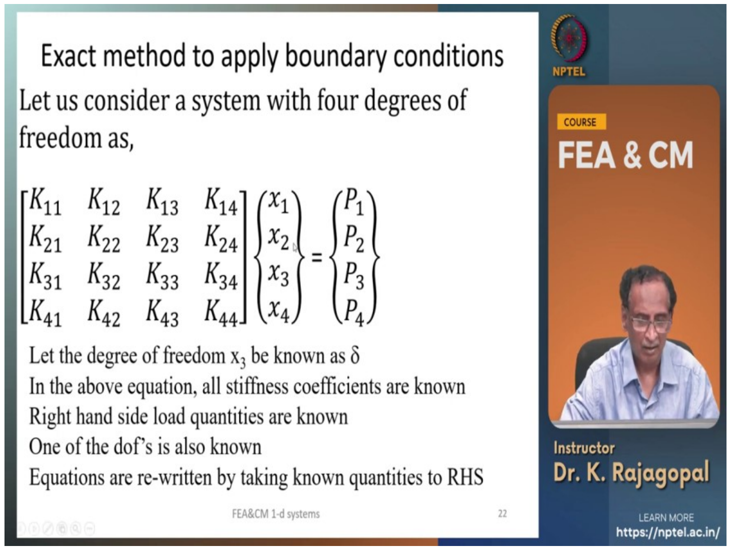

Let us look at the exact method first and let us say that we have a system of four simultaneous equations with four unknowns x 1 x 2 x 3 x 4 and then the applied forces are P1 P 2 P 3 P 4 and let us say that one of these degrees of freedom x 3 is known to us or it is constrained to be Delta and after solving our Delta x 3 should be equal to Delta. And in here if we see these are all the known quantities K 11 K 12 K 13 and so on.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}$$

Because once you know the geometric and material properties we can determine K 11 K 12 and other things and then the right hand side P 1 P 2 P 3 P 4 these are the applied forces. So, this also we know but only thing that are unknown are x 1 x 2 x 3 and x 4 out of this x 3 is known. So, we can actually modify this system of equations and send all the known quantities to the right hand side.

And so, that we are left with only the unknown quantities on the left hand side. And so, we can rewrite this system of equations by sending all the known quantities to the right hand side. So, since x 3 is known and K 13 is known we can take a product of K 13 and x 3 and send this to the right hand side. And that will belong to degree of freedom 1. So, it should go to the right hand side and in row one.

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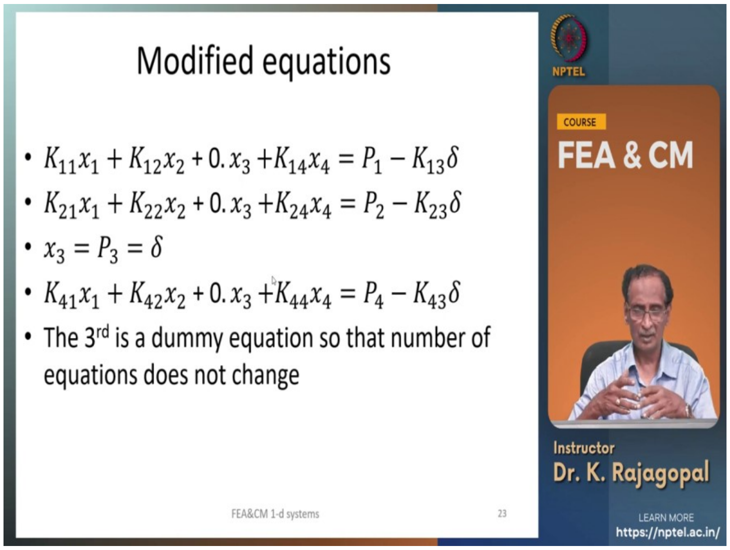

And so, our first equation will be

$$K_{11}x_1 + K_{12}x_2 + 0.x_3 + K_{14}x_4 = P_1 - K_{13}\delta$$

K 11 x 1 + K 12 x 2 + 0 times x 3 because K 13 we have taken to the right hand side + K 14 x 4 is equal to P 1 - K 13 Delta.

Similarly the second equation

$$K_{21}x_1 + K_{22}x_2 + 0.x_3 + K_{24}x_4 = P_2 - K_{23}\delta$$

$$x_3 = P_3 = \delta$$

and then the third equation is actually it is a trivial equation x 3 is equal to Delta and and then the fourth equation is

$$K_{41}x_1 + K_{42}x_2 + 0.x_3 + K_{44}x_4 = P_4 - K_{43}\delta$$

a K 41 x 1 Plus K 42 x 2 + 0 times x 3 + K 44 x 4 is P 4 - K 43 Delta. And our third equation is actually it is a trivial equation that x 3 is equal to Delta.

So, you might ask since we already know x 3 why do we need to have this in the system of equations? Yeah, the reason is very simple because we started with four unknowns and four degrees of freedom and one of the degrees of freedom is known x 3 is known and it is very difficult to program to eliminate one degree of freedom suddenly. We started with four degrees of freedom and it is more easy to continue with the same number of degrees of freedom.

So, for that reason we retained the third equation but with a very trivial equation x 3 is equal to Delta and then our job becomes more simple our writing the program becomes very simple.

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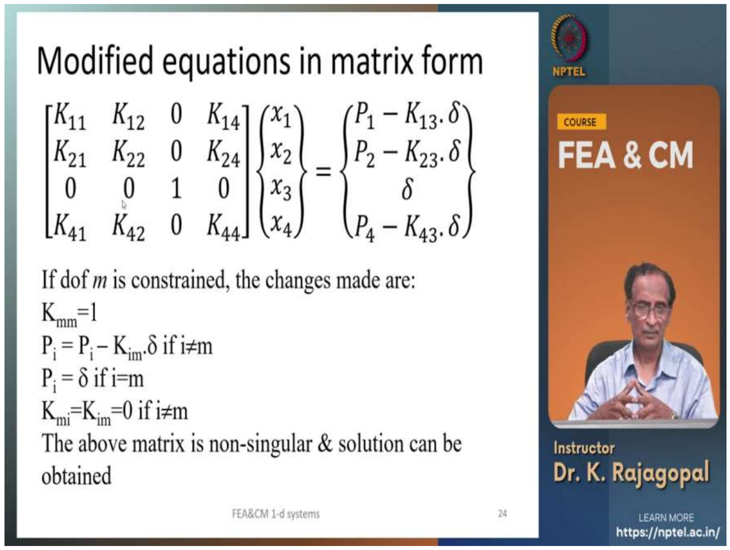

And and then if you rewrite those equations in a matrix form we get like this

$$\begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} \\ K_{21} & K_{22} & 0 & K_{24} \\ 0 & 0 & 1 & 0 \\ K_{41} & K_{42} & 0 & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 - K_{13}.\delta \\ P_2 - K_{23}.\delta \\ \delta \\ P_4 - K_{43}.\delta \end{Bmatrix}$$

K 11 K 12 0 K 14 and so on. And the right hand side is P 1 - K 13 Delta whereas the third equation is just simply x 3 is equal to Delta. So, in general if degree of freedom m is constrained the changes to be made are K mn is 1 and P i is P i - K im times Delta if i is not equal to m and P i is Delta if I is equal to m and K mi is K i K im is 0.

If dof *m* is constrained, the changes made are:
$K_{mm}=1$
$P_i = P_i - K_{im}.\delta$ if $i \neq m$
$P_i = \delta$ if $i=m$
$K_{mi}=K_{im}=0$ if $i \neq m$
The above matrix is non-singular & solution can be obtained

That is a half diagonal terms once you make these changes this stiffness Matrix the modified stiffness Matrix is not singular it will have some if it will have some determinant and we should be able to invert this Matrix. So, these are the changes that we make and then we eliminate these degrees of freedom sorry by eliminating this degree of freedom x 3 we can do this.

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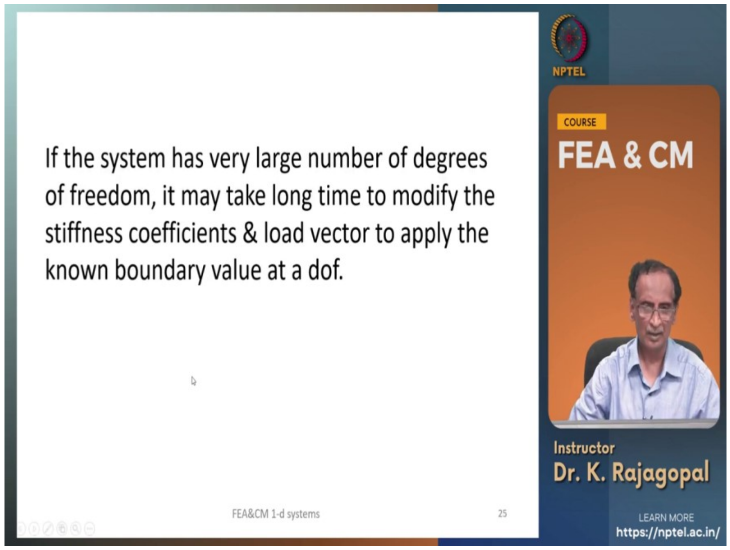

So, if the system has very large number of degrees of freedom may take a very long time to modify each of these stiffness coefficients and the load vector.

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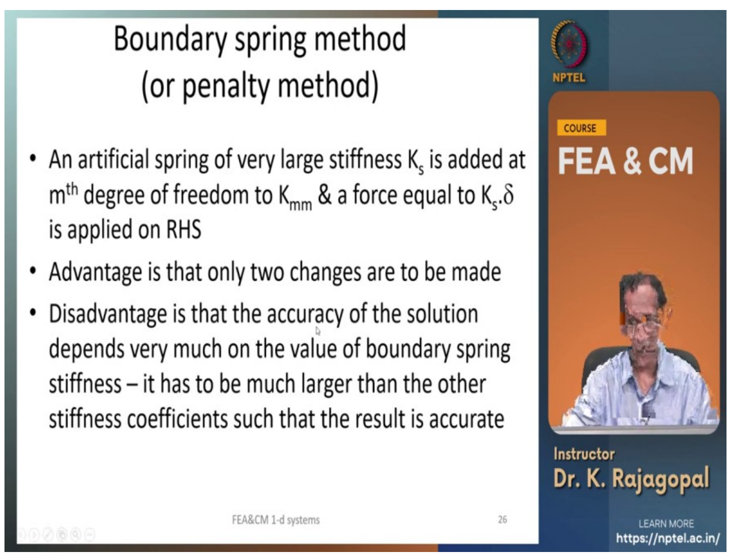

And so, we should have some other method for applying a similar constraints and that is called as a boundary spring method. And so it is also called as a penalty method because on the system we apply some penalty so, that we can enforce what we want. And what we do is we take an artificially an artificial spring of very very large stiffness case and add it at the empty degree of freedom where we want to apply some constraint and then apply a force equal to KS times Delta on the right hand side.

So, the advantage is that we make only 2 changes one on the right hand side and one in the stiffness coefficient. But the disadvantage is that the accuracy of the solution depends very much and the value of the boundary spring and the value of the boundary spring stiffness has to be much much larger than the all other stiffness coefficients. So, that our result is accurate so, that we can see with an example.

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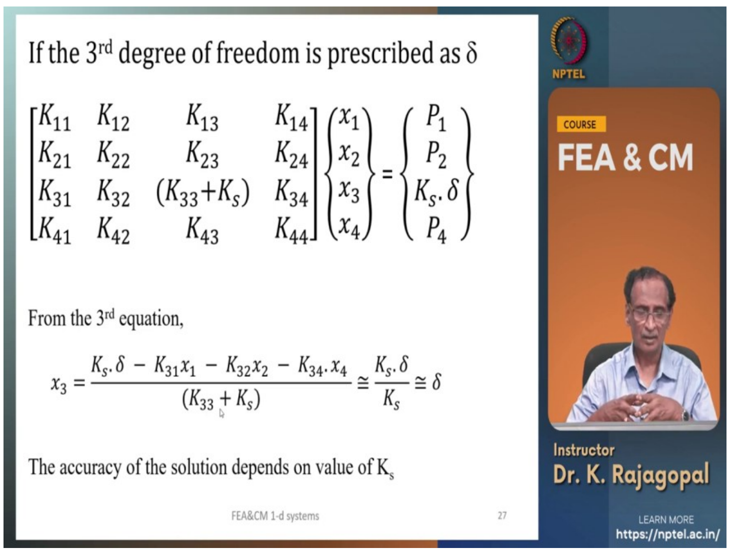

So, on the same problem say we are constraining the third degree of freedom to sum Delta and we can add the KS to the diagonal element of this stiffness Matrix and then on the right hand side we can apply a force equal to KS times Delta and this K is so, large that compared to all the other forces that are developed the K 31 times x 1 K 32 times x 2 K 34 times x 4 they are very small and negligible.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & (K_{33}+K_s) & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ K_s.\delta \\ P_4 \end{Bmatrix}$$

And so, our x 3 after doing the calculations will be approximately equal to Delta if provided our KS is very very large the calculated Delta will be the same as what we want.

From the 3rd equation,

$$x_3 = \frac{K_s.\delta - K_{31}x_1 - K_{32}x_2 - K_{34}.x_4}{(K_{33} + K_s)} \cong \frac{K_s.\delta}{K_s} \cong \delta$$

But if our spring stiffness KS is not of very large quantity then we could have some errors that we can check with a numerical example. And the accuracy of this solution depends very much on KS and if the case is very very large let us say the ideally case should be infinite.

So, that at the end of this solution this K is in the numerator and denominator they cancel out and all other quantities are very small because this is infinite times Delta it is a very large quantity infinite and but then if we make it very very large we could have some round of Errors when we are doing the computation. So, we have to have a balance between the computational accuracy and then this boundary spring the stiffness that we give.

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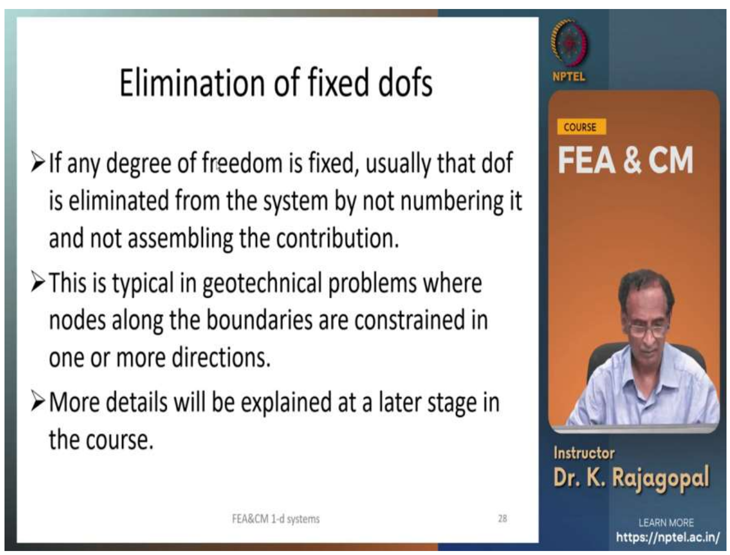

So, now by doing one of these methods we can we can enforce the constraint or the other method is if any of the fixed degrees of freedom is there we can directly eliminate that right from the beginning and we do not number that particular degree of freedom and assign any stiffness coefficient. And especially this is typical of geotechnical problems where we have very very long boundaries where our displacements are zero.

And at the time of the number in the equations itself we do not number those degrees of freedom and that is how we can eliminate those fixed degrees of freedom. That we will see later not at this stage.

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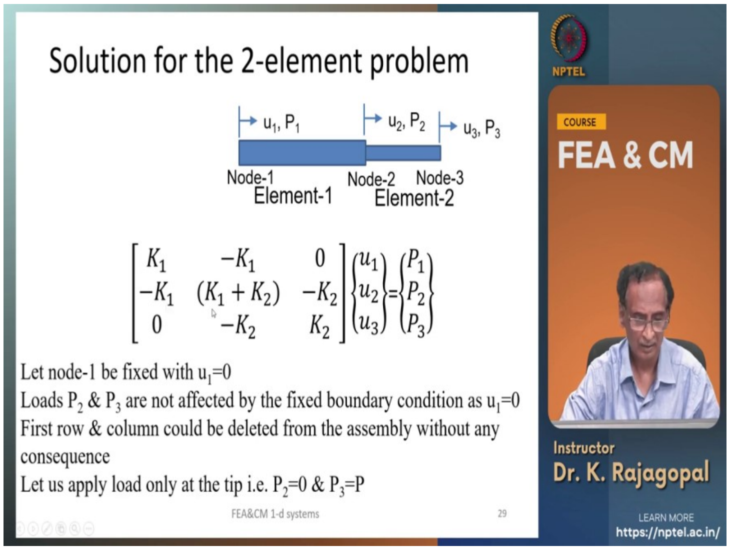

And so, here let us go back to our 2 element problem and let us say that our u 1 is 0 and P 2 and P 3 are not affected because our u 1 is zero. So, it is the first row and first column can be can be eliminated because the P 1 is is not affected because this displacement is zero. And so, we can eliminate the first row and the first column and we end up with a 2 by 2 Matrix like this.

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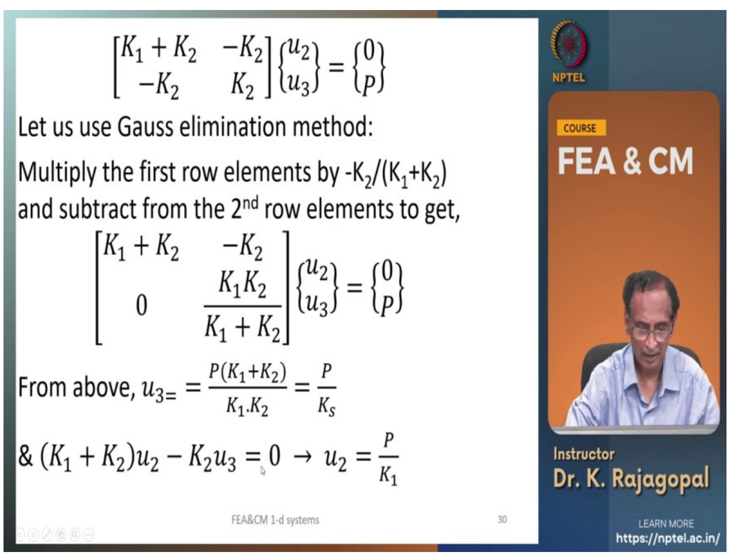

$$\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

K 11 K 1 + K 2 - K 2 - K 2 K 2 u 2 u 3 and 0 and P because we applied only one force P at this at this point three. So, we can solve it either by Gauss elimination method or by by matrix inversion method and we see that our u 3

$$u_{3=} = \frac{P(K_1+K_2)}{K_1.K_2} = \frac{P}{K_s} \quad K_s = \text{combined stiffness} = \frac{K_1.K_2}{K_1+K_2}$$

is P by KS and P times K 1 + K 2 by K 1 K 2. And actually we have seen that this is equal to the spring constant KS when we have 2 strings in parallel to each other and are in series sorry and our K 1 + K 2 times u 2 - K2 times u3 is 0.

That is once you get u 3 we can get u 2 as a 3 by K 1 and the KS is our combined stiffness K 1 K 2 by K 1 + K 2.

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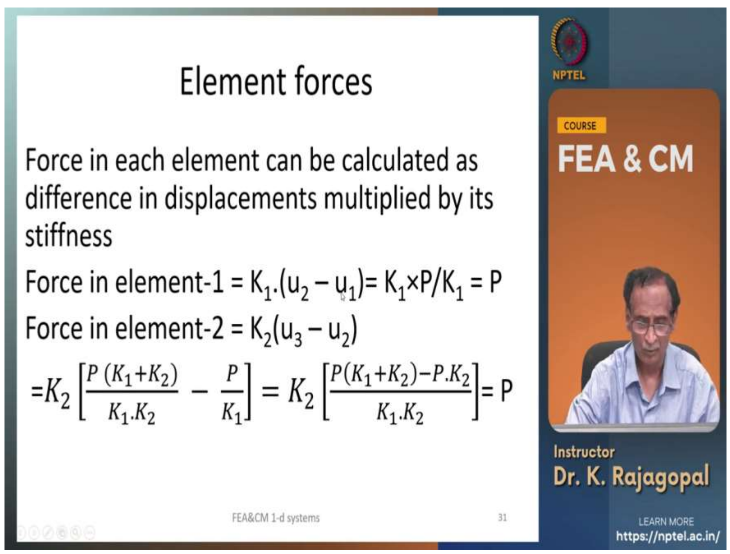

And once we get the deformations we can find the forces in the 2 elements the force in element one is a K 1 times u 2 - u 1.

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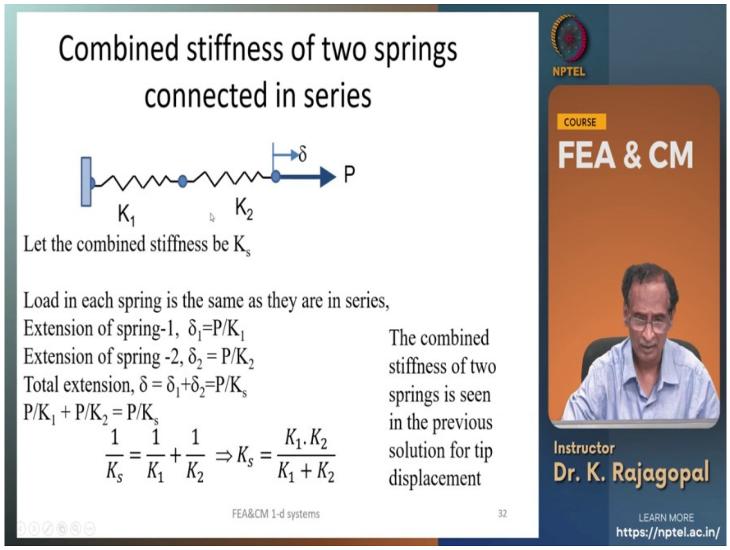

And the force in element 2 is K 2 times u 3 - u 2 and actually let us look at one small numerical example to illustrate this actually this combined stiffness of 2 parallel 2 Springs in series is explained here and that I think you can leisurely see.

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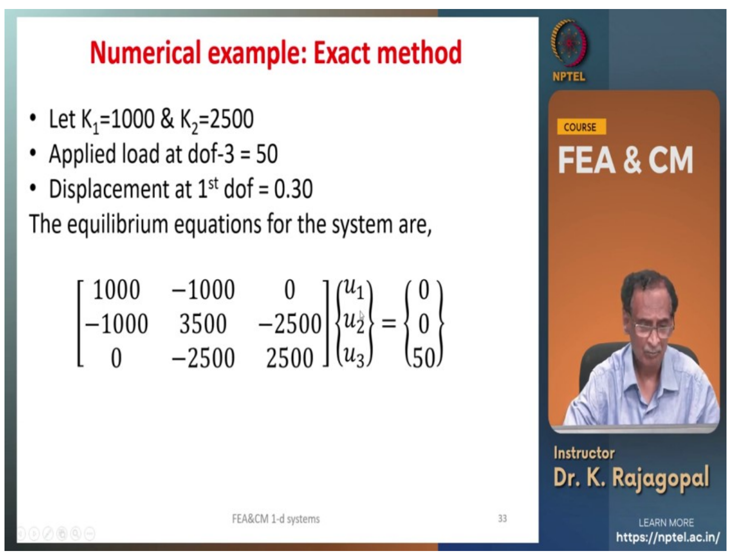
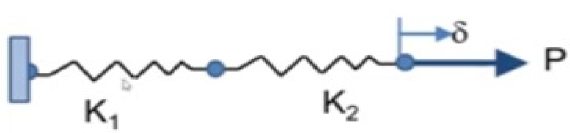


Now let us look at one numerical example so, that we can give some values and then understand the calculations better let us say in this previous example of 2 elements with the spring 1 K 1 and spring 2 K 2 we applied a load of 50 a degree of freedom 3 and a degree of freedom 1 we constrained the displacement to point three the server equilibrium equations are like this see this K 1 is thousand and K 2 is 2500.



$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

So, this is our assembled equations K 1 - K 1 - K 1 K 1 + K 2 - K 2 and so on and here we know that our u 1 is 0.3.

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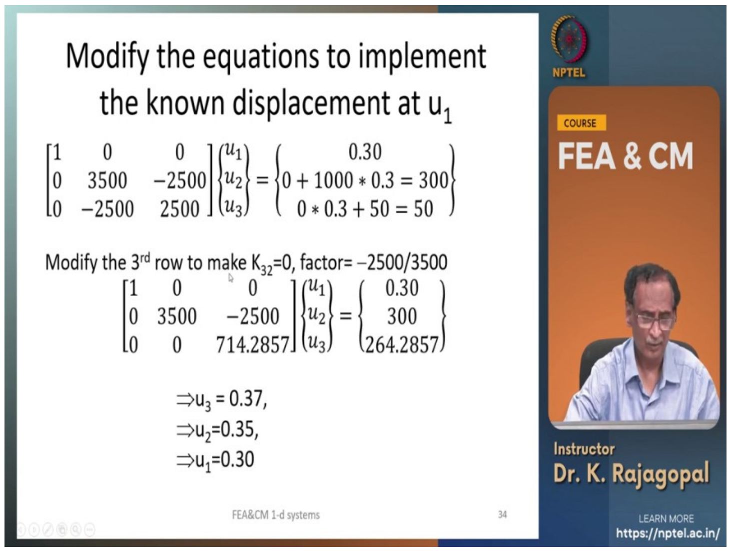

And our known quantities we can take to the right hand side and here it was - 1000. So, - 1000 times 0.3 if you send it to the right hand side we get 1000 times 0.3 that is 300 and then at the degree of freedom 3 we have applied a force of 50 and we can solve this by converting this to an upper diagonal matrix and we get the u 3 as 0.37 u 2 is 0.35 and u 1 is 0.3.

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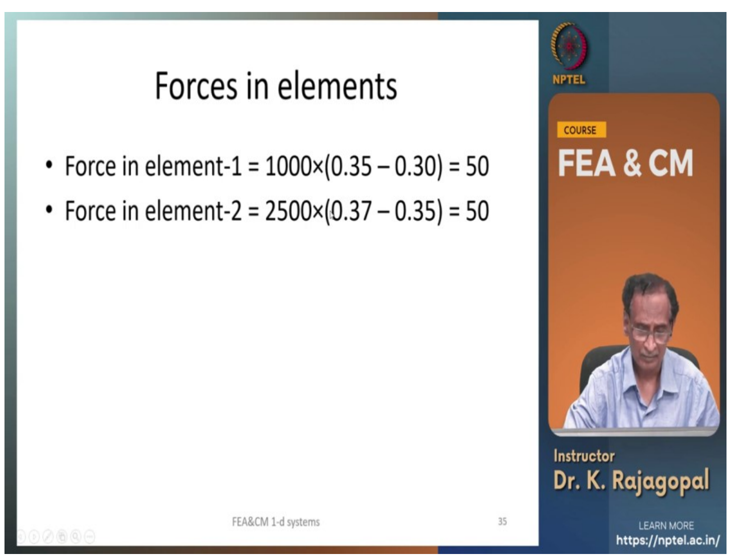

And that is what we want to do 3 is a fixed as a 0.3 and the force in element 1 is 1000 times u 3 - u 2 that is the relative deformation that is that comes out as 50. and the force in element 2 is 2500 multiplied by 0.37 - 0.35 and that comes to once again 50 because they are in in series.

$$\text{Force in element-1} = 1000\times(0.35 - 0.30) = 50$$
$$\text{Force in element-2} = 2500\times(0.37 - 0.35) = 50$$

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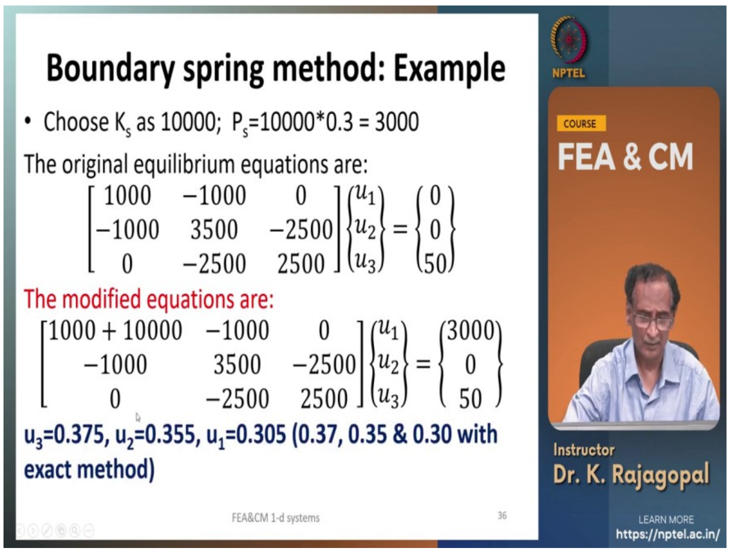

And let us look at the boundary string method to illustrate and let us take the boundary spring stiffness is ten thousand. So, the PS is a three thousand one thousand ten thousand times point three it is our original equations are like this

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

and the modified equations are like this

The modified equations are:

$$\begin{bmatrix} 1000 + 10000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ 0 \\ 50 \end{Bmatrix}$$

and u 1 we want as 0.3. So, on the right hand side we apply the force of 3000 and then the left hand side we have added the boundary spring of 10000 to the first degree of freedom.

If you solve this system of equations we see that u 3 is 0.375 u 2 is 0.355 and u 1 is 0.305 and with the exact method these were the displacements that we got which are very close to these values.

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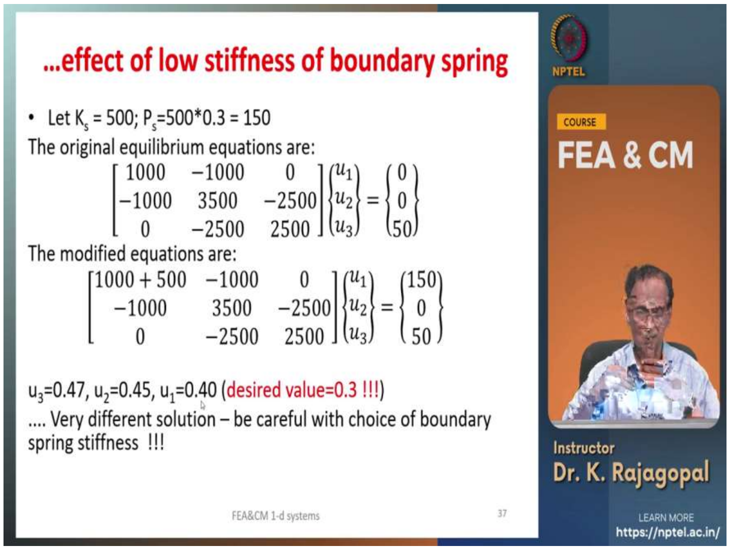

And now let us look at another example by assigning only 500 to the boundary spring KS is 500. And if you solve we see that our u3 is 0.47 u 2 is 0.45 and u 1 is 0.4.

- Let $K_s = 500$; $P_s = 500*0.3 = 150$

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

The modified equations are:

$$\begin{bmatrix} 1000+500 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 0 \\ 50 \end{Bmatrix}$$

$u_3=0.47$, $u_2=0.45$, $u_1=0.40$ (desired value=0.3 !!!)

See we wanted a u 1 of 0.3 but we end up with 0.4 which is totally different. And so, actually whenever we use boundary spring method we have to be very careful in the boundary spring value that we assign and if we assign a large enough value then we can get similar to the previous solution that is very close like 0.375, 0.355, 0.305.

So, 0.305 is very close to 0.3 and instead of 10 000 let us say we give a KS of 1 million then these displacements may be more closer to the to the exact values that we calculated earlier

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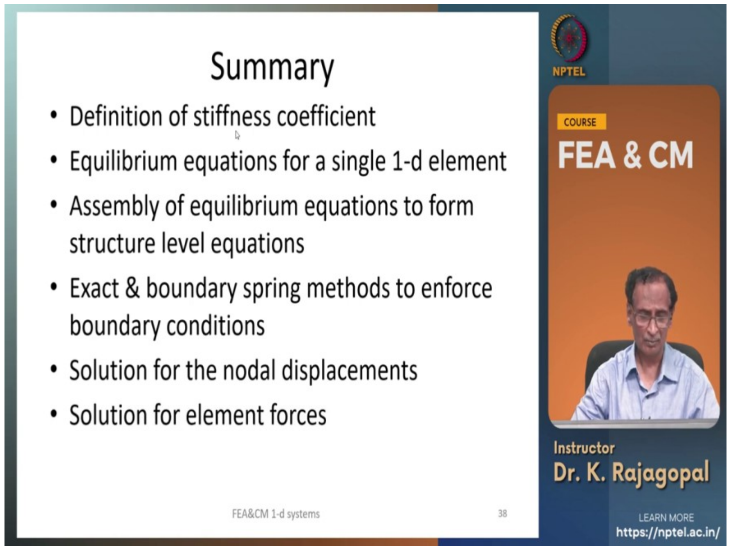

Just to summarize in this lecture we have defined the stiffness coefficient for a one dimensional element. Then we form the equations for one for a single element then assemble them for multiple elements and then we we have seen how to apply the boundary constraints either by exact method or by the boundary spring method and then we can once we apply this boundary constraints we can solve the equations then we got the solution for nodal displacements and then the element forces.

So, actually this we have seen all the steps that we will undergo in any finite element analysis we get the displacements and then we get the forces. So, this we will apply for larger systems in the subsequent lectures. So, that we understand the concepts. So, this is my last slide and if you have any questions please send an email to this graph profkrg@gmail.com and I will be able to respond to your queries. So, thank you very much.