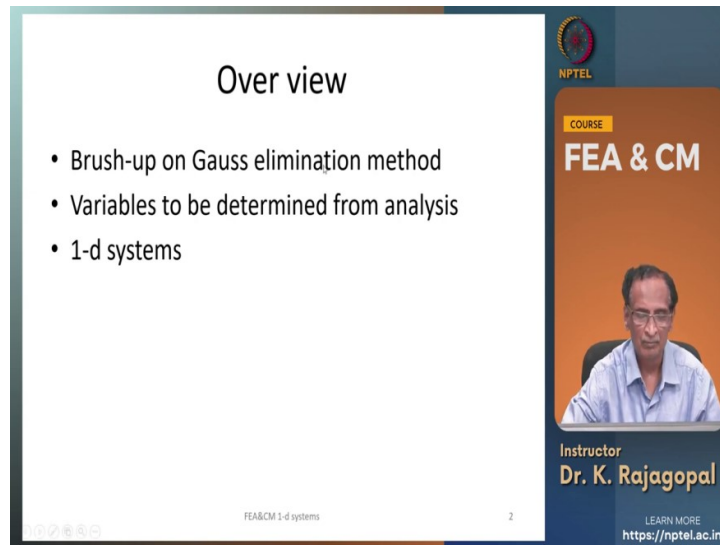


FEM and Constitutive Modelling in Geomechanics
Prof. K. Rajagopal
Department of Civil Engineering
Indian Institute of Technology - Madras

Lecture: 3
Development of Equilibrium Equations for 1-D Systems

(Refer Slide Time: 00:49)



The slide is titled "Over view" and contains a bulleted list of topics:

- Brush-up on Gauss elimination method
- Variables to be determined from analysis
- 1-d systems

The slide is part of an NPTEL course titled "FEA & CM" (Finite Element Analysis and Constitutive Modelling) taught by Dr. K. Rajagopal. The slide number is 2, and the course title "FEA&CM 1-d systems" is visible at the bottom left of the slide content area. The NPTEL logo is in the top right corner, and the URL "https://nptel.ac.in/" is at the bottom right.

So, hello students let us continue from our previous lecture. In the previous lecture we had seen some basics of the Matrix algebra. We require the Matrix operations for understanding all our subsequent calculations and in today's lecture we will look at the axially loaded one-dimensional systems we will see how to analyze them and just a brief brush up on the Gauss elimination method; before we go with the today's topic.

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Gauss Elimination Method

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{23} & K_{33} & K_{34} \\ K_{41} & K_{24} & K_{43} & K_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$

$$\begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{K}_{14} \\ 0 & \bar{K}_{22} & \bar{K}_{23} & \bar{K}_{24} \\ 0 & 0 & \bar{K}_{33} & \bar{K}_{34} \\ 0 & 0 & 0 & \bar{K}_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \end{pmatrix}$$

$$x_4 = \frac{\bar{P}_4}{\bar{K}_{44}} \qquad x_3 = \frac{\bar{P}_3 - \bar{K}_{34} \times x_4}{\bar{K}_{33}}$$


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So, we had seen that if you have system of simultaneous equations with variables x_1 , x_2 , x_3 , and x_4 we can solve it by either Matrix inversion or by Gauss elimination method.

And between these 2 are the Gauss elimination method we had discussed is more suitable for finite element calculations because we have a bandwidth Matrix where significant number of stiffness coefficients away from the diagonal are zero.

Gauss Elimination Method

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{23} & K_{33} & K_{34} \\ K_{41} & K_{24} & K_{43} & K_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$

$$\begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} & \bar{K}_{14} \\ 0 & \bar{K}_{22} & \bar{K}_{23} & \bar{K}_{24} \\ 0 & 0 & \bar{K}_{33} & \bar{K}_{34} \\ 0 & 0 & 0 & \bar{K}_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \end{pmatrix}$$

$$x_4 = \frac{\bar{P}_4}{\bar{K}_{44}} \qquad x_3 = \frac{\bar{P}_3 - \bar{K}_{34} \times x_4}{\bar{K}_{33}}$$

So, we first convert our system of equations into an upper diagonal matrix like this. So, that in the last equation we have only one coefficient and then we can get the last variable x_4 as \bar{P}_4 / \bar{K}_{44} and once we determine the x_4 we can determine x_3 from the previous equation because we have 2 unknown 2 quantities here x_3 and x_4 and x_4 was already determined and now we can determine x_3 like this and continue the process x_2 x_1 and so on.

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Quantities of interest

- Displacements
- Pore pressures
- Strains
- Stresses
- Forces in support systems, etc.

➤ Primary quantities are the displacements (and pore pressures)

➤ Strains from displacements

➤ Stresses from strains

➤ **Equilibrium equations need to be set up to determine the displacements**

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So, now let us see what are all the quantities that we solve for in any geotechnical problem? The first quantity is a displacements. Displacements could be only one when we are dealing with one dimensional problems are 2 at each node when we are dealing with the 2 dimensional and three when we are dealing with the three dimensional problems. And then we are we have the pore pressures and then we have the strains and then stresses and then the forces in the support systems are in the soil and so on.

And the primary quantities that we determine are the displacements than in the problems with water we also have the pore pressure and we determine the strains from the displacements using the compatibility equations. And then we can determine the stresses from the strains using the constituted equations that we will see later in some other lectures. And we need to set up some equilibrium equations.

So, that we can determine the displacements that is some stiffness multiplied by displacement is equal to force and what we mean by equilibrium equation is the applied force should be exactly equal to the reaction force and that is the basis on which we will develop all our equations and initially we will discuss only the problems with the only displacements that is the elasticity problems and later we will introduce water and we discuss the poor elastic problems later.

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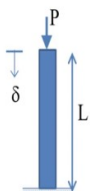
Starting from strength of materials,

If cross-sectional area of column is A, Young's modulus is E,
 Stress, $\sigma = P/A$
 Strain, $\epsilon = \sigma/E = (P/A)/E = P/(A.E)$
 Tip deformation, $\delta = L.\epsilon = L. (P/A.E)$
 Combining all the above, tip deformation is:

$$\delta = \frac{P.L}{A.E} \quad \text{or}$$

$$\frac{A.E}{L} . \delta = P \quad \text{or} \quad \frac{A.E}{L} = \frac{P}{\delta}$$

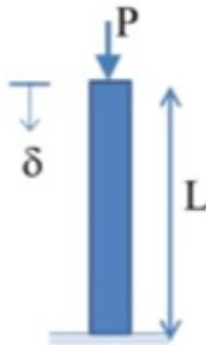
Units of $A.E/L$ are $L^2.F/L^2/L = F/L = \text{force per unit deformation}$
 which is denoted as stiffness, same as P/δ units



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So, let us start from the most fundamental strength of materials because all of us are familiar with this famous equation Delta is equal to PL by AE that is the the equation that we study the first thing in any of the strength of materials courses.



Let us look at columnar element of length L and area A and let its Young's modulus be E and let us say that we applied a load of P and we would like to know what is the displacement at the top.

And for doing that we can do the following we can determine the stress as the load divided by the area P by A

$$\sigma = P / A$$

The strain is the stress divided by the Young's modulus and the stress is P by A and E is the Young's modulus.

$$\epsilon = \sigma / E = P / (A.E)$$

So, P by AE and once we get the strain the tip deformation Δ is L times ϵ where L is the length of the element. So, it is L times $\Delta \epsilon$ is PL by AE .

$$\delta = L \cdot \epsilon = L \cdot (P / A \cdot E)$$

And so, our Δ the tip displacement is PL by $A \cdot E$ or we can also write this in a slightly different form as $A \cdot E$ by L times Δ is P or $A \cdot E$ by L is P by Δ .

$$\delta = \frac{P \cdot L}{A \cdot E} \quad \text{or}$$

$$\frac{A \cdot E}{L} \cdot \delta = P \quad \text{or} \quad \frac{A \cdot E}{L} = \frac{P}{\delta}$$

And let us look at the units for these quantities. Say the units of this product $A \cdot E$ by L , A has the units of area L^2 P is the force per unit area of the stress F by L^2 and the whole thing divided by L . So, that is in the units of F by L that is the force per unit deformation.

And this $A \cdot E$ by L is a quantity that we call as the stiffness and that has the units of force per unit deformation. And on the right hand side also we have the P by Δ , P has the units of force and Δ has the units of displacement or the length. So, it is P by Δ . So, both the left hand side and the right hand side they have the same units. And so, $A \cdot E$ by L is our stiffness and that multiplied by some displacement is equal to the force.

And actually we can think of it in some other manner on the right hand side we have the applied force that is the that is what we are applying on the on the system and $A \cdot E$ by L times Δ is the reaction force. So, at some point the reaction force should be exactly equal to the applied force and that is what we call as equilibrium.

(Refer Slide Time: 07:45)

The slide is titled "Development of equilibrium equations" and contains the following text:

- External (applied) Force \equiv Internal force (reaction)
- Any force applied on the system will deform the body until a force equal and opposite to the applied force is developed
- If the body is soft, equal and opposite reaction force is developed at a large deformation
- If the body is stiff, equal and opposite reaction is developed at a small deformation

At the bottom of the slide, it says "FEA&CM 1-d systems" and "6". On the right side, there is a video thumbnail of Dr. K. Rajagopal, with the text "COURSE FEA & CM" and "Instructor Dr. K. Rajagopal". Below the thumbnail, it says "LEARN MORE https://nptel.ac.in/".

And so, what we mean by the equilibrium is externally applied force should be exactly equal to the internal force of the reaction that is a reaction is developed by the body to resist the applied forces. And so, any force that we apply on the system will produce some deformations that we may be able to measure with a ruler or with a micrometer or something depending on the magnitude of these deformations.

And and this body will go on deforming until an equal and opposite force is applied a force is developed to oppose the applied forces right. And if the body is very soft equal and the opposite reaction force is developed at a large deformation and if the body is stiff equal and opposite reaction is developed at a small deformation. So, that explains why if you have a very stiff spring and you apply some force it will deform very little but if you have a soft spring it will deform a large deformed by a larger amount.

And the governing principle is they will deform until an equal and opposite force is developed to oppose the opposed applied force.

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Stiffness coefficient


Stiffness \times displacement = Force

Or

Stiffness = Force/displacement

Stiffness coefficient K_{ij} is defined as the force developed in i^{th} degree of freedom (dof) due to a unit deformation applied in j^{th} dof while all other degrees of freedom are restrained


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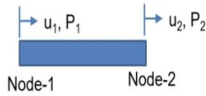
So, this stiffness coefficient is required because we require to calculate the reaction force that is calculated as stiffness times displacement and that is equal to the applied force on the right hand side. The stiffness is a force divided by the displacement and in general we Define stiffness coefficient K_{ij} is actually we are going to deal with the multi degree of freedom system.

So, we have to develop these systems for large number of displacements or the degrees of freedom and our stiffness coefficient K_{ij} is defined as the force developed in the i^{th} degree of freedom dof due to a unit deformation applied in the j^{th} degree of freedom. So, we apply some unit displacement in one direction and then what is the reaction force developed in the i direction that is called as the as the stiffness coefficient and while we are doing this we constrain all the other degrees of freedom that is the basic definition for K_{ij} .

And let us try to develop these coefficients for simple systems before we go on to the continuum that is our soil.

(Refer Slide Time: 10:55)

2-node one-dimensional bar element




- Element has one node at each end, totally two nodes for each element
- Element has only one degree of freedom at each node, totally two degrees of freedom for the element (u_1, u_2) **(axial)**
- Two forces are applied, one at each node (P_1, P_2)
- Element will have 2 equilibrium equations
- Let the stiffness of the element be AE/L

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And let us start with one dimensional bar element that is purely axial all the deformations are only in the axial Direction and also the forces are in the axial Direction either tension or compression.

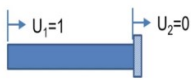


Now let us consider a simple 2 dimensional sorry one dimensional 2 node bar element with the 2 nodes node one and node 2 and 2 deformations u_1 and u_2 and 2 applied forces P_1 and P_2 and this element has nodes at 1 at each end.

So, Node 1 on the Left End node 2 at the right hand and this element has 2 axial degrees of freedom u_1 and u_2 and then 2 axial forces P_1 and P_2 and since we have 2 degrees of freedom this element will have 2 equilibrium equations that is a reaction at Node 1 R_1 should be exactly equal to P_1 and the reaction at node 2 should be exactly equal to the applied force P_2 and let us say that our stiffness is AE by L where A is the cross-sectional area of the element E is the young's modulus and L is the length of the of the element.

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Equilibrium equations of a 2-node bar element




- Let a unit axial displacement be applied at node-1, $u_1=1$ while the other node is fixed, $u_2=0$
- This unit displacement will produce a force of $AE/L \times 1$ at dof-1. By definition, the corresponding stiffness coefficient is K_{11} as unit deformation is applied in dof-1 and force is developed at dof-1 in the same direction, $\Rightarrow K_{11}=AE/L$
- At node 2, equal and opposite force should develop to maintain equilibrium
- i.e. $K_{21} = -AE/L$, force developed at dof-2 in the opposite direction due to a unit displacement at dof-1

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So, let us apply unit deformations in different directions and then measure the forces. So, that we can define our stiffness coefficients K_{ij} and to start with let us apply a unit deformation a degree of freedom 1 that is u_1 is equal to 1. And while we are doing that we constrain this node or this degrees degree of freedom u_2 to 0. And so, the unit deformation at 1 will produce a force equal to $A E$ by L times 1.

Because $A E$ by L is our stiffness that is relate that is related to the force through the displacement. So, at degree of freedom 1 the force developer is AE by L and by definition the corresponding stiffness coefficient is K_{11} that is the force developed in degree of freedom 1 when a unit displacement is applied in the same degree of freedom one and the force developed is K_{11} that is AE by L .

$$K_{11} = A.E / L$$

And at node 2 equal and the opposite Force should develop to maintain the equilibrium and at node 2 at the degree of freedom 2 now the stiffness coefficient will be K_{21} . The force developed at the degree of freedom 2 because of a unit displacement applied a degree of freedom one. So, that K_{21} will be our needs to be equal to $- a$ by L . So, that we have equal and opposite term reaction force at the other end.

$$K_{21} = - A.E / L$$

So, the force developed at degree of freedom 2 is $-AE/L$ due to a unit displacement at degree of freedom 1.

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Equilibrium equations of a 2-node bar element

$U_1=0$ $U_2=1$

- Let a unit displacement be applied at node-2, $u_2=1$ while the other node is fixed, $u_1=0$
- This unit displacement will produce a force of $AE/L \cdot 1$ at dof-2. By definition, the corresponding stiffness coefficient is K_{22} as unit deformation is applied at dof-2 and force is measured at dof-2, $K_{22}=AE/L$
- At node 1, equal and opposite force should develop to maintain equilibrium
- i.e. $K_{12} = -AE/L$, force developed at dof-1 due to a unit displacement at dof-2

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And similarly we can apply a unit displacement at a degree of freedom 2 while we constrain this degree of freedom from moving. So, by doing this we can write that K_{22} is AE/L that is the force developed in degree of freedom 2 because of a unit displacement at degree of freedom 2. And the force developed at degree of freedom 1 because of a unit displacement at degree of freedom 2 will be K_{12} and that should be equal to $-AE/L$ to maintain our equilibrium.

And otherwise the system will undergo some undue deformations that we call as rigid body deformations. So, here our K_{12} is $-AE/L$ that is the force developed at a degree of freedom one because of a unit displacement at degree of freedom 2.

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By definition, internal force developed at dof-1 due to nodal displacements u_1, u_2

$$K_{11} \cdot u_1 + K_{12} \cdot u_2$$

Internal force developed at dof-2 will be,


$$K_{21} \cdot u_1 + K_{22} \cdot u_2$$

These internal forces should be equal to the applied forces at the respective degrees of freedom,

$$K_{11} \cdot u_1 + K_{12} \cdot u_2 = P_1$$


$$K_{21} \cdot u_1 + K_{22} \cdot u_2 = P_2$$

In matrix form, these equations could be written as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}, \quad \begin{bmatrix} \frac{A.E}{L} & -\frac{A.E}{L} \\ -\frac{A.E}{L} & \frac{A.E}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$


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So, by combining these the internal forces developed can be written like this the internal Force developed a degree of freedom 1 because of displacements u_1 and u_2 will be

$$K_{11} \cdot U_1 + K_{12} \cdot U_2$$

and as we have seen K_{11} is the force developed because of a unit displacement. So, that is the stiffness coefficient and if you apply a deformation of u_1 the force developed is K_{11} times u_1 .

And similarly the force developed in degree of freedom 1 because of a unit displacement to degree of freedom 2 is K_{12} . So, K_{12} times u_2 is the reaction force development in degree of freedom 1. And similarly the internal Force developed in a degree of freedom 2 will be $K_{21} u_1 + K_{22} u_2$ and these are the internal forces or the reaction forces and these should be exactly equal to the applied forces P_1 and P_2 .

$$\begin{aligned} K_{11} \cdot u_1 + K_{12} \cdot u_2 &= P_1 \\ K_{21} \cdot u_1 + K_{22} \cdot u_2 &= P_2 \end{aligned}$$

So, $K_{11} u_1 + K_{12} u_2 = P_1$ and $K_{21} u_1 + K_{22} u_2 = P_2$ and we can write this in a matrix form like this

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}, \quad \begin{bmatrix} \frac{A.E}{L} & -\frac{A.E}{L} \\ -\frac{A.E}{L} & \frac{A.E}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

K_{11} K_{12} K_{21} K_{22} u_1 u_2 is equal to P_1 P_2 , P_1 P_2 are the applied forces and K_{11} K_{12} and all these things are numerical equal to AE by L and K_{11} and K_{22} they are positive quantities because they refer to their own degrees of freedom that is K_{12} K_{21} are the cross

terms the force developed at some other degree of freedom because of unit displacement at this degree of freedom and by definition that should be -.

So, there you go this is the set of equilibrium equations and then if you have the material properties the cross-sectional area Young's modulus and then the length of the element we can determine the stiffness coefficients and then get our stiffness Matrix and P_1 P_2 are the applied forces these are known and u_1 u_2 are the displacements. So, now the question is say if I give you the material properties and then the applied load can you determine u_1 and u_2 .

That is can we invert this Matrix basically that is what we mean if you are able to invert this Matrix we can determine u_1 and u_2 . So, we can easily find out if we can invert this or not by doing what by calculating the determinant of this Matrix. So, if you calculate the determinant $A E$ by L multiplied by $A E$ by L - of these 2 quantities and we see that the determinant is 0. So, that means that although we have the system of equations that we need we cannot solve. Before we solve we need to do something else that we will see later.

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The slide is titled "Symmetry of stiffness matrices" and contains the following text:

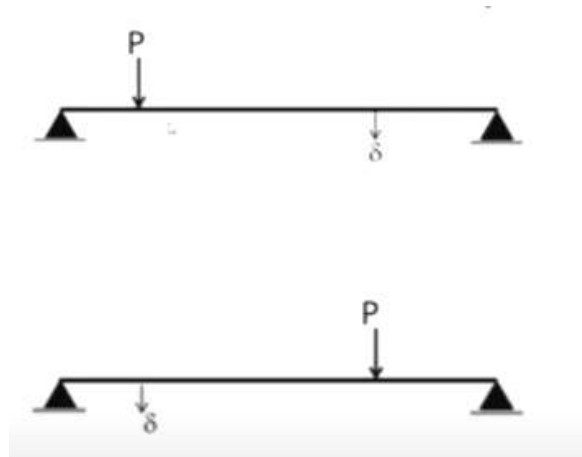
- Arises from Maxwell's reciprocal theorem or Betti's theorem from structural analysis

Below the text are two diagrams of a beam of length L supported by two pin supports. In the first diagram, a downward load P is applied at the left end, and a downward displacement δ is shown at the right end. In the second diagram, the load P is applied at the right end, and the displacement δ is shown at the left end. This illustrates that the displacement at one end due to a load at the other end is the same as the displacement at the other end due to a load at the first end.

On the right side of the slide, there is a vertical banner with the NPTEL logo at the top, followed by "COURSE" and "FEA & CM". Below this is a photo of the instructor, Dr. K. Rajagopal, with the text "Instructor Dr. K. Rajagopal". At the bottom of the banner, it says "LEARN MORE" and provides the URL "https://nptel.ac.in/".

At the bottom left of the slide, it says "FEA&CM 1-d systems" and at the bottom right, it says "12".

And now see we see that our system of equations gives us symmetric stiffness Matrix. See K_{12} and K_{21} are the same - $A E$ by L and this symmetry we can imagine by looking at the reciprocal theorem Maxwell reciprocal theorem.



That is so, if we have a structure with a load applied at one point and you measure the displacement at some other point or apply the load at the other point and measure the displacement at this point the Delta should be the same.

Whether you apply the load here and measure the displacement here or apply the load here and measure the displacement here that is the Maxwell's reciprocal theorem and because of that we see that all our stiffness matrices they are symmetric.

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Assembly of two bar elements connected between three nodes: structure has 3 nodes & 2 elements

- Let stiffness (AE/L) of elements be K_1 and K_2
- Let the displacements at different nodes be u_1, u_2 & u_3
- Let the externally applied forces at the three nodes be P_1, P_2 & P_3

Equilibrium equations for the two elements can be written separately as,

$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad \&$$

$$\begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}$$

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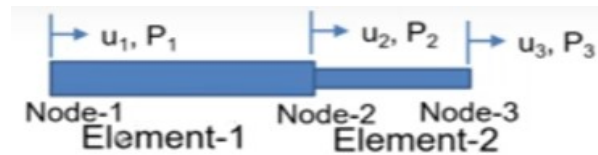
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So, now let us continue for a larger body of our structure let us consider now 2 elements element one and element two and it is not necessary that they have the same cross-sectional area under same properties and other things.



Let it let each one have their own properties and now we have three nodes node one node 2 node three and three displacements u_1 u_2 and u_3 and the three applied forces P_1 P_2 and P_3 .

And let the stiffness of these 2 elements be K_1 and K_2 because theoretically each element can be made of different material they may have different length and different cross sectional areas and so on. And let the displacements be u_1 u_2 and u_3 and the applied forces are P_1 P_2 and P_3 . And the equilibrium equations for the 2 elements can be written separately for element 1 and element 2 separately like this.

$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$


$$\begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}$$

See for element 1 that is connected to degrees of freedom u_1 and u_2 we can write like this $K_1 - K_1 - K_1 K_1 u_1 u_2$ is equal to $P_1 P_2$. Similarly for element 2 it is connected to degrees of freedom u_2 and u_3 and the applied forces are P_2 and P_3 . So, this is our system of equations for element 2. And but then our structure has three degrees of freedom.

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How to develop equilibrium equations for the entire structure?

- Need to assemble the equilibrium equations of individual elements to form the equations for the entire structure
- How?
- Dof-2 is common to both elements 1 & 2 at node-2
- We can think of two springs connected in parallel as an analogy



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And now we have to see how we can develop the equilibrium equations for the entire structure. See now we have determined the equilibrium equations at the element level at for element 1 and element 2 separately. And we need to form the equilibrium equations for the entire structure and that we can do by assembling assembly of the contributions from different elements. And so, if we assemble we get the equations for the entire structure but how.

So, if you look at the degree of freedom 2 is common for element 1 and element 2. So, the displacement is the same at these 2 nodes for both elements one and element 2 right. And so, it is actually we can think of 2 Springs connected in parallel because these axial elements you can imagine them as the spring elements just string because that is what they are.

(Refer Slide Time: 24:07)

Equivalent stiffness of two springs connected in parallel

Diagram: A horizontal rigid bar is supported by two vertical springs. A downward force P is applied to the bar. The displacement of the bar is δ . The left spring has stiffness K_1 and carries load P_1 . The right spring has stiffness K_2 and carries load P_2 .

- Two springs are connected rigidly between two rigid platens
- Both springs will undergo the same axial deformation, δ
- Let the combined stiffness of the system be K_s
- Applied force P is shared by both the springs in proportion to their stiffness

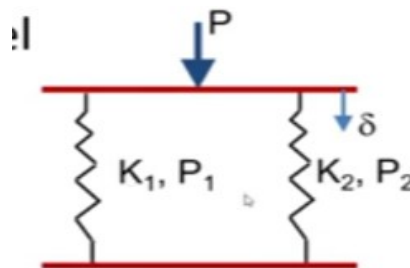
$$P = P_1 + P_2 = K_1 \delta + K_2 \delta = (K_1 + K_2) \delta = K_s \delta$$

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And let us consider that 2 elements K 1 and sorry element one and element 2 are connected rigidly between 2 patterns and the stiffness of element one is K 1 and stiffness of element 2 is K 2.



And when we apply any Force both will undergo the same deformation because they are connected with the rigid pattern the same displacement Delta in both the elements. And let us

say that the combined stiffness is K_S that is the contribution from element 1 and contribution from element 2.

And so, the applied force P is shared by both the Springs in proportion to their stiffness. So, the force developed in Spring 1 is k_1 times Δ and the force developed in Spring 2 is K_2 times Δ and the total Force P is $P_1 + P_2$ that is K_1 times Δ + K_2 times Δ . And let us say that our combined stiffness is K_S and if you look at this the P is K_S times Δ the K_S is the combined the stiffness of the system and P_1 P_2 are $K_1 \Delta$ and $K_2 \Delta$.

$$P = P_1 + P_2 = K_1 \cdot \delta + K_2 \cdot \delta = (K_1 + K_2) \delta \equiv K_s \cdot \delta$$

So, if you look at this the combined stiffness is $K_1 + K_2$ that should give us the clue on how to assemble the contribution from these two from these 2 elements.

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By using the analogy of the parallel springs undergoing the same deformation at common point, the equilibrium equations of both elements can be assembled as,

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

The stiffness matrix now corresponds to the entire structure. This process of combining the contributions from different elements is called as assembly.

Notice that the stiffness matrix is symmetric at element level and also at the structure level

Notice the banded nature of the stiffness matrix.

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So, if you assemble a degree of freedom to the contribution is added up. So, we have K_1 and K_2 , $K_1 + K_2$.

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

And so, this is how we do. And so, this $K_1 - K_1 - K_1$ is the contribution from element 1 and from element 2 we have $K_2 - K_2 - K_2$ and K_2 and at this degree of freedom 2 the contributions of both the elements are added together. So, we have this $K_1 + K_2$. So, now the stiffness Matrix and the equations they correspond to the entire structure.

And we if we are able to solve this system of equations we can get our unknowns u_1 u_2 u_3 . And so once again we see that our stiffness Matrix is banded and unsymmetric - K_{11} - K_{12} - K_{22} K_{23} and you see a 0 here. And it is not difficult to imagine why we got 0 here. If you look at if you look at the structure see this degree of freedom 3 is not connected to degree of freedom 1.

So, if I apply some unit displacement or in some force here only this the u_2 might react because it is connected to u_3 but u_1 is far away and it is not directly connected to u_3 server stiffness Matrix will look like this K_{11} - K_{12} 0 this 0 just simply means that if we apply anything at degree of freedom 1 the degree of freedom 3 is not affected.

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Let us consider more number of nodes & elements

$$\begin{bmatrix}
 K_1 & -K_1 & 0 & 0 & 0 & 0 & 0 \\
 -K_1 & K_1 + K_2 & -K_2 & 0 & 0 & 0 & 0 \\
 0 & -K_2 & K_2 + K_3 & -K_3 & 0 & 0 & 0 \\
 0 & 0 & -K_3 & K_3 + K_4 & -K_4 & 0 & 0 \\
 0 & 0 & 0 & -K_4 & K_4 + K_5 & -K_5 & 0 \\
 0 & 0 & 0 & 0 & -K_5 & K_5 + K_6 & -K_6 \\
 0 & 0 & 0 & 0 & 0 & -K_6 & K_6
 \end{bmatrix}
 \begin{Bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6 \\
 u_7
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 P_1 \\
 P_2 \\
 P_3 \\
 P_4 \\
 P_5 \\
 P_6 \\
 P_7
 \end{Bmatrix}$$

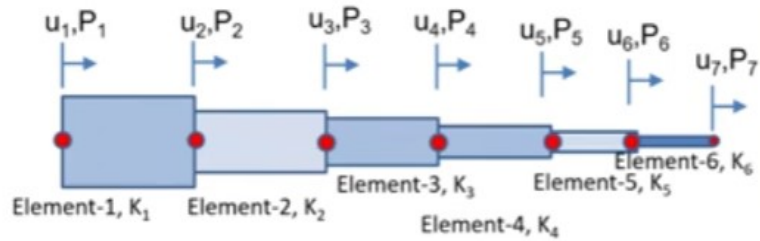
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And now let us consider a bigger element bigger structure. So, now with 6 elements and 7 degrees of freedom u_1 u_2 u_3 u_4 u_5 u_6 u_7 and now you see the banded nature of this stiffness Matrix K_{11} - K_{12} - K_{13} K_{14} + K_2 and so on.



$$\begin{bmatrix}
 K_1 & -K_1 & 0 & 0 & 0 & 0 & 0 \\
 -K_1 & K_1 + K_2 & -K_2 & 0 & 0 & 0 & 0 \\
 0 & -K_2 & K_2 + K_3 & -K_3 & 0 & 0 & 0 \\
 0 & 0 & -K_3 & K_3 + K_4 & -K_4 & 0 & 0 \\
 0 & 0 & 0 & -K_4 & K_4 + K_5 & -K_5 & 0 \\
 0 & 0 & 0 & 0 & -K_5 & K_5 + K_6 & -K_6 \\
 0 & 0 & 0 & 0 & 0 & -K_6 & K_6
 \end{bmatrix}
 \begin{Bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6 \\
 u_7
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 P_1 \\
 P_2 \\
 P_3 \\
 P_4 \\
 P_5 \\
 P_6 \\
 P_7
 \end{Bmatrix}$$


And these are the the displacements $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ and then these are the applied forces and um the determinant of this bigger Matrix is also zero that we do not need to doubt.

Because that one element level we have seen then we can extrapolate the same thing and for even for 2 element Case where we have a three by three Matrix we can theoretically calculate and see that the determinant is zero. And if you take lot of effort and find the determinant of the center Matrix we will see that it is zero. We do not need to actually calculate.

(Refer Slide Time: 29:29)


Some observations

- Notice the highly banded nature of these stiffness matrices
- Can these matrices be inverted to determine the displacements?
- Determinant of these matrices is zero !!!!
- As such, the system is not constrained from undergoing rigid body deformations
- At least one of the displacements should be constrained, i.e. fixed to some known value such that the system does not undergo rigid body deformations
- RIGID Body deformations – equal deformation at all degrees of freedom
- RIGID body deformations do not create any internal strains & hence body is not stressed – **hence reaction forces are not developed**



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So, the first thing that we need to notice is that the stiffness Matrix is highly banded and most of the elements are zero meets column. So, if you apply a Gauss elimination procedure for this system we just need to eliminate one value below the diagonal in each of these columns.

So, it becomes very simple but then if you have to invert this Matrix you need to spend a lot of time and the determinant of these matrices is also zero.

So, we cannot really do anything we cannot invert these matrices and find our displacements and as such these systems are not constrained and they may undergo rigid body deformations a rigid body deformation is something very simple. So, I take this pen and move it in the air somewhere and all the points on this and this pen are moving by the same amount. So, that means that there is no relative deformation between the 2 points within the pen and there is no strain within this within this pen.

And if there are no strains they will not be any internal forces or internal strains. So, to make this system stable we need to apply some constraint at least one constraint because in a one-dimensional system we just require one constraint and one of the displacements we can fix to some known value it could be zero or it could be something else. So, that we can prevent the system from undergoing rigid body deformations.


And the rigid body deformation is a deformation such that all the degrees of freedom undergo the same deformation equal deformation. And the rigid body deformations as such they do not create an internal strains and hence the body is not stressed our body is not stressed. So, if I keep the pen here or somewhere else the forces that are developed within the body are zero and if the body is not stressed it will not be able to produce any reaction force.

So, we can say that there is it body a deformation does not satisfy our equilibrium equations because we are applying some external Force but the internal reaction is zero. So, that means that any body that is undergoing rigid body deformations will undergoing infinite displacements because there is no limit on the on the on the displacements because the reaction force is not developed.

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Can we solve this problem by computer program?

Bar loaded axially in tension at both ends




> Solution is obvious to us, but not to the computer programs
 > singular matrix is encountered & hence solution cannot be obtained as such

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So, let us look at conceptually very simple problem. Let us take one axial element and then go on pulling at the 2 sides by an equal amount. And we know that this system is in equilibrium because we have applied equal and opposite forces at the 2 ends. And the solution is obvious to us the axial force is equal to P that is the tensile force and the stress is P by Δl P by A .

Bar loaded axially in tension at both ends

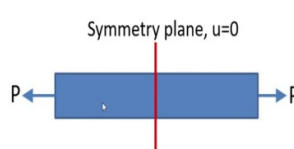


But then if you pose this problem to any computer it will say that no it is this system is unstable because there is no constraint because you cannot really invert the matrices and these matrices are called a singular matrices and as such we cannot solve this. And we can do a small trick to make this table see at this neutral point at the exactly at the mid length of this element what will be the displacement.

See here at this end let us say that there is a positive displacement and at this end it will have an equal displacement but in the negative direction and exactly at the midpoint the displacement should be zero because that is the Symmetry point and in between we will have a smaller displacement of either positive or negative magnitudes.

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Enforcing symmetry to enable solution



Symmetry plane, $u=0$

By enforcing symmetry boundary condition, either left half or right half of the structure can be analyzed

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So, the Symmetry plane our displacement is zero. So, this is what we can enforce on the system that at this point my displacement is 0 and then you apply the force and then find the deformations. And once we apply this you know deformation constraint on the Symmetry plane then we can either solve the right hand side the half of the problem or the left hand side it does not matter because both will give the same result.

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Techniques for applying boundary constraints

- Exact method
- Boundary spring method (or penalty method)

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And but then how do we apply some constraints that we call as the boundary constraints and there are 2 methods one is the exact method and the other is the boundary spring method at the penalty method. And I will explain both of them very simple.

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Exact method to apply boundary conditions


Let us consider a system with four degrees of freedom as,

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$

Let the degree of freedom x_3 be known as δ
 In the above equation, all stiffness coefficients are known
 Right hand side load quantities are known
 One of the dof's is also known
 Equations are re-written by taking known quantities to RHS

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Let us look at the exact method first and let us say that we have a system of four simultaneous equations with four unknowns x_1, x_2, x_3, x_4 and then the applied forces are P_1, P_2, P_3, P_4 and let us say that one of these degrees of freedom x_3 is known to us or it is constrained to be Δ and after solving our Δ, x_3 should be equal to Δ . And in here if we see these are all the known quantities K_{11}, K_{12}, K_{13} and so on.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$


Because once you know the geometric and material properties we can determine K_{11}, K_{12} and other things and then the right hand side P_1, P_2, P_3, P_4 these are the applied forces. So, this also we know but only thing that are unknown are x_1, x_2, x_3 and x_4 out of this x_3 is known. So, we can actually modify this system of equations and send all the known quantities to the right hand side.

And so, that we are left with only the unknown quantities on the left hand side. And so, we can rewrite this system of equations by sending all the known quantities to the right hand side. So, since x_3 is known and K_{13} is known we can take a product of K_{13} and x_3 and send this to the right hand side. And that will belong to degree of freedom 1. So, it should go to the right hand side and in row one.

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Modified equations


- $K_{11}x_1 + K_{12}x_2 + 0 \cdot x_3 + K_{14}x_4 = P_1 - K_{13}\delta$
- $K_{21}x_1 + K_{22}x_2 + 0 \cdot x_3 + K_{24}x_4 = P_2 - K_{23}\delta$
- $x_3 = P_3 = \delta$
- $K_{41}x_1 + K_{42}x_2 + 0 \cdot x_3 + K_{44}x_4 = P_4 - K_{43}\delta$
- The 3rd is a dummy equation so that number of equations does not change



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And so, our first equation will be

$$K_{11}x_1 + K_{12}x_2 + 0 \cdot x_3 + K_{14}x_4 = P_1 - K_{13}\delta$$

$K_{11}x_1 + K_{12}x_2 + 0 \cdot x_3 + K_{14}x_4 = P_1 - K_{13}\delta$ because K_{13} we have taken to the right hand side + $K_{14}x_4$ is equal to $P_1 - K_{13}\delta$.

Similarly the second equation

$$K_{21}x_1 + K_{22}x_2 + 0 \cdot x_3 + K_{24}x_4 = P_2 - K_{23}\delta$$

$$x_3 = P_3 = \delta$$

and then the third equation is actually it is a trivial equation x_3 is equal to δ and then the fourth equation is

$$K_{41}x_1 + K_{42}x_2 + 0 \cdot x_3 + K_{44}x_4 = P_4 - K_{43}\delta$$

$K_{41}x_1 + K_{42}x_2 + 0 \cdot x_3 + K_{44}x_4 = P_4 - K_{43}\delta$. And our third equation is actually it is a trivial equation that x_3 is equal to δ .

So, you might ask since we already know x_3 why do we need to have this in the system of equations? Yeah, the reason is very simple because we started with four unknowns and four degrees of freedom and one of the degrees of freedom is known x_3 is known and it is very difficult to program to eliminate one degree of freedom suddenly. We started with four degrees of freedom and it is more easy to continue with the same number of degrees of freedom.

So, for that reason we retained the third equation but with a very trivial equation x_3 is equal to Delta and then our job becomes more simple our writing the program becomes very simple.

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Modified equations in matrix form

$$\begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} \\ K_{21} & K_{22} & 0 & K_{24} \\ 0 & 0 & 1 & 0 \\ K_{41} & K_{42} & 0 & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 - K_{13} \cdot \delta \\ P_2 - K_{23} \cdot \delta \\ \delta \\ P_4 - K_{43} \cdot \delta \end{Bmatrix}$$


If dof m is constrained, the changes made are:

- $K_{mm}=1$
- $P_i = P_i - K_{im} \cdot \delta$ if $i \neq m$
- $P_i = \delta$ if $i=m$
- $K_{mi}=K_{im}=0$ if $i \neq m$

The above matrix is non-singular & solution can be obtained



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And and then if you rewrite those equations in a matrix form we get like this

$$\begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} \\ K_{21} & K_{22} & 0 & K_{24} \\ 0 & 0 & 1 & 0 \\ K_{41} & K_{42} & 0 & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 - K_{13} \cdot \delta \\ P_2 - K_{23} \cdot \delta \\ \delta \\ P_4 - K_{43} \cdot \delta \end{Bmatrix}$$

$K_{11} \ K_{12} \ 0 \ K_{14}$ and so on. And the right hand side is $P_1 - K_{13} \Delta$ whereas the third equation is just simply x_3 is equal to Delta. So, in general if degree of freedom m is constrained the changes to be made are K_{mm} is 1 and P_i is $P_i - K_{im}$ times Delta if i is not equal to m and P_i is Delta if i is equal to m and K_{mi} is K_{im} is 0.

If dof m is constrained, the changes made are:

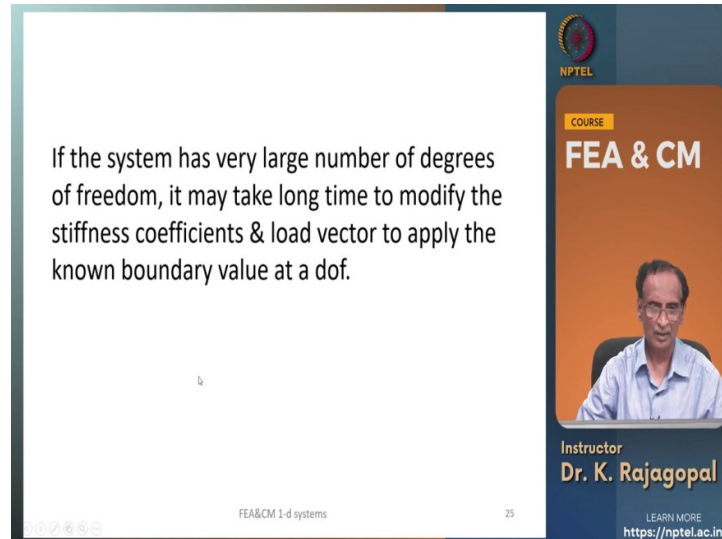
- $K_{mm}=1$
- $P_i = P_i - K_{im} \cdot \delta$ if $i \neq m$
- $P_i = \delta$ if $i=m$
- $K_{mi}=K_{im}=0$ if $i \neq m$

The above matrix is non-singular & solution can be obtained

That is a half diagonal terms once you make these changes this stiffness Matrix the modified stiffness Matrix is not singular it will have some if it will have some determinant and we

should be able to invert this Matrix. So, these are the changes that we make and then we eliminate these degrees of freedom sorry by eliminating this degree of freedom $\times 3$ we can do this.

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If the system has very large number of degrees of freedom, it may take long time to modify the stiffness coefficients & load vector to apply the known boundary value at a dof.

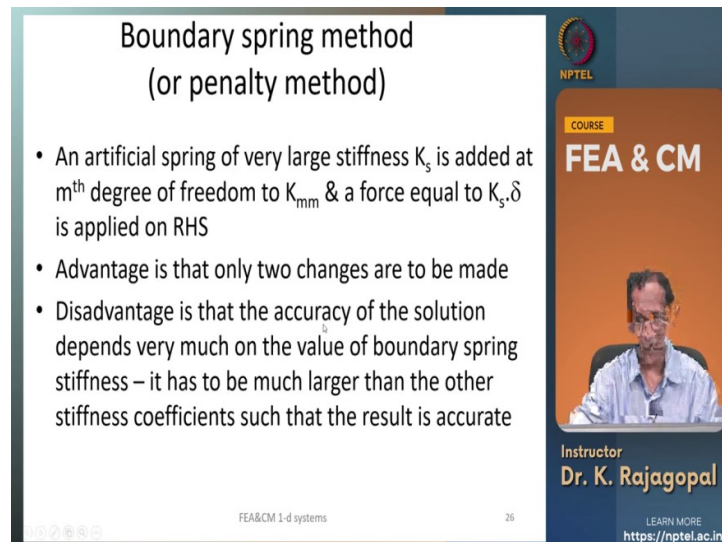
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So, if the system has very large number of degrees of freedom may take a very long time to modify each of these stiffness coefficients and the load vector.

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Boundary spring method
(or penalty method)

- An artificial spring of very large stiffness K_s is added at m^{th} degree of freedom to K_{mm} & a force equal to $K_s \cdot \delta$ is applied on RHS
- Advantage is that only two changes are to be made
- Disadvantage is that the accuracy of the solution depends very much on the value of boundary spring stiffness – it has to be much larger than the other stiffness coefficients such that the result is accurate

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And so, we should have some other method for applying a similar constraints and that is called as a boundary spring method. And so it is also called as a penalty method because on the system we apply some penalty so, that we can enforce what we want. And what we do is we take an artificially an artificial spring of very very large stiffness case and add it at the

empty degree of freedom where we want to apply some constraint and then apply a force equal to K_S times Δ on the right hand side.

So, the advantage is that we make only 2 changes one on the right hand side and one in the stiffness coefficient. But the disadvantage is that the accuracy of the solution depends very much and the value of the boundary spring and the value of the boundary spring stiffness has to be much much larger than the all other stiffness coefficients. So, that our result is accurate so, that we can see with an example.

(Refer Slide Time: 42:01)

If the 3rd degree of freedom is prescribed as δ

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & (K_{33}+K_s) & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ K_s \cdot \delta \\ P_4 \end{Bmatrix}$$

From the 3rd equation,

$$x_3 = \frac{K_s \cdot \delta - K_{31}x_1 - K_{32}x_2 - K_{34}x_4}{(K_{33} + K_s)} \cong \frac{K_s \cdot \delta}{K_s} \cong \delta$$

The accuracy of the solution depends on value of K_s

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So, on the same problem say we are constraining the third degree of freedom to sum Δ and we can add the K_S to the diagonal element of this stiffness Matrix and then on the right hand side we can apply a force equal to K_S times Δ and this K is so, large that compared to all the other forces that are developed the K_{31} times x_1 K_{32} times x_2 K_{34} times x_4 they are very small and negligible.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & (K_{33}+K_s) & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ K_s \cdot \delta \\ P_4 \end{Bmatrix}$$

And so, our x_3 after doing the calculations will be approximately equal to Δ if provided our K_S is very very large the calculated Δ will be the same as what we want.

From the 3rd equation,

$$x_3 = \frac{K_s \cdot \delta - K_{31}x_1 - K_{32}x_2 - K_{34} \cdot x_4}{(K_{33} + K_s)} \cong \frac{K_s \cdot \delta}{K_s} \cong \delta$$

But if our spring stiffness K_s is not of very large quantity then we could have some errors that we can check with a numerical example. And the accuracy of this solution depends very much on K_s and if the case is very very large let us say the ideally case should be infinite.

So, that at the end of this solution this K is in the numerator and denominator they cancel out and all other quantities are very small because this is infinite times Delta it is a very large quantity infinite and but then if we make it very very large we could have some round of Errors when we are doing the computation. So, we have to have a balance between the computational accuracy and then this boundary spring the stiffness that we give.

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The slide is titled "Elimination of fixed dofs" and contains three bullet points:

- If any degree of freedom is fixed, usually that dof is eliminated from the system by not numbering it and not assembling the contribution.
- This is typical in geotechnical problems where nodes along the boundaries are constrained in one or more directions.
- More details will be explained at a later stage in the course.

The slide also features a sidebar on the right with the NPTEL logo, the course title "FEA & CM", a photo of the instructor "Dr. K. Rajagopal", and the URL "https://nptel.ac.in/".

So, now by doing one of these methods we can we can enforce the constraint or the other method is if any of the fixed degrees of freedom is there we can directly eliminate that right from the beginning and we do not number that particular degree of freedom and assign any stiffness coefficient. And especially this is typical of geotechnical problems where we have very very long boundaries where our displacements are zero.

And at the time of the number in the equations itself we do not number those degrees of freedom and that is how we can eliminate those fixed degrees of freedom. That we will see later not at this stage.

(Refer Slide Time: 45:02)

Solution for the 2-element problem

$$\begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1 + K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Let node-1 be fixed with $u_1=0$
 Loads P_2 & P_3 are not affected by the fixed boundary condition as $u_1=0$
 First row & column could be deleted from the assembly without any consequence
 Let us apply load only at the tip i.e. $P_2=0$ & $P_3=P$

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And so, here let us go back to our 2 element problem and let us say that our u_1 is 0 and P_2 and P_3 are not affected because our u_1 is zero. So, it is the first row and first column can be eliminated because the P_1 is not affected because this displacement is zero. And so, we can eliminate the first row and the first column and we end up with a 2 by 2 Matrix like this.

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$$\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Let us use Gauss elimination method:
 Multiply the first row elements by $-K_2/(K_1+K_2)$
 and subtract from the 2nd row elements to get,

$$\begin{bmatrix} K_1 + K_2 & -K_2 \\ 0 & \frac{K_1 K_2}{K_1 + K_2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

From above, $u_3 = \frac{P(K_1+K_2)}{K_1 K_2} = \frac{P}{K_2}$
 & $(K_1 + K_2)u_2 - K_2 u_3 = 0 \rightarrow u_2 = \frac{P}{K_1}$

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$$\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

$K_1 u_1 + K_2 u_2 - K_2 u_3 = 0$ and P because we applied only one force P at this at this point three. So, we can solve it either by Gauss elimination method or by matrix inversion method and we see that our u_3

$$u_3 = \frac{P(K_1 + K_2)}{K_1 K_2} = \frac{P}{K_s} \quad K_s = \text{combined stiffness} = \frac{K_1 K_2}{K_1 + K_2}$$

is P by K_s and P times $K_1 + K_2$ by $K_1 K_2$. And actually we have seen that this is equal to the spring constant K_s when we have 2 strings in parallel to each other and are in series sorry and our $K_1 + K_2$ times $u_2 - K_2$ times u_3 is 0.

That is once you get u_3 we can get u_2 as $\frac{3}{K_1}$ and the K_s is our combined stiffness $K_1 K_2$ by $K_1 + K_2$.

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Element forces

Force in each element can be calculated as difference in displacements multiplied by its stiffness

Force in element-1 = $K_1(u_2 - u_1) = K_1 \times P / K_1 = P$

Force in element-2 = $K_2(u_3 - u_2)$

$$= K_2 \left[\frac{P(K_1 + K_2)}{K_1 K_2} - \frac{P}{K_1} \right] = K_2 \left[\frac{P(K_1 + K_2) - P.K_2}{K_1 K_2} \right] = P$$

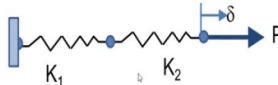
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And once we get the deformations we can find the forces in the 2 elements the force in element one is a K_1 times $u_2 - u_1$.

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Combined stiffness of two springs connected in series





Let the combined stiffness be K_s

Load in each spring is the same as they are in series,
 Extension of spring-1, $\delta_1 = P/K_1$
 Extension of spring -2, $\delta_2 = P/K_2$
 Total extension, $\delta = \delta_1 + \delta_2 = P/K_s$
 $P/K_1 + P/K_2 = P/K_s$

$$\frac{1}{K_s} = \frac{1}{K_1} + \frac{1}{K_2} \Rightarrow K_s = \frac{K_1 \cdot K_2}{K_1 + K_2}$$

The combined stiffness of two springs is seen in the previous solution for tip displacement


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And the force in element 2 is $K_2 \times (u_3 - u_2)$ and actually let us look at one small numerical example to illustrate this actually this combined stiffness of 2 parallel 2 Springs in series is explained here and that I think you can leisurely see.



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Numerical example: Exact method

- Let $K_1=1000$ & $K_2=2500$
- Applied load at dof-3 = 50
- Displacement at 1st dof = 0.30

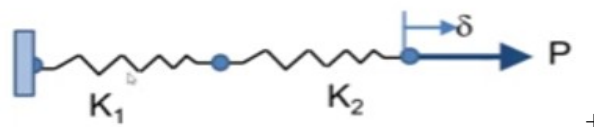
The equilibrium equations for the system are,

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$


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Now let us look at one numerical example so, that we can give some values and then understand the calculations better let us say in this previous example of 2 elements with the spring 1 K_1 and spring 2 K_2 we applied a load of 50 a degree of freedom 3 and a degree of freedom 1 we constrained the displacement to point three the server equilibrium equations are like this see this K_1 is thousand and K_2 is 2500.



$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

So, this is our assembled equations $K_1 - K_1 - K_1$ $K_1 + K_2 - K_2$ and so on and here we know that our u_1 is 0.3.

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Modify the equations to implement the known displacement at u_1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.30 \\ 0 + 1000 * 0.3 = 300 \\ 0 * 0.3 + 50 = 50 \end{Bmatrix}$$

Modify the 3rd row to make $K_{32}=0$, factor = $-2500/3500$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3500 & -2500 \\ 0 & 0 & 714.2857 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.30 \\ 300 \\ 264.2857 \end{Bmatrix}$$

$\Rightarrow u_3 = 0.37,$
 $\Rightarrow u_2 = 0.35,$
 $\Rightarrow u_1 = 0.30$

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And our known quantities we can take to the right hand side and here it was - 1000. So, - 1000 times 0.3 if you send it to the right hand side we get 1000 times 0.3 that is 300 and then at the degree of freedom 3 we have applied a force of 50 and we can solve this by converting this to an upper diagonal matrix and we get the u_3 as 0.37 u_2 is 0.35 and u_1 is 0.3.

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Forces in elements

- Force in element-1 = $1000 \times (0.35 - 0.30) = 50$
- Force in element-2 = $2500 \times (0.37 - 0.35) = 50$

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And that is what we want to do 3 is a fixed as a 0.3 and the force in element 1 is 1000 times $u_3 - u_2$ that is the relative deformation that is that comes out as 50. and the force in element 2 is 2500 multiplied by $0.37 - 0.35$ and that comes to once again 50 because they are in series.

$$\text{Force in element-1} = 1000 \times (0.35 - 0.30) = 50$$

$$\text{Force in element-2} = 2500 \times (0.37 - 0.35) = 50$$

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Boundary spring method: Example

- Choose K_s as 10000; $P_3 = 10000 \times 0.3 = 3000$

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

The modified equations are:

$$\begin{bmatrix} 1000 + 10000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ 0 \\ 50 \end{Bmatrix}$$

$u_3 = 0.375$, $u_2 = 0.355$, $u_1 = 0.305$ (0.37, 0.35 & 0.30 with exact method)

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And let us look at the boundary spring method to illustrate and let us take the boundary spring stiffness is ten thousand. So, the PS is a three thousand one thousand ten thousand times point three it is our original equations are like this

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

and the modified equations are like this

The modified equations are:

$$\begin{bmatrix} 1000 + 10000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 3000 \\ 0 \\ 50 \end{Bmatrix}$$

and u_1 we want as 0.3. So, on the right hand side we apply the force of 3000 and then the left hand side we have added the boundary spring of 10000 to the first degree of freedom.

If you solve this system of equations we see that u_3 is 0.375 u_2 is 0.355 and u_1 is 0.305 and with the exact method these were the displacements that we got which are very close to these values.

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...effect of low stiffness of boundary spring

- Let $K_s = 500$; $P_s = 500 \cdot 0.3 = 150$

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

The modified equations are:

$$\begin{bmatrix} 1000 + 500 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 0 \\ 50 \end{Bmatrix}$$

$u_3 = 0.47$, $u_2 = 0.45$, $u_1 = 0.40$ (**desired value = 0.3 !!!**)
 Very different solution – be careful with choice of boundary spring stiffness !!!

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And now let us look at another example by assigning only 500 to the boundary spring K_S is 500. And if you solve we see that our u_3 is 0.47 u_2 is 0.45 and u_1 is 0.4.

- Let $K_s = 500$; $P_s = 500 \cdot 0.3 = 150$

The original equilibrium equations are:

$$\begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

The modified equations are:


$$\begin{bmatrix} 1000 + 500 & -1000 & 0 \\ -1000 & 3500 & -2500 \\ 0 & -2500 & 2500 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 0 \\ 50 \end{Bmatrix}$$

$u_3 = 0.47$, $u_2 = 0.45$, $u_1 = 0.40$ (**desired value = 0.3 !!!**)

See we wanted a u_1 of 0.3 but we end up with 0.4 which is totally different. And so, actually whenever we use boundary spring method we have to be very careful in the boundary spring value that we assign and if we assign a large enough value then we can get similar to the previous solution that is very close like 0.375, 0.355, 0.305.

So, 0.305 is very close to 0.3 and instead of 10 000 let us say we give a K_S of 1 million then these displacements may be more closer to the to the exact values that we calculated earlier

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Summary

- Definition of stiffness coefficient
- Equilibrium equations for a single 1-d element
- Assembly of equilibrium equations to form structure level equations
- Exact & boundary spring methods to enforce boundary conditions
- Solution for the nodal displacements
- Solution for element forces

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Just to summarize in this lecture we have defined the stiffness coefficient for a one dimensional element. Then we form the equations for one for a single element then assemble them for multiple elements and then we we have seen how to apply the boundary constraints either by exact method or by the boundary spring method and then we can once we apply this boundary constraints we can solve the equations then we got the solution for nodal displacements and then the element forces.

So, actually this we have seen all the steps that we will undergo in any finite element analysis we get the displacements and then we get the forces. So, this we will apply for larger systems in the subsequent lectures. So, that we understand the concepts. So, this is my last slide and if you have any questions please send an email to this graph profkr@gmail.com and I will be able to respond to your queries. So, thank you very much.