Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 30
Bilinear Elastic Models

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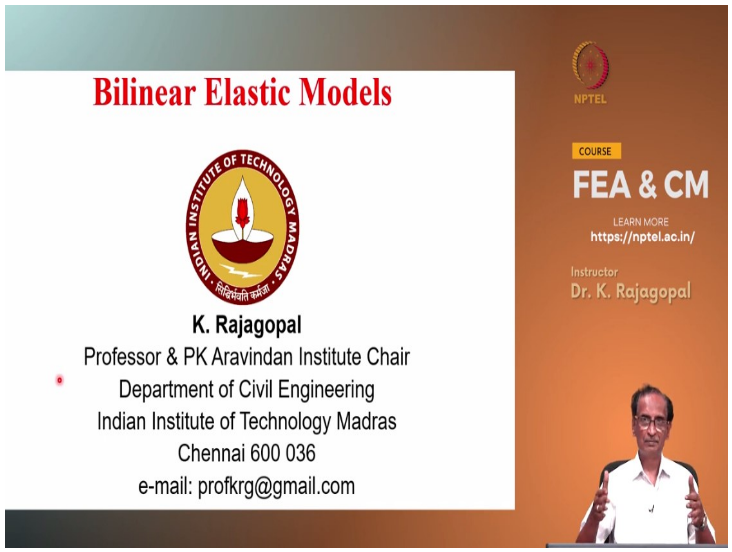

So hello students now let us start doing some constitutive to modeling and one of the simplest constitutive model that we can think of is a bilinear elastic model and let us see how we can do some computations by using this bi-linear elastic model.

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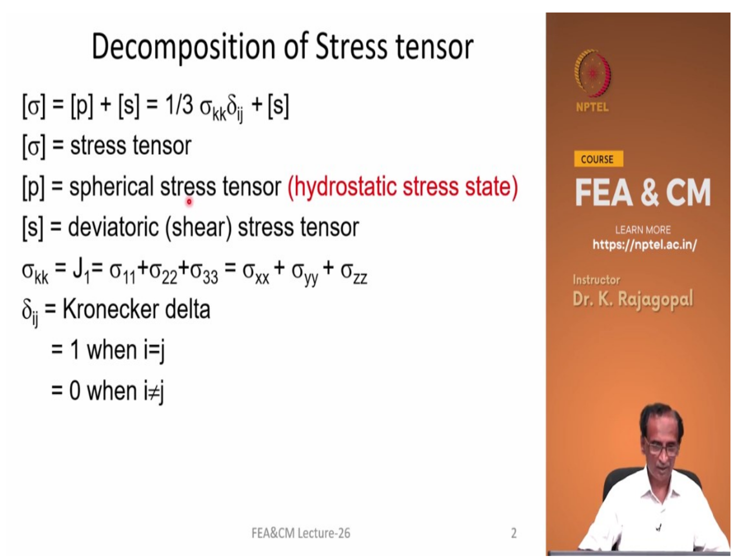

And the basis for this is our decomposition of the stress tensor. In one of our previous classes we had seen that the total stress tensor can be decomposed into some spherical stress tensor p and then the deviatoric stress tensor as the p is our hydrostatic pressure state that does not cause any failure because it is compression from all the directions and the soil is extremely strong in compression in fact in pure compression it will not fail.

$[\sigma] = [p] + [s] = 1/3\ \sigma_{kk}\delta_{ij} + [s]$

$[\sigma]$ = stress tensor

$[p]$ = spherical stress tensor (hydrostatic stress state)

$[s]$ = deviatoric (shear) stress tensor

$\sigma_{kk} = J_1 = \sigma_{11}+\sigma_{22}+\sigma_{33} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

δ_{ij} = Kronecker delta

= 1 when i=j

= 0 when i≠j

And then the failure is only because of our shear stress and then we can actually control our parameters. So, there are different modulus terms like the bulk modulus that controls the volumetric strains and then the spherical stress tensor and then the shear stresses are controlled by the shear modulus. The shear stress is at the g times gamma where gamma is our shear strain and by playing with these numbers we can approximately simulate our strength of the material and so on and.

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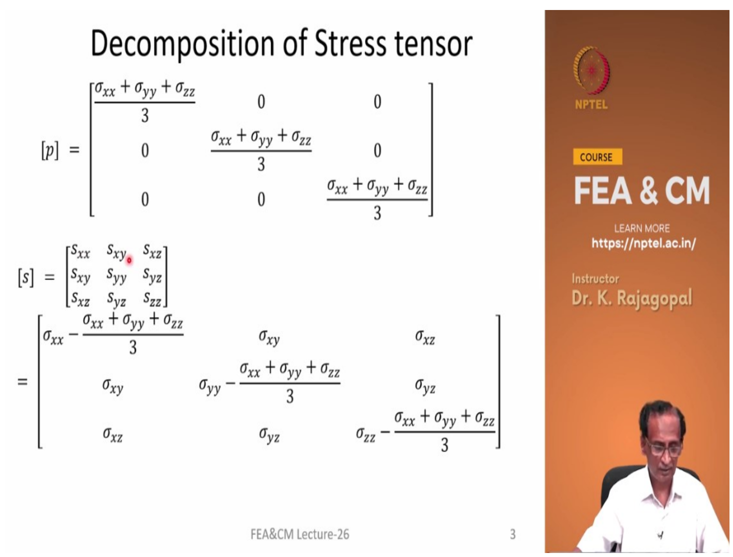

So this is how we decomposed our total stress tensor into a hydrostatic stress tensor or the spherical stress tensor and then and then the deviatoric stress tensor S xx, S yx, X z and so on.

$$[p] = \begin{bmatrix} \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & 0 & 0 \\ 0 & \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & 0 \\ 0 & 0 & \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} \end{bmatrix}$$

$$[s] = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx} - \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3} \end{bmatrix}$$

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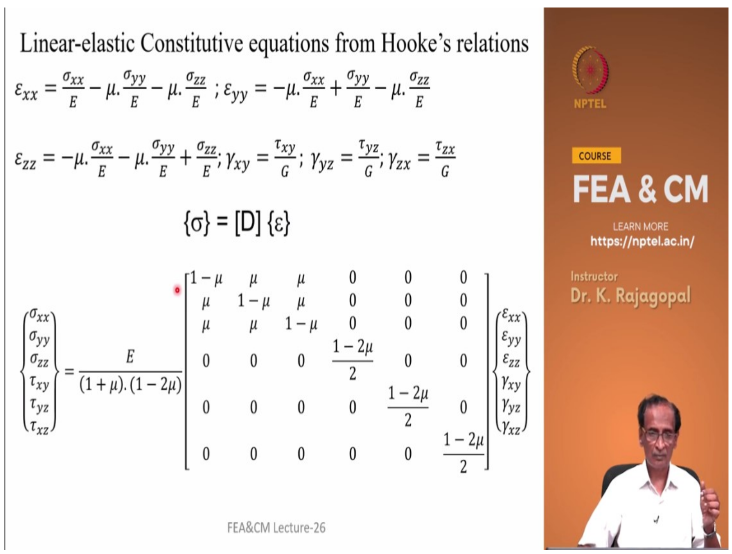

And our constitutive equations we have seen in terms of the Young's modulus and Poisson's ratio. We can think of the same constitutive equation in terms of other parameters the bulk modulus and shear modulus or the bulk modulus and Young's modulus and so on. So, that we can actually get more flexibility.

Linear-elastic Constitutive equations from Hooke's relations

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \mu.\frac{\sigma_{yy}}{E} - \mu.\frac{\sigma_{zz}}{E}\ ; \varepsilon_{yy} = -\mu.\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \mu.\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{zz} = -\mu.\frac{\sigma_{xx}}{E} - \mu.\frac{\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E}; \gamma_{xy} = \frac{\tau_{xy}}{G}; \gamma_{yz} = \frac{\tau_{yz}}{G}; \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\mu).(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

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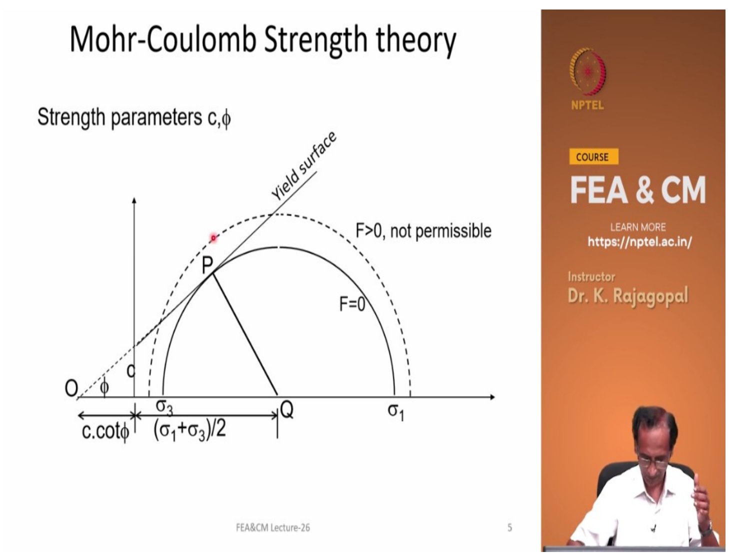

And when it comes to the soils the simplest strength equation can be obtained by using the Mohr Coulomb Strength theory where we can say that our tau max is c + sigma and tan phi that is an equation to apply on a particular failure plane, but in more general sense we cannot apply that c + sigma and tan phi we can only apply it in terms of our principle stresses sigma 1 and sigma 3.

And our Mohr Coulomb yield surface in the tau sigma and plane on the x axis we have the shear stress sorry the normal stress sigma and then on the y axis we have the shear stress tau and in this space this is our shear surface or the yield surface. The intercept on the y axis is your cohesion at the cohesive strength and the slope of this line is the friction angle phi and if you draw any more circle based on the stress state that we have if the Mohr circle is within the yield surface we say that the stress is in the elastic state.

And if it is just tangent to the yield surface we say that it is at the limit state and anything anymore circle that is cutting across the yield surface is not admissible it is not possible then we have to do something whenever that happens and from this Mohr circle that is just tangent to the yield surface we can by looking at this triangle OPQ we can derive a relation between the sigma 1 and sigma 3 in terms of the c and phi.

The sigma 1 is the maximum principle stress, sigma 3 is the is the minor principle stress and sin phi is PQ by OQ. PQ is the opposite and OQ is the diagonal. OQ is this c times cotangent phi the small length up to the y axis and then this length that is the mean normal stress sigma 1 + sigma 3 by 2.

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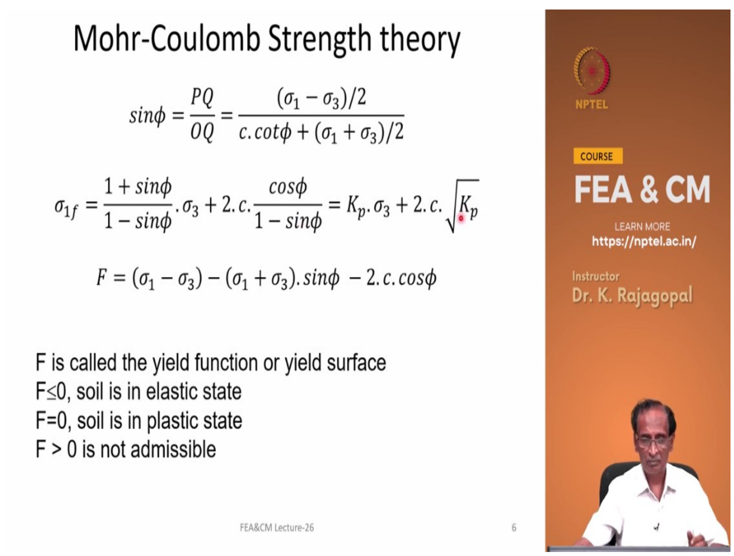

And sin phi is PQ by OQ and by simplifying this equation we can write the sigma 1 f that is the maximum principle stress sigma 1 is 1 plus sin phi by 1 - sin phi times sigma 3 + 2 c cosine phi by 1 - sin phi that is equal to K p. This 1 + sin phi by 1 - sin phi is our passive pressure coefficient K p and this term cosine phi by 1 - sin phi is the square root of K p and we can define a yield function or yield surface F in terms of this by slightly reformulating this like this sigma 1 - sigma 3 - of sigma 1 + sigma 3 sin phi – 2 c cosine phi.

$$sin\phi = \frac{PQ}{OQ} = \frac{(\sigma_1 - \sigma_3)/2}{c.cot\phi + (\sigma_1 + \sigma_3)/2}$$

$$\sigma_{1f} = \frac{1 + sin\phi}{1 - sin\phi}.\sigma_3 + 2.c.\frac{cos\phi}{1 - sin\phi} = K_p.\sigma_3 + 2.c.\sqrt{K_p}$$

$$F = (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3).sin\phi - 2.c.cos\phi$$

F is called the yield function or yield surface
$F \leq 0$, soil is in elastic state
$F = 0$, soil is in plastic state
$F > 0$ is not admissible

Actually this type of drawing the more circle we can do it on a graph sheet, but we cannot implement this in a computer program and we need some other method for seeing whether we are in the elastic state or the plastic state and for that this type of equation they help us. So, if your F is less than 0 the soil is in the elastic state and F is 0 and the soil is in the plastic state or the limit state and F greater than 0 is not admissible. We cannot accept any stress state that is crossing the yield surface.

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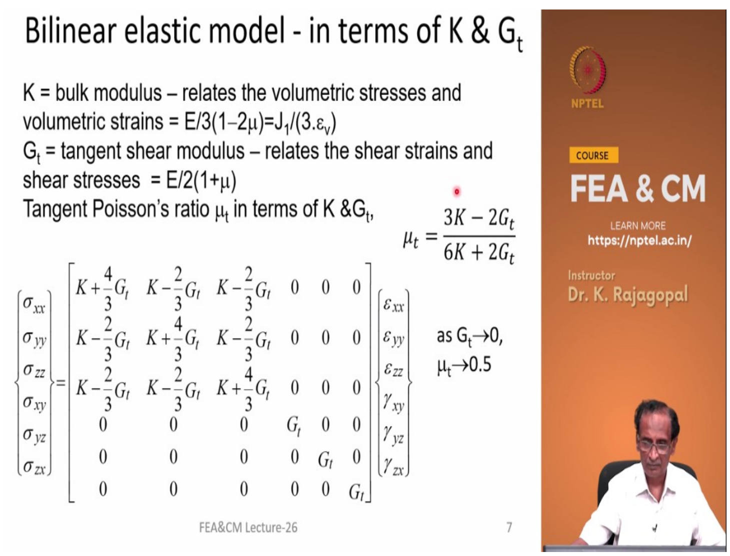

And based on these observations we can develop one model called as bilinear elastic model in terms of the bulk modulus K and the shear modulus G and the K is the bulk modulus that relates the bulk stresses or the volumetric stresses and then the volumetric strains and that is K is E by 3 times 1 - 2 mu and we can also write it as the mean normal stress divided by the corresponding epsilon v.

Bilinear elastic model - in terms of K & G_t

K = bulk modulus – relates the volumetric stresses and volumetric strains = $E/3(1-2\mu)=J_1/(3.\varepsilon_v)$
G_t = tangent shear modulus – relates the shear strains and shear stresses = $E/2(1+\mu)$
Tangent Poisson's ratio μ_t in terms of K &G_t,

$$\mu_t = \frac{3K - 2G_t}{6K + 2G_t}$$

as $G_t \rightarrow 0$,
$\mu_t \rightarrow 0.5$

G t is the tangent shear modulus it relates the shear strain and the shear stresses and the G t is related to Young's modulus and Poisson's ratio is E by 2 times 1 + mu and if you formulate our constitutive matrix in terms of K and G we can calculate our tangent Poisson's ratio mu in terms of the K and G as mu is a 3 K 2 G by 6 K + 2 G and the subscripted t for the shear modulus is to say that our shear modulus will change during the analysis.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} K+\frac{4}{3}G_t & K-\frac{2}{3}G_t & K-\frac{2}{3}G_t & 0 & 0 & 0 \\ K-\frac{2}{3}G_t & K+\frac{4}{3}G_t & K-\frac{2}{3}G_t & 0 & 0 & 0 \\ K-\frac{2}{3}G_t & K-\frac{2}{3}G_t & K+\frac{4}{3}G_t & 0 & 0 & 0 \\ 0 & 0 & 0 & G_t & 0 & 0 \\ 0 & 0 & 0 & 0 & G_t & 0 \\ 0 & 0 & 0 & 0 & 0 & G_t \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

And our constitutive matrix in terms of the K and G is written like this. Here the stress is equal to constitutive matrix multiplied by strain vector and if you look at this say at the limit state as the shear modulus tends to 0 our Poisson's ratio will tend to 0.5.

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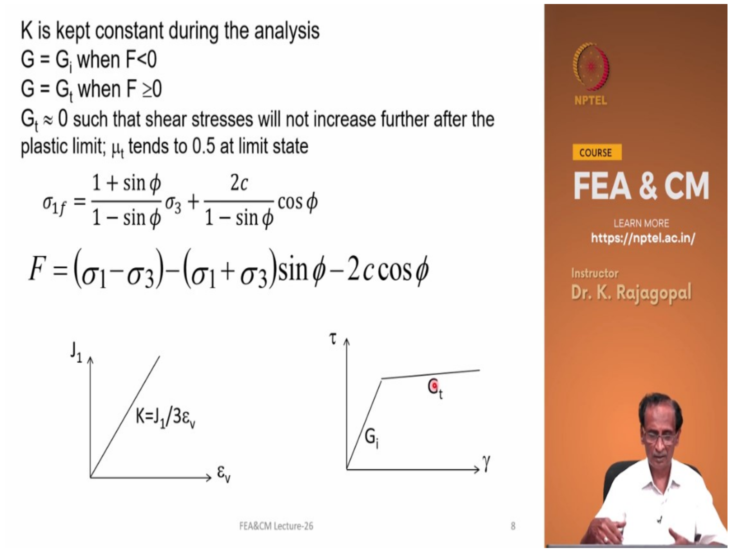
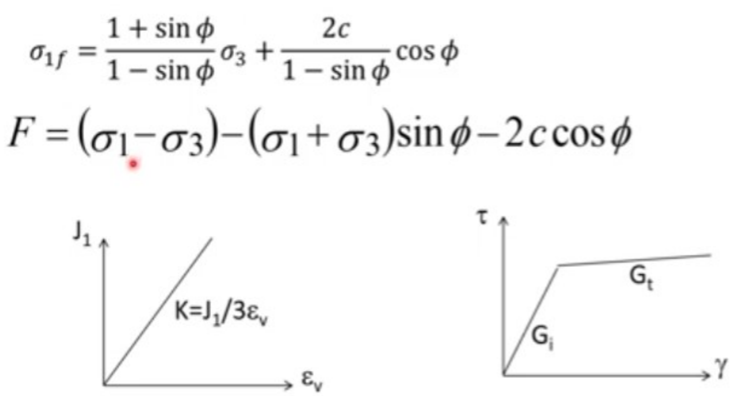


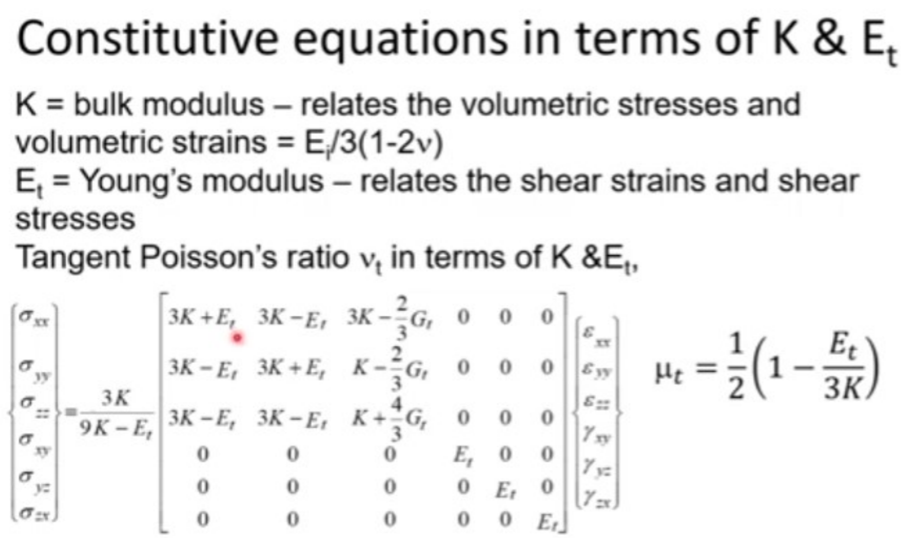
And our shear modulus G we can control based on whether our Mohr circle is in the elastic state or at the limit state and our bulk modulus we can keep it as constant because there is not going to be any failure because of the spherical stresses and the shear modulus G is set to some initial value when our yield function is less than 0 and when the yield function is greater than or equal to 0 we set the G to a small value G t very close to 0.

But not 0 so that any further increase in the shear stresses will not happen. So, beyond that plastic limit even if you increase the shear strains the shear stresses will remain more or less constant and in this state our Poisson's ratio tends towards 0.5 and this is what we have seen earlier and if you plot a graph between the J 1 and epsilon v it is a straight line that is the K and then the tau versus gamma is initially the shear modulus is G i up to some limit.



When your yield function becomes 0 or greater than 0 and beyond that the G is set to some small value G t it could be 1 percent of the initial value or 0.1 percent and so on like that we can decide based on the problem that we have and there is another method of forming the constitutive matrix in terms of bulk modulus and then the Young's modulus instead of the shear modulus we can directly formulate in terms of the Young's modulus.

And this is the equation of the constitutive matrix in terms of our bulk modulus and then the Young's modulus and then the tangent Poisson's ratio is one half of 1 - E by 3 K and the yield limit our E could be very small and so at that stage our Poisson's ratio will tend towards 0.5.



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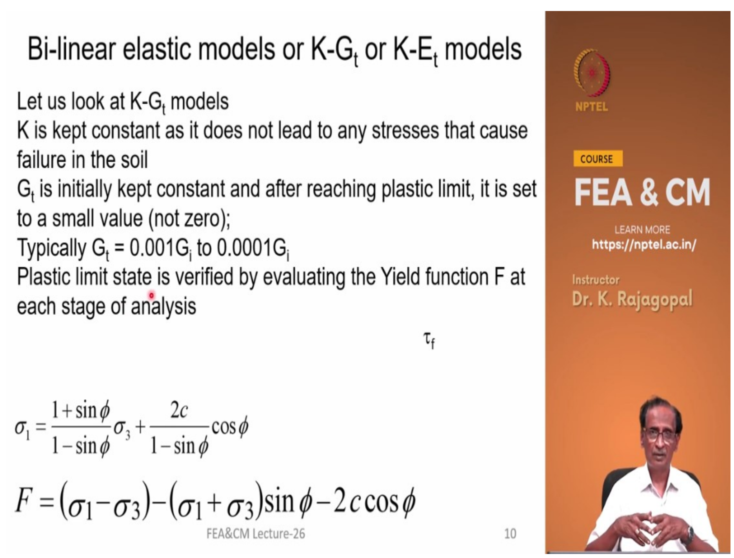
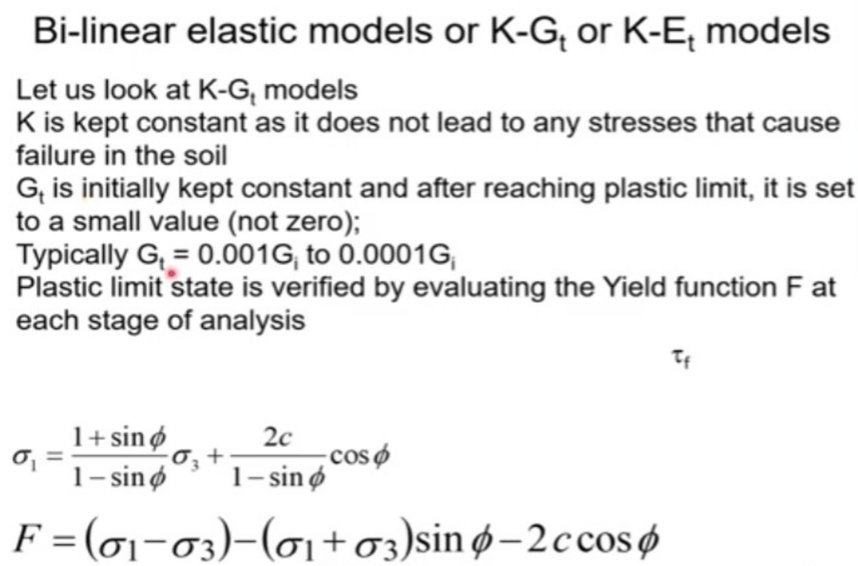


So, let us look at our K G type model the bilinear elastic model. Initially our G the shear modulus is set to some finite value and then once the yield function becomes positive that means that our Mohr circle is just crossing the yield surface. We set the tangent shear modulus to a small value as 0.001 G i or 0.0001 G I and then we continue the analysis and at any stage to check whether our stress state is within the elastic state or the plastic state we look at the yield function value.



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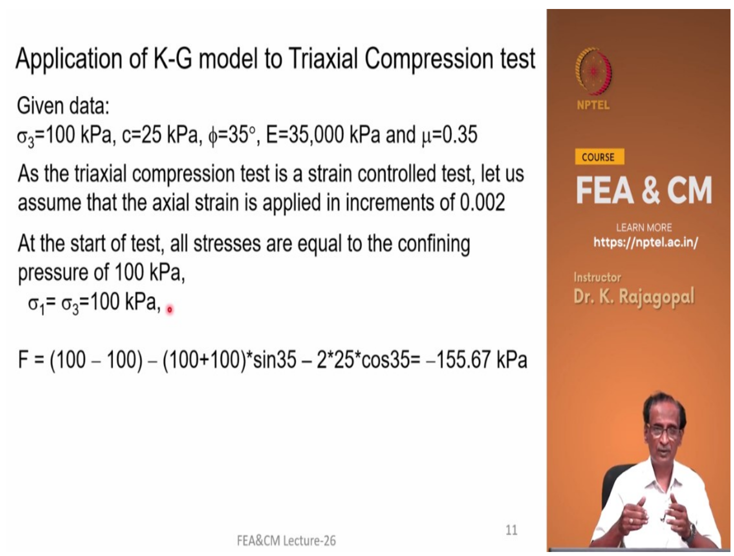

Let us apply this model the K G model or the bilinear elastic model to performing a triaxial compression test and this demonstration is suitable for hand calculations like whatever I am demonstrating is only for hand calculation so that you can understand the procedure, but the same thing we can apply it in any finite element program and there are several calculations that we cannot imagine or we cannot imagine doing that by hand.

Let us take the triaxial compression test performed at a confining pressure sigma 3 of 100 kPa and let our cohesive strength be 25 and the friction angle is 35 degrees and our Young's modulus is 35,000 and then the Poisson's ratio is 0.35 and our triaxial compression test is a strain control test. We are applying some strain and then measuring the reaction force through our proving ring.

Application of K-G model to Triaxial Compression test

Given data:

σ_3=100 kPa, c=25 kPa, ϕ=35°, E=35,000 kPa and μ=0.35

As the triaxial compression test is a strain controlled test, let us assume that the axial strain is applied in increments of 0.002

At the start of test, all stresses are equal to the confining pressure of 100 kPa,

$\sigma_1 = \sigma_3$=100 kPa,

F = (100 – 100) – (100+100)*sin35 – 2*25*cos35= –155.67 kPa

Let us say that we are applying axial straining in increments of 0.002 it could be anything like just applying this in some reasonable value so that within short number of cycles we can complete the analysis. So, our epsilon Z that is in the vertical direction is 0.002 and at the start of the test all the stresses sigma 1 sigma 3 they are equal and it is equal at 100. So, if you substitute the sigma 1 and sigma 3 of 100.

And then the c and phi in our yield function equation it is - 155.67 and obviously the stress state corresponding to the all round pressure state is at dot on the normal stress axis and its within the yield limit. So, our F the yield function value is negative. So, that means that we are in the elastic state and here in the triaxial compression test we have only two stresses. One is the confining pressure.

And that is constant it is actually it is like an axis symmetric case where we have the radial stress sigma 3 and then the vertical stress sigma 1 and we can work in terms of these two stresses that will completely describe the stress state within the triaxial compression test.

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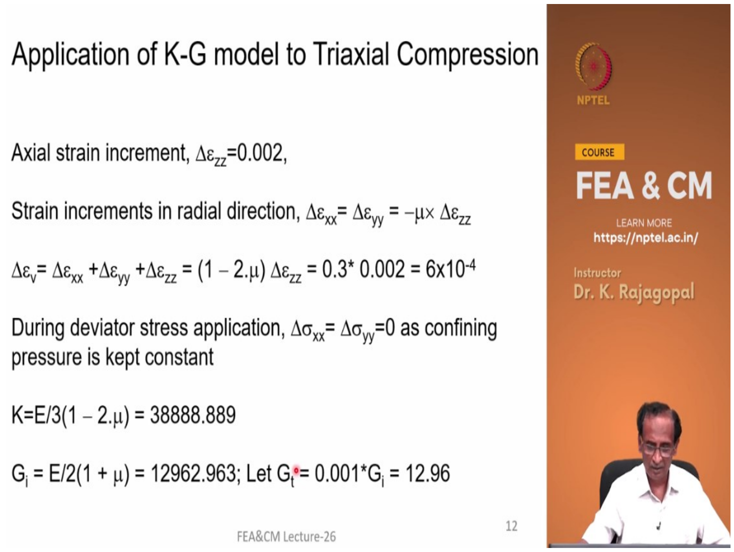

Now we are applying axial strain increment delta epsilon zz as 0.002 and then the strain increments in the radial direction in the x and y directions are - mu times delta epsilon z and our incremental volumetric strain delta epsilon v is delta epsilon xx + delta epsilon yy + delta epsilon zz. Actually I could have also written it as a delta epsilon r in that case delta epsilon v is a two times delta epsilon r + delta epsilon z.

Axial strain increment, $\Delta\varepsilon_{zz}$=0.002,

Strain increments in radial direction, $\Delta\varepsilon_{xx} = \Delta\varepsilon_{yy} = -\mu \times \Delta\varepsilon_{zz}$

$\Delta\varepsilon_v = \Delta\varepsilon_{xx} + \Delta\varepsilon_{yy} + \Delta\varepsilon_{zz} = (1 - 2.\mu)\, \Delta\varepsilon_{zz} = 0.3^* \, 0.002 = 6\text{x}10^{-4}$

Just to make it a bit more clear I am using x and y coordinates but it should be just an odd coordinate, but multiplied with 2. So, delta epsilon v is 1 - 2 mu times delta epsilon z and that is 6 times 10 to the power of - 4 and if delta epsilon z is compression delta epsilon v is also compression because it has the same sign and during the deviator stress application that is in the second phase we are applying axial strain.

During deviator stress application, $\Delta\sigma_{xx} = \Delta\sigma_{yy} = 0$ as confining pressure is kept constant

$K = E/3(1 - 2.\mu) = 38888.889$

$G_i = E/2(1 + \mu) = 12962.963$; Let $G_t = 0.001^* G_i = 12.96$

And our confining pressure remains constant sigma 3 remains constant at 100. So, that means that our delta sigma xx and delta sigma yy are 0 and our initial bulk modulus is E by 3 times

1 - 2 mu that is a 38888.89 and our initial modulus a shear modulus is E by 2 times 1 + mu that is 12,962.96 and let us say that the tangent shear modulus after exceeding the yield limit is 0.001 times G i that is for 12.96.

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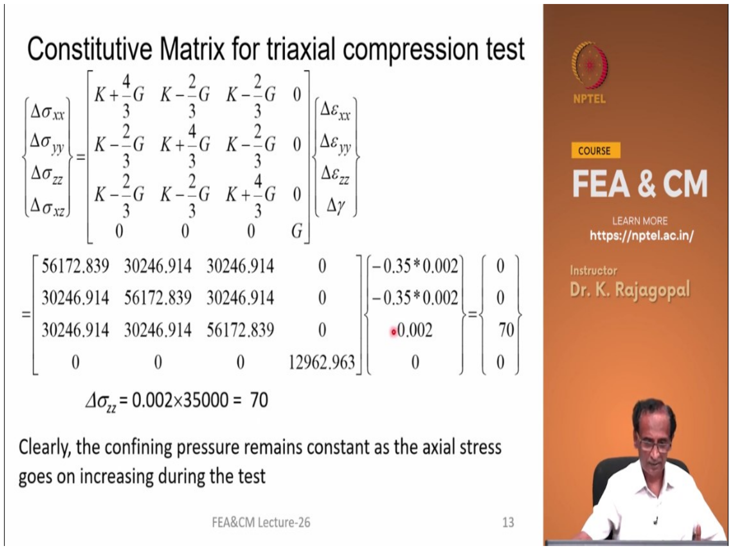

And the constitutive matrix for the triaxial compression test can be written like this delta sigma xx delta sigma yy delta sigma zz delta sigma xz. Actually this is you can also take it as delta sigma r delta sigma theta and then delta tau, but I am just writing it in terms of x and y for better clarity and our K + 4 by 3 G is this the previous step we have calculated both K and G.

Constitutive Matrix for triaxial compression test

$$\begin{Bmatrix} \Delta\sigma_{xx} \\ \Delta\sigma_{yy} \\ \Delta\sigma_{zz} \\ \Delta\sigma_{xz} \end{Bmatrix} = \begin{bmatrix} K+\frac{4}{3}G & K-\frac{2}{3}G & K-\frac{2}{3}G & 0 \\ K-\frac{2}{3}G & K+\frac{4}{3}G & K-\frac{2}{3}G & 0 \\ K-\frac{2}{3}G & K-\frac{2}{3}G & K+\frac{4}{3}G & 0 \\ 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_{xx} \\ \Delta\varepsilon_{yy} \\ \Delta\varepsilon_{zz} \\ \Delta\gamma \end{Bmatrix}$$

$$= \begin{bmatrix} 56172.839 & 30246.914 & 30246.914 & 0 \\ 30246.914 & 56172.839 & 30246.914 & 0 \\ 30246.914 & 30246.914 & 56172.839 & 0 \\ 0 & 0 & 0 & 12962.963 \end{bmatrix} \begin{Bmatrix} -0.35*0.002 \\ -0.35*0.002 \\ 0.002 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 70 \\ 0 \end{Bmatrix}$$

So, if you calculate K + 4 by 3 G is 56,172 K – 2 by 3 G is this and our strain increment is - 0.35 times 0.002 in both x and y directions and then delta epsilon z is 0.002 because if you are compressing in the axial direction there has to be an expansion in the radial direction. So, we have - and if you do this our radial stress increments are coming out as 0 then the axial stress increment is 70.

$$\Delta\sigma_{zz} = 0.002 \times 35000 = 70$$

And actually that delta sigma z is just simply E times that is Young's modulus is given as 35,000 multiplied by delta epsilon z 0.002 that is 70 that shows that all these calculation whatever we have done is correct and so we see that as we are applying the axial compression during the second phase of the test our confining pressure remains constant.

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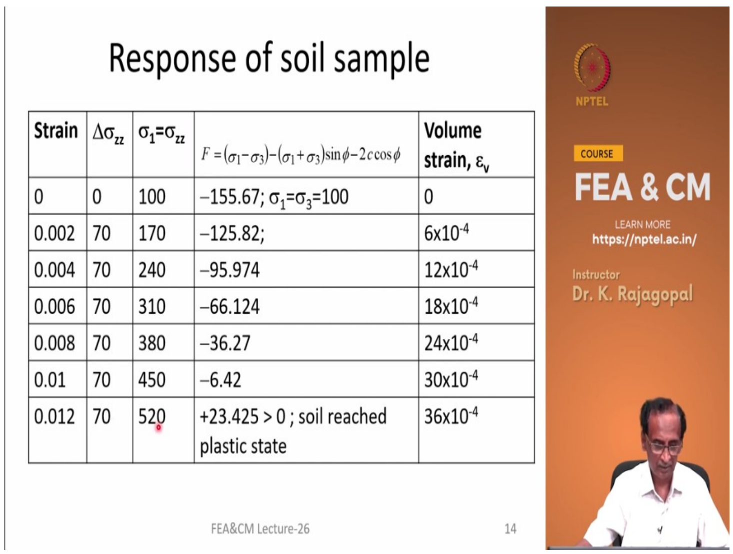

Response of soil sample

Strain	$\Delta\sigma_{zz}$	$\sigma_1=\sigma_{zz}$	$F=(\sigma_1-\sigma_3)-(\sigma_1+\sigma_3)\sin\phi-2c\cos\phi$	Volume strain, ε_v
0	0	100	−155.67; $\sigma_1=\sigma_3=100$	0
0.002	70	170	−125.82;	6×10^{-4}
0.004	70	240	−95.974	12×10^{-4}
0.006	70	310	−66.124	18×10^{-4}
0.008	70	380	−36.27	24×10^{-4}
0.01	70	450	−6.42	30×10^{-4}
0.012	70	520	+23.425 > 0 ; soil reached plastic state	36×10^{-4}

So, now let us do some calculations so initially our sigma 1 and sigma 3 are the same 100 and the F is - 155 and the volume strain is 0 that is the start and let us increase the axial strain to 0.002 and delta sigma z is 70. So, our sigma 1 is 170 sigma 1 is the vertical stress that is sigma z and our F value is - 125 and the volumetric strain increment is 6 times 10 to power of - 4 and then let us increase the axial strain to 0.004.

Response of soil sample

Strain	$\Delta\sigma_{zz}$	$\sigma_1=\sigma_{zz}$	$F=(\sigma_1-\sigma_3)-(\sigma_1+\sigma_3)\sin\phi-2c\cos\phi$	Volume strain, ε_v
0	0	100	–155.67; $\sigma_1=\sigma_3=100$	0
0.002	70	170	–125.82;	6×10^{-4}
0.004	70	240	–95.974	12×10^{-4}
0.006	70	310	–66.124	18×10^{-4}
0.008	70	380	–36.27	24×10^{-4}
0.01	70	450	–6.42	30×10^{-4}
0.012	70	520	+23.425 > 0 ; soil reached plastic state	36×10^{-4}

Once again since we are in the elastic state we use our delta sigma z as E i multiplied by delta epsilon z. So, that is 70 so our sigma 1 is 240 and our yield function is - 95 and our volumetric strain is 12 times 10 to the power - 4 and then we continue and you see at 0.01 our sigma 1 has reached 450 and our yield function value is - 6.42 and the volumetric strain is a 30 times 10 to the power – 4.

And since our yield function value is negative when we apply the next strain increment we get stress increment of the same thing 70. So, sigma 1 is 520 and now we see that when we calculate the yield function value it has become positive this is + 23.425 which is greater than 0. So, that means that the soil has reached the plastic limit and now we have to be alert and see what we should do so that any further increase in the axial strain will not increase the shear stresses.

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Behaviour after plastic state

After reaching plastic state, G_t is set to 12.96

Tangent Poisson's ratio

= (3*38888.888-2*12.96)/(6*38888.888+2*12.96) = 0.499

Incremental volume strain = $(1 - 2.\mu).\Delta\varepsilon_{zz} = 2x10^{-3} \times \Delta\varepsilon_{zz} = 4x10^{-6}$

Increase in axial stress =0.07 with further increments in axial strain

Vertical Stress and volumetric strain changes after plasticity are as follows:

Strain	$\Delta\sigma_{zz}$	$\sigma_1=\sigma_{zz}$	Yield function value, F	Volume strain, ε_v
0.012	70	520	+23.425 > 0 ; soil reached plastic state	$36x10^{-4}$
0.014	0.07	520.07	+23.454	$36.06x10^{-4}$
0.016	0.07	520.14	+23.483	$36.12x10^{-4}$
0.018	0.07	520.21	+23.512	$36.18x10^{-4}$
0.020	0.07	520.28	+23.541	$36.24x10^{-4}$

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So, at this stage what we do is we can reset the shear modulus to a small value 0.001 times G i that is 12.96 and then our tangent Poisson's ratio is 3 K – 2 G by 6 K + 2 G that comes to 0.499 and our incremental volumetric strain is 4 times 10 to the power - 6 then increase in the axial stress is 0. 07 which is a very small compared to the previous value of 70. So, the vertical stress.

Behaviour after plastic state

After reaching plastic state, G_t is set to 12.96

Tangent Poisson's ratio

= (3*38888.888-2*12.96)/(6*38888.888+2*12.96) = 0.499

Incremental volume strain = $(1 - 2.\mu).\Delta\varepsilon_{zz} = 2x10^{-3} \times \Delta\varepsilon_{zz} = 4x10^{-6}$

Increase in axial stress =0.07 with further increments in axial strain

And the volumetric strain changes after the plasticity is; see previously we were here at a strain of 0.012 and our sigma 1 was 520 and the yield function value is a 23.425 and our volumetric strain was 36 times 10 to the power of - 4 and now let us increase the strain to 0.014 that is we applied further strain increment of 0.002 and now our delta sigma z increases by only 0.07 because our shear modulus is reduced to a small value of 12.96.

Strain	$\Delta\sigma_{zz}$	$\sigma_1=\sigma_{zz}$	Yield function value, F	Volume strain, ε_v
0.012	70	520	+23.425 > 0 ; soil reached plastic state	$36x10^{-4}$
0.014	0.07	520.07	+23.454	$36.06x10^{-4}$
0.016	0.07	520.14	+23.483	$36.12x10^{-4}$
0.018	0.07	520.21	+23.512	$36.18x10^{-4}$
0.020	0.07	520.28	+23.541	$36.24x10^{-4}$

Previously it was 12,960 now it is 12.96. So, our delta sigma z is only 0.07 so our sigma 1 is 520.07 and then our volumetric strain increment now is only 4 times 10 to the power of – 6. So, this is our volumetric strain and if you increase the strain by another 0.002 your strain increases to 0.016. So, our sigma 1 is more or less remaining constant at 520 and our volumetric strain is also remaining constant more or less constant.

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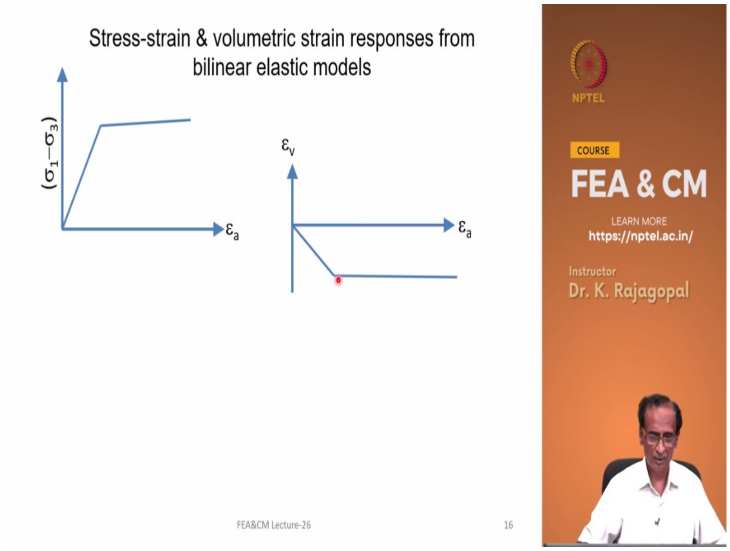

So, if you plot the stress-strain graphs with increasing axial strain initially your deviator stress will go on increasing, but after the yield limit it remains more or less constant that is what we have seen here. Once it has reached at 520 it has remained more or less constant, it is only increasing by a small value and similarly after you reach the plastic limit your volume has more or less remained constant because our Poisson's ratio now is a very close to 0.5 that shows that the soil is incompressible.

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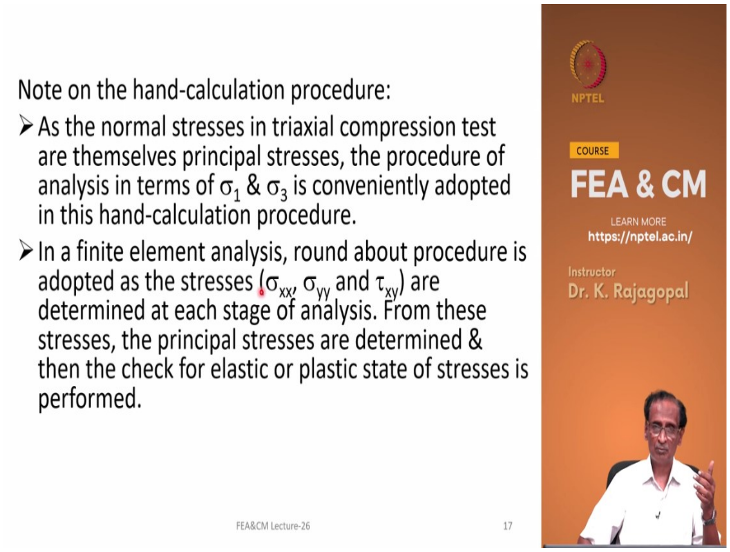

And just a note so as our normal stresses in the triaxial compression test they are both principal stresses sigma 1 and sigma 3. So, that we can directly apply our Mohr Coulomb relation and just for the purpose of hand calculation that was simple to illustrate that beyond our yield limit when our yield function value is greater than 0 we set the shear modulus to a small value and then we see that any further increase in the shear stresses does not happen.

But in a real finite element analysis is actually it is a very cumbersome process because we solve first for the displacement strains, stresses and then the stresses are the Cartesian stresses sigma xx, sigma yy and tau xy and then from these we need to estimate the principles as sigma 1 and sigma 3 and then check for the yield limit in terms of our sigma 1 and sigma 3 c and phi.

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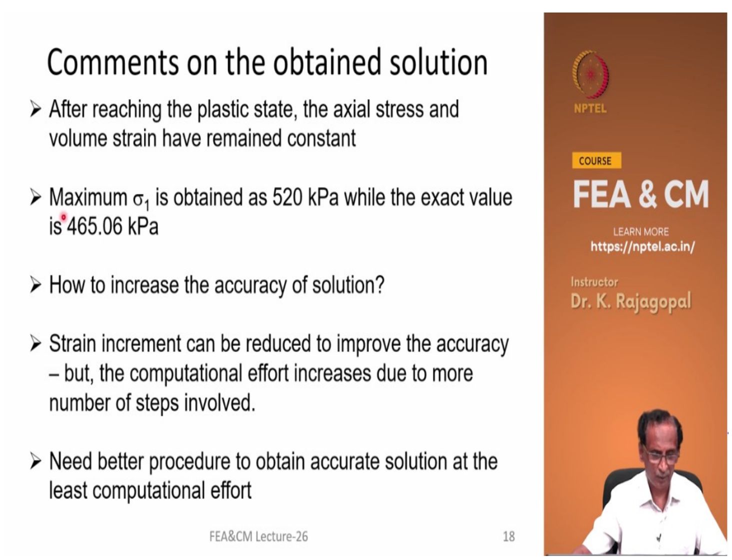

So, actually after reaching the plastic state the axial stress and the volumetric strain have remained more or less constant. So that way we should be happy that we are able to simulate some limit on the shear stress. It is not increasing without any bound and then our volume is remaining constant after the failure, but then in this case our sigma 1 obtained is 520 while the exact value is 465.

There is a huge difference. In fact in terms of percentage it might work out about 20 percent and if this happens at a small element level if you apply it to any bearing capacity problem or something your bearing capacity estimate could be totally different or if you apply it to any slip circle analysis we will see that our factor of safety is very, very high. Instead of getting 1.3 you might end up getting 1.7, 1.8 that is because of our wrong estimation of our failure stresses.

So, how do we increase the accuracy? See the strain increment is something that we can control. So, instead of applying the strain increment in terms of 0.002 let us say we apply this in 0.001 your solution accuracy might increase a little bit that we will see through the excel program, but then if you reduce this strain increment the number of computations is going to increase and the solution time will be much longer.

So, that also we have to keep in mind and so we need a better procedure to obtain more accurate result like can we directly estimate our 465 by doing something else that we will see later.

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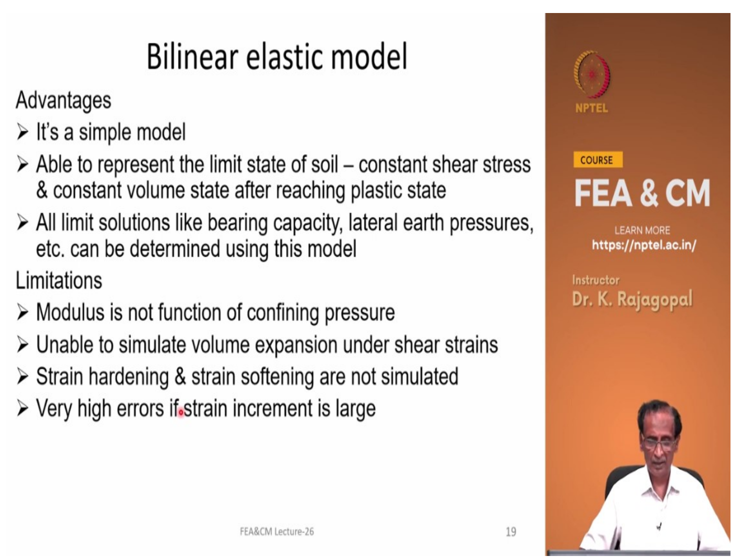

And the advantage of this bilinear elastic model it is a very simple model and we are able to utilize the shear strength parameter c and phi and then put a limit on the stresses that we generate and it is able to represent the limit state of the soil and after reaching the plastic limit our shear stresses have more or less remained constant and then we are able to represent the constant volume state after reaching the plastic state.

So, that is one of our observations from our laboratory test. At critical state our shear stress remains constant and then volume also remains constant. So, that we are able to simulate by using a simple bilinear elastic model and all the limit solutions like our bearing capacity or the lateral earth pressures or the factor of safety of the slope on against the slip circle failure we can simulate by using the simple bilinear elastic model.

But the limitation is the modulus is not a function of confining pressure and then not only that as the stress is increasing your modulus should go on decreasing, but here we have only a bilinear elastics. So, our initial modulus remains constant up to shear failure and then after that once again it remains constant at a very small value, but in our stress strain test we see that the stress and strain are curvilinear graphs not straight lines.

And another limitation of this bilinear elastic model is we are not able to simulate volume expansion under the shear strains and the strain hardening and strain softening are also not simulated. We need to go in for better models and if our strain increment is very large then we could have very high errors. So, instead of using a strain increment of 0.002 let us say we used 0.1 or 0.005 what would have happened that we will see through the excel program.

And our address could be, very, very large depending on the strain increment that we use. So, let me just illustrate some applications for the bilinear elastic model.

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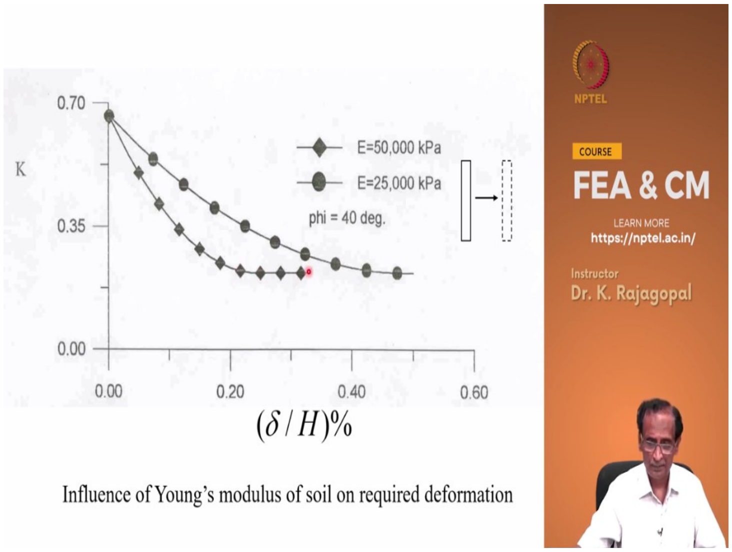

And here is an application for predicting the lateral pressure coefficient under active deformations. So, we have a retaining wall, we have subjected to lateral translation and we know that our lateral pressures will go on reducing and the y axis we have the K that is the net lateral force divided by 1 half gamma h square. The net lateral force is so we are applying equal horizontal displacement to all the nodes on the retaining wall.

And when you apply any displacement the programs they calculate a reaction force from our integral B transpose sigma we calculate reaction force and we can sum up all those reaction forces at these nodes and then that divided by 1 half gamma h square will give you the K and so for Young's modulus of 25,000 you need more deformation whereas with Young's modulus of 50,000 you need a lesser deformation and our phi is 40 degrees.

So, our K active is about 0.217 which is approximately represented by both these graphs and with the modulus of 50 000 we reach the active state faster and the percentage deformation that you need is about 0.2 percent, but when your Young's modulus is only 25000 you need almost 0.4 percent.

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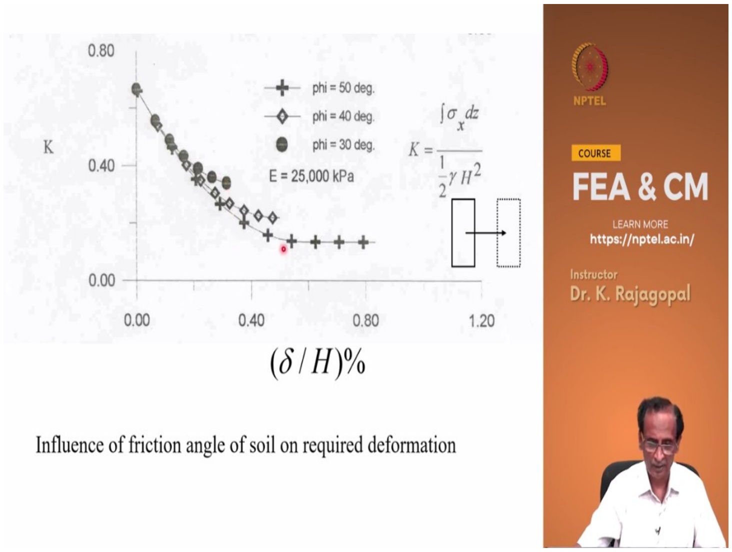

And this is with three different friction angles 30 degrees, 40 degrees, 50 degrees and with 30 degrees your K is the one third and when it is 40 degrees it is 0.217 and with 50 degrees it is very small and you see that with the 30 degrees you need a lesser deformation compared to 50 degrees to reach the active state and the Young's modulus is the is the same for all the three cases just to illustrate what is the effect of friction angle on the lateral earth pressures this example was worked out.

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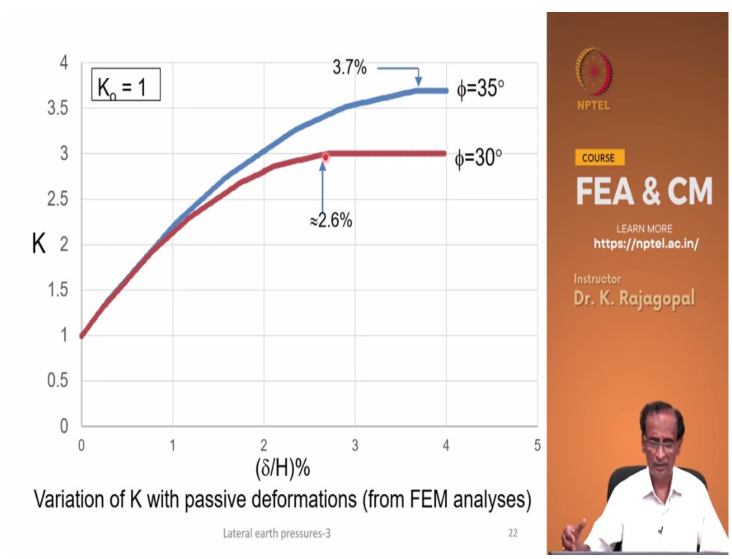

And let us look at the passive state. So, this is on the x axis we have the passive deformation that is the wall deformations into the soil and the K 0 is 1 and as the wall is pushed into the soil your lateral pressure will go on increasing and for 30 degrees your K P is 3 and the lateral pressure coefficient will go on increasing, increasing and at lateral strain of 2.6 percent the soil has reached the plastic limit of K P of 3.

And when friction angle is 35 degrees it required 3.7 percent. So, you compare this 2.6 and 3.7 to the deformation that you require in the active state. See for 30 degrees your required amount of deformation is only about 0.35 percent this is for a E of 25,000 and for 40 degrees we need about maybe 0.4, 0.5 or something, but in the case of passive you require much larger deformations 2.6 and the 3.75.

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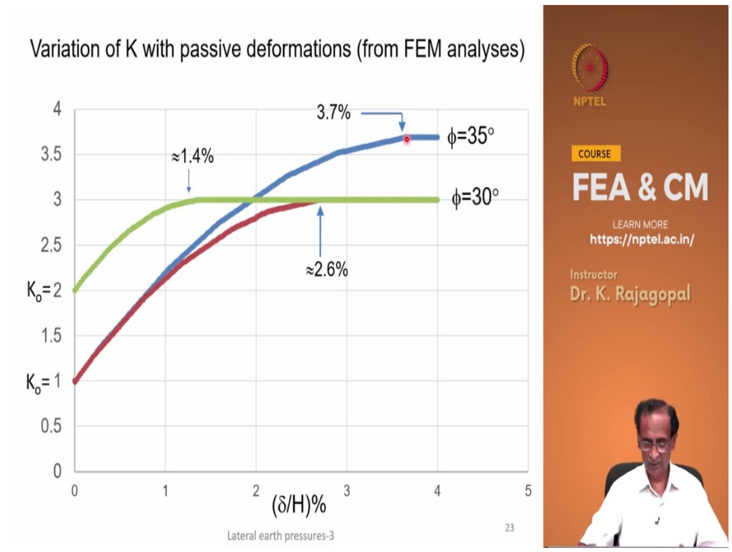

And this is the result with two different K 0 values K 0 of 1, K 0 of 2. See when the K 0 is higher you require smaller deformation because the K P is 3 and K 0 is 2 and so that means that as you are pushing the soil it can reach faster to reach a K of 3 as compared to reaching a K of 3 from a K 0 of 1. So, here this both red and green lines are corresponding to a friction angle of 30 degrees.

The K P is 3 when your initial K 0 is 1 you require a deformation of almost 2.6 percent, but when your K 0 is 2 you require only 1.4 percent. You see the drastic reduction that is because your initial stress state itself is closer to the passive pressure state and when friction angle is 35 degrees and the K of 1 you require about 3.7 percent of the wall deformation to reach the passive state.

So, this the bilinear elastic model is a very simple one and this is one of the first constitutive models that we have worked at and it is easy to implement because all we are doing is controlling the shear modulus values and the shear modulus initially is very high, but then after reaching the limit state the G is set to a small value so that your further increase of the shear stresses is more or less 0.

So, with this simplistic model we can predict our bearing capacity solutions or any limit solution like your lateral earth pressures behind the retaining walls or the slip circle analysis of slopes and prediction of the factor of safety. So, now let me show you one excel spreadsheet program. **(Video Starts: 38:45).** And we can do some interesting calculations. This is a bi-linear elastic model in terms of shear modulus and bulk modulus.

And our initial sorry the Young's modulus is 35,000 and the Poisson’s ratio is a 0.35 and this particular one is for simulating the triaxial compression test and let us take the same values as we have done in the class. Let us take a c of 25 the friction angle of 35 degrees and the sigma 3 is 100 and the maximum axial stress is 465 and then let us apply axial strain in increments of 0.002 and the reduction factor for the shear modulus is a 0.001.

So, the tangent shear modulus is 12.963 and then the tangent shear modulus is 0.49 and our calculations are like this. So, the axial strain of 0.002 the incremental radial strain is a – 0.0007 and then the volumetric strain is 2 times the radial strain + epsilon z that is 6 times 10 to the power of - 4 and the increment in the axial stress is 70 and this is our yield function value - 125 and our axial stress will go on increasing from 100 to 170, 240, 310, 380, 450, 520.

At this stage we see that our yield function value is positive. So, the further calculations should use our tangent shear modulus instead of the initial shear modulus. So, that is done our dsz that is the axial stress increment is only 0.07 and our incremental volumetric strain is also very small and so you see here our volumetric strain is more or less remaining constant and our axial stress is remaining very close to 520.

So, with increase in the axial strain it will continue at the same value. Now let us see what happens see if I apply this in faster increments 0.005 and see our predicted yield stress is 625. So, it is about 625 whereas our maximum stress is 465. So, we see that the accuracy of the solution depends very much on the axial strain increment. Let us now apply this in very small increment 0.001.

And now we see that our predicted stress is 485 which is better than 520, but then our number of load steps has increased. So, that means that we have to spend more time and more computational effort to improvise our solution. So, we can further increase this let us say 0.0005. So, our incremental stress is only 17.5. So, our yield stress is 467.5 which is very close to 465.

But then the penalty that we are paying is in terms of the number of steps of analysis. So, our stress strain graph is like this. So, our yield stresses are very close to 467 and then the volumetric strain is also very good like after initial volumetric compression the volume will remain more or less constant during the critical state or in the plastic state. So, at least these two aspects we are able to satisfy that after failure the shear stress should remain constant.

And then after failure your volumetric strains should remain constant. So, the bilinear elastic model is able to represent some aspects of our stress-strain relations that we have developed that beyond the plastic limit the stress will remain constant and then the volume will remain constant and since we are applying Mohr Coulomb relation we should be able to predict very accurately all the limit bearing capacities or the lateral earth pressures or our factor of safety against the slope failure in terms of the slip circle and so on.

(Video Ends: 44:45) So, that is a brief introduction to our bilinear elastic model and if you have any questions please send an email to this address profkrg@gmail.com and then i will respond back to you. So, thank you very much. We will meet next time.