Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 37
Nonassociated Elastic- Plastic Joint Element

Hello all, I welcome you all for the next lecture on FEA and constitutive modelling in geomechanics. So, in the previous lecture we have discussed on the importance of constitutive models and how it forms a key a point in the numerical framework and what are how to model these things using the experimental studies and also the mathematical framework behind using these constitutive models.

So, we have explained about the basic tenets of plasticity and how it will be useful in constituting the mathematical models.

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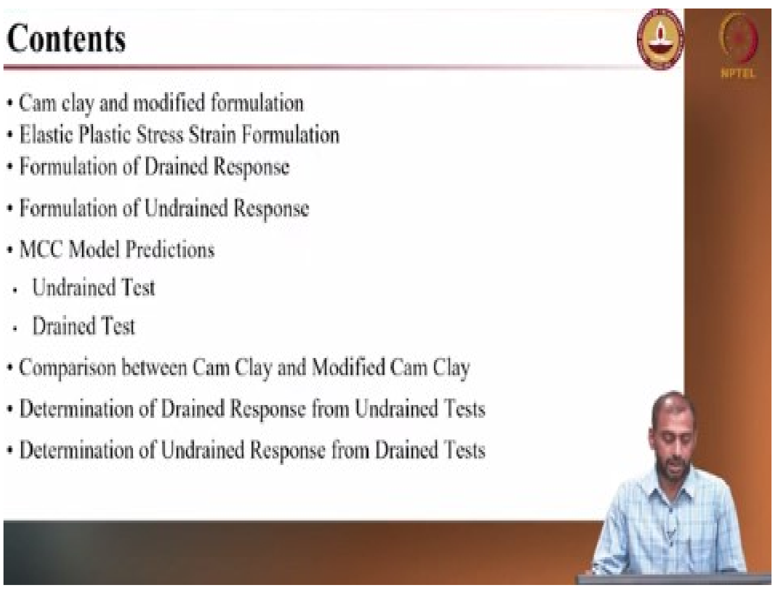

So, in this lecture I will be giving a detailed description on the cam clay and the modified cam clay models and its formulation a brief description on the formulation and also, we will discuss more on the advantages and limitation of this original cam clay and modified cam clay models followed by we will see the drained and undrained response predictions. So, using these models how we use these models and predict the stress strain response.

So, we will see how the models predict the volume change, work pressure and the stress strain response and comparison between the modified cam clear and original cam clay will also be done. Finally, if you have a drained response how to predict the undrained response and if you have an undrained response how to predict the drained response using these cam clay based models will be detailed.

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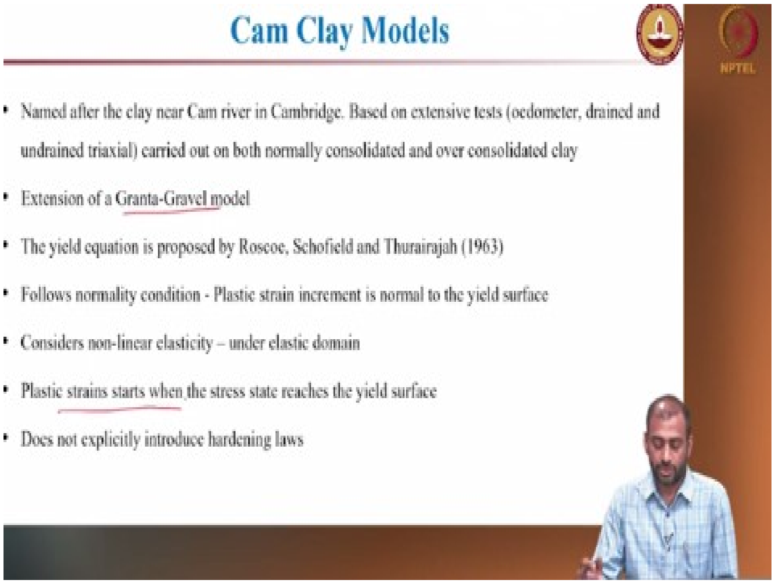

So, first we will start with what are cam clay models. So, again it is based on the cam clay, the name is given based on a river in Cambridge and so the clay near the river the Cam river. So, the clay from that river is used to perform a lot of test conventional tests such as odometer test, drained and undrained triaxial test. Under both normally consolidated state and over consolidated state. So, based on this extensive experimental study on this particular clay they try to come up with this cam clay model.

So, the cam clay model is an extension of Granta Gravel model. As you can see here it is a Granta Gravel model. So, this Granta Gravel model also was developed in Cambridge, Granta was again a upstream part of this particular river and so they developed this model and later it was actually modified into a original cam clay model. So, Roscoe, Schofield and Thurairajah in 1963 they have actually proposed an yield equation based on a internal work the energy principles they have proposed this yield equation.

We will see the yield equation in the next slide. But this yield equation forms the basis of the cam claim model and this model follows a normality condition that is which is nothing but as we discussed in the previous lecture the plastic strain increment is always normal to the yield surface. And it considers the non-linear elasticity under elastic domain and the plastic strain starts when the stress state reaches the yield surface.

So, it does not explicitly uses an hardening law but in the yield equation it considers the pre-consolidation pressure which controls the hardening aspects of the material response. So, this is about the introduction to a cam clay model.

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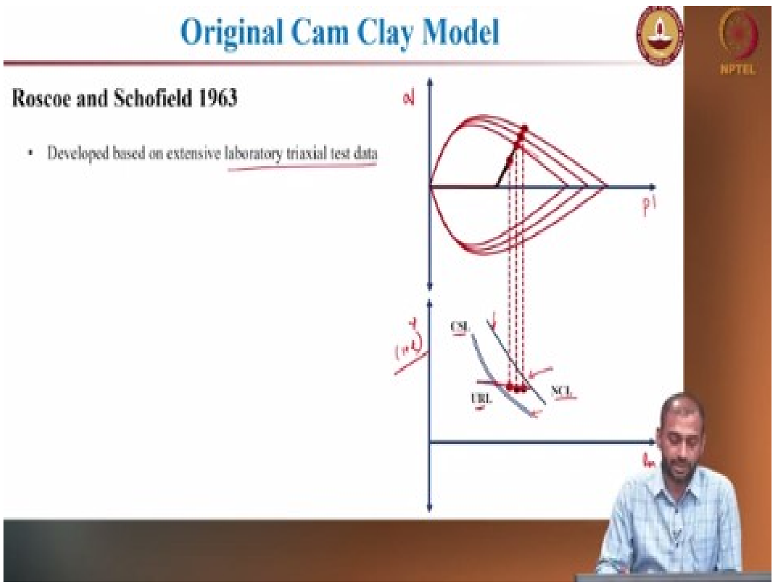

We will start discussing about the original cam clay model. So, as you see here this is between p prime q space and this is l and p prime versus specific volume which is nothing but 1 + e. So, this is your normally consolidated line and this is your critical straight line. So, this is your NCL and this is your CSL and this is your unload, reload. So, as you isotropically consolidate and then start sharing this is your volume or the void ratio when you start shearing.

But as you start sharing further you reach the yield locus. So, this is the yield locus? Why this yield locus is in this particular shape we will discuss later. But this as you go as you start sharing further you will reach this particular yield surface and this yield surface as I said starts expanding. So, once it starts expanding the volume or the void ratio also decreases and reaches the critical state.

So, this is a typical response that you observe and this has been observed using an extensive triaxial test data by Roscoe and Schofield.

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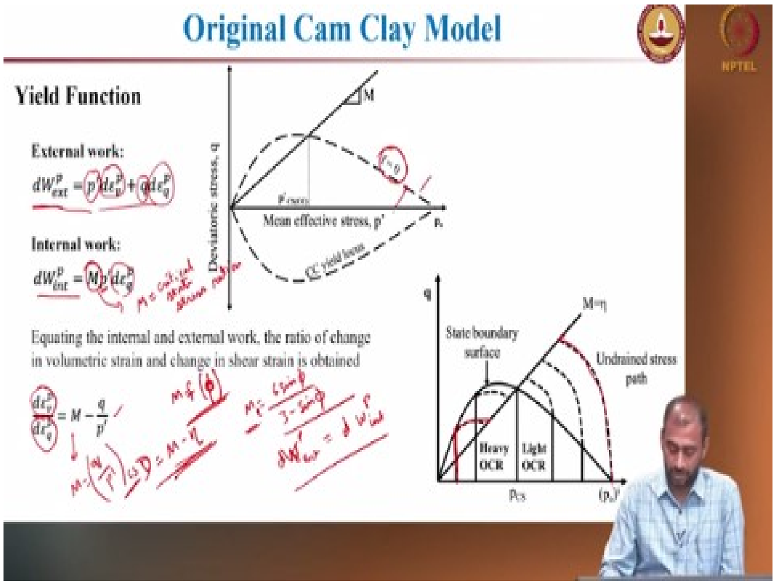

Based on these results they proposed the yield function. So, as I said this is based on energy principles. So, what they did is they tried to write the external work. So, this is external work is nothing but the mean effective stress that you are applying and the corresponding plastic volumetric strain increment and the deviated stress you are applying in the corresponding D matrix strain increment.

External work:

$$dW_{ext}^{p} = p' d\varepsilon_{v}^{p} + q d\varepsilon_{q}^{p}$$

Internal work:

$$dW_{int}^{p} = Mp' d\varepsilon_{q}^{p}$$

So, if you sum it up you will get the external work done see this is what you are applying on to the specimen. What is the internal work that is getting generated when you are applying this external work? So, you have this material property here. This is M is nothing but the critical state stress ratio and M is equal to is a function of friction angle. So, M = 6 sin phi 3 - sin phi. So, this is under triaxial compression so you have this M c value.

So, this is based on the internal work that is because of the friction between the particle. So, this is getting generated when you apply a confining pressure on the shear strain that is happening. So, you have an equation for internal work you have an equation for an external work. When you equate this internal work is equal to d epsilon external sorry d w external = d w internal and then you sum simplify this equation you will finally come up with the ratio of plastic volumetric strain and plastic shear string.

So, as we discussed in the previous lecture the ratio of plastic volumetric strain increment to plastic shear strain increment is nothing but dilatancy. So, from equating this energy or internal work and external work you will get the dilatancy which is nothing but M - eta. This is M z; M is nothing but M = q by p prime at critical state but this is at any stress ratio. So, we get this dilatancy relation from original cam clay model and you can see this is a linear relationship M - eta.

$$\frac{d\varepsilon_v^p}{d\varepsilon_q^p} = M - \frac{q}{p'}$$

So, once you get this so this is just a representation of the state boundary locus. So, when you are normally consolidating your stress path is like this to reach the critical state when you have a highly over consolidated soil it vertically increases and then the failure locus contracts to reach this critical state. So, this is your failure condition f = 0 which is when the stress state is on the failure envelope.

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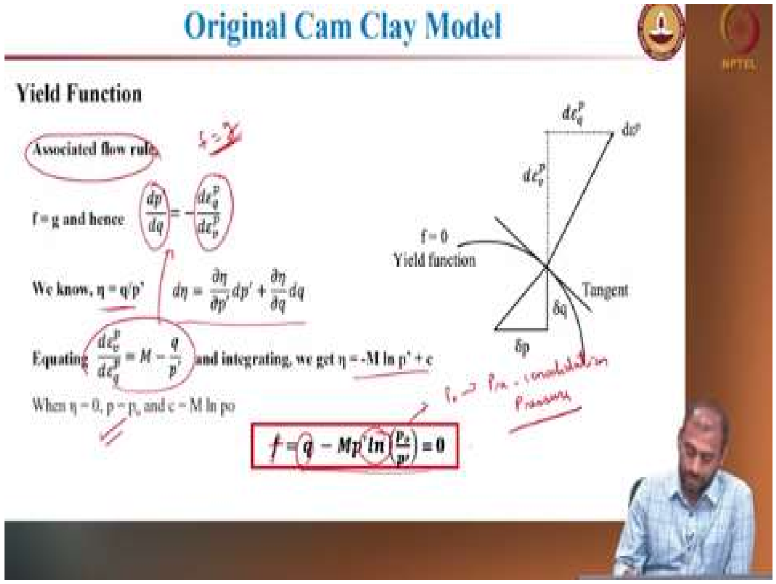
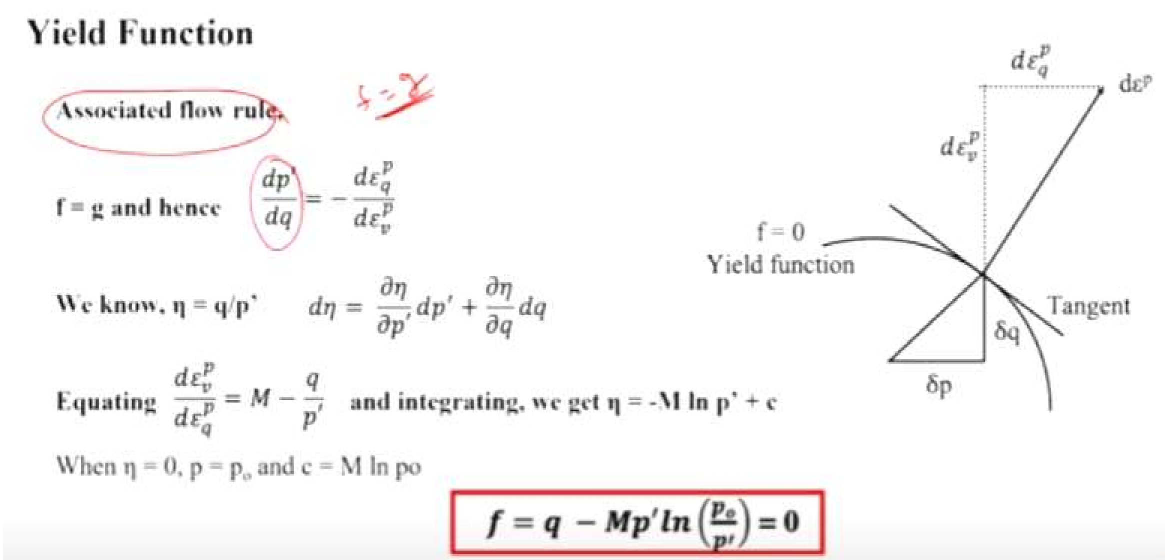


Once you know this then you again as I said the original cam clay and modified cam clay uses associated flow rule where f = g. So, you know that and you can relate stress and strain using this relation. We also know eta is nothing but q by p prime we differentiate this we also equate the dilatancy relation into this equation we integrate and finally we try to get the failure equation. The failure condition is nothing but q - M p ln p 0 by p for this particular boundary condition you can get this failure equation.



This is a function of p 0 this the pre consolidation pressure so you have this particular p 0. So, you have this particular failure equation. So, since it has a logarithmic term in it the shape is like a log spiral or not a log spiral, I think the shape is like a leaf-like shape which is a logarithmic shape of the failure locus. So, this having a logarithmic term into your failure equation actually helps in predicting certain response better.

But actually, we will see the limitation because of this logarithmic term in this particular failure equation in the later slides but you will get this failure equation.

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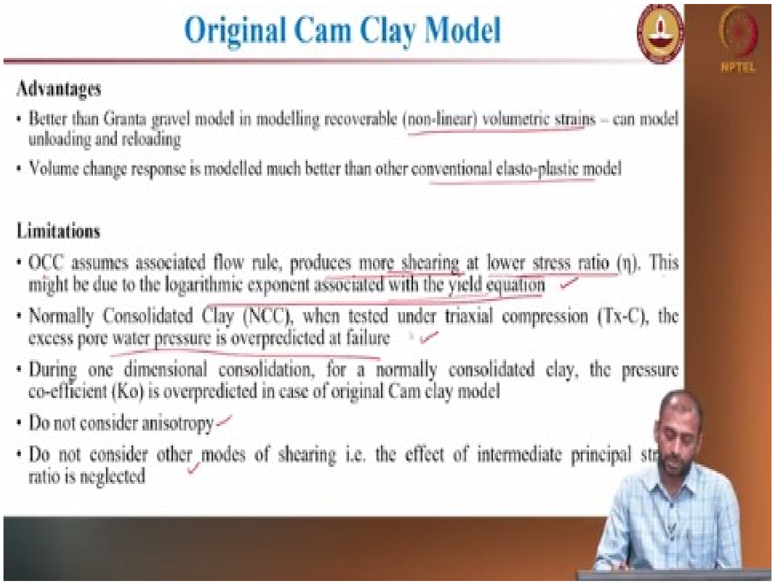

So, because of this I think the shearing response is different. So, what actually the problems since it is having a logarithmic term. So, this might be due to the logarithmic exponent associated with the yield equation. So, it produces more shear stress at lower stress ratio. So, since it produces more shear stress at lower stress ratio it is sort of over predicts then so that is avoided using when you start using a modified cam clay model. So, these are some of the limitations.

So, OCC assumes associated flow rule and because of the logarithmic exponent present in the yield equation, it produces more shearing in lower stress ratio. Again, for a normally consolidated soil the pore pressure is over predicted. So, the pore pressure is over predicted, the shear stress is less in case of original cam clay model. So, again we do not predict the we do not take into account the effect of anisotropy other more modes of failure or other modes of sharing and the effect of intermediate principle stress ratio is not considered.

So, these are the issues with the original cam clay model. But of course, it is better than a Granta Gravel model in predicting the non-linear volumetric strains and the volume change that response is much better than the conventional elastoplastic models. When you use an elastic perfectly plastic models the volume change response is not well predicted because especially the dilation is not well predicted.

Whereas in case of original cam clay model it was able to predict the dilation. However, the stress ratio and so at lower stress ratio the it produces more shearing and also it over predicts the pore pressures. So, these are the issues with original cam clay model.

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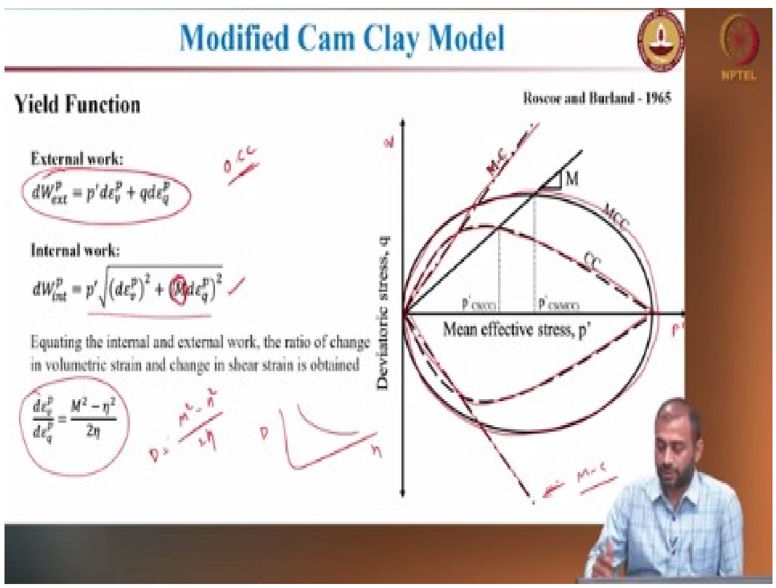
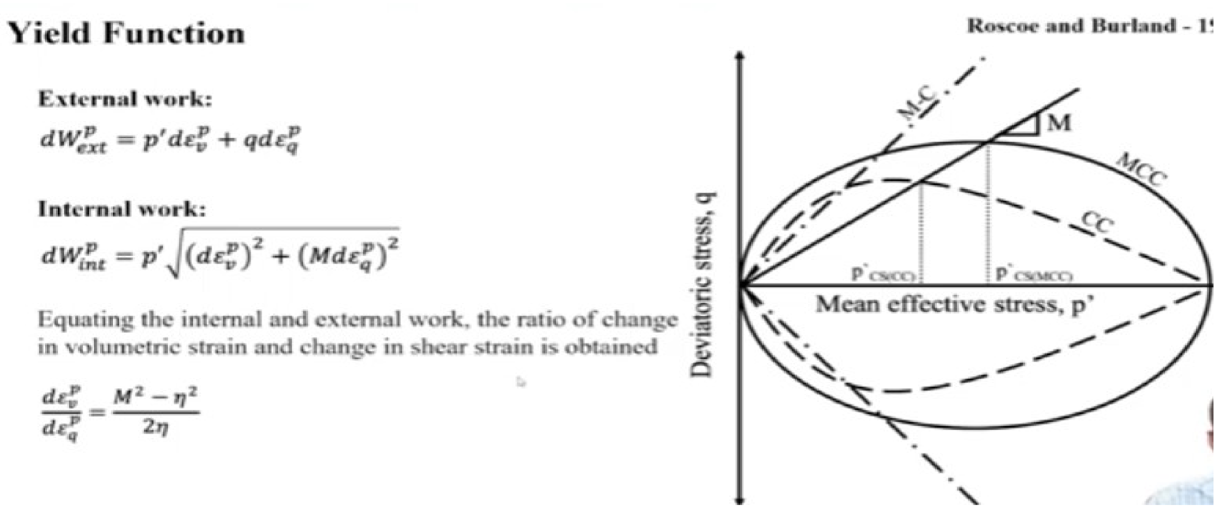


To avoid this in 1965 Roscoe and Burland came up with a modified version of this cam clay model. So, what they did? They use the same external work similar to the original cam clay but they modified the internal work. So, still they use the internal work M so the material property M in addition to that they slightly modified the internal work equation. Modifying this resulted in a dilatancy equation which is not linear.

So, as you can see here it is M square - eta square by 2 eta. So, it will produce a non-linear curve non-linear relationship between stress dilatancy D and eta you will get a non-linear curve. So, this is slightly better so this equation also represents so again if you follow the same procedure like what you did for original cam clay you get a failure equation which looks like a equation of an ellipse. So, the failure locus is somewhat elliptical in shape in p prime q space.



So, this is your original cam clay model failure locus and this is the failure locus of here modified cam clay and I have also put for your reference this is the value locus that you obtained for Mohr Coulomb model so it keeps expanding. So, this is the failure locus you generally observe in case of a modified cam clay model using this particular failure equation.

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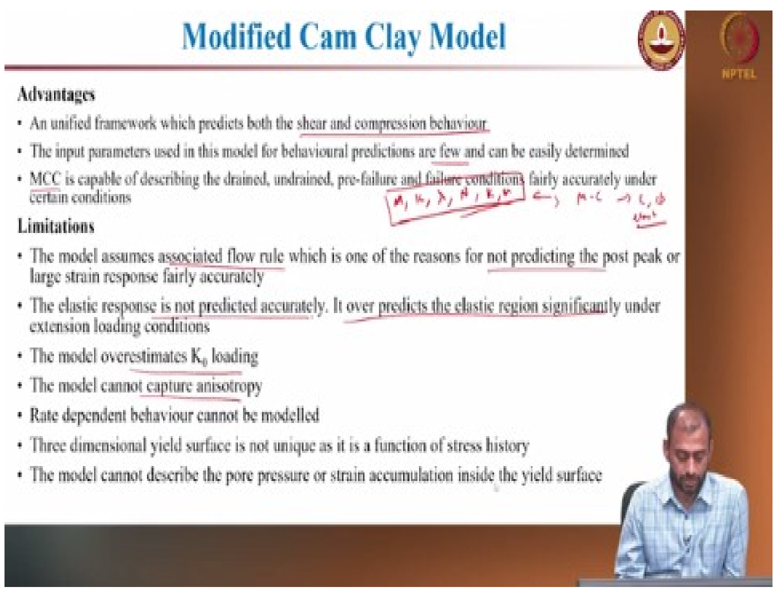
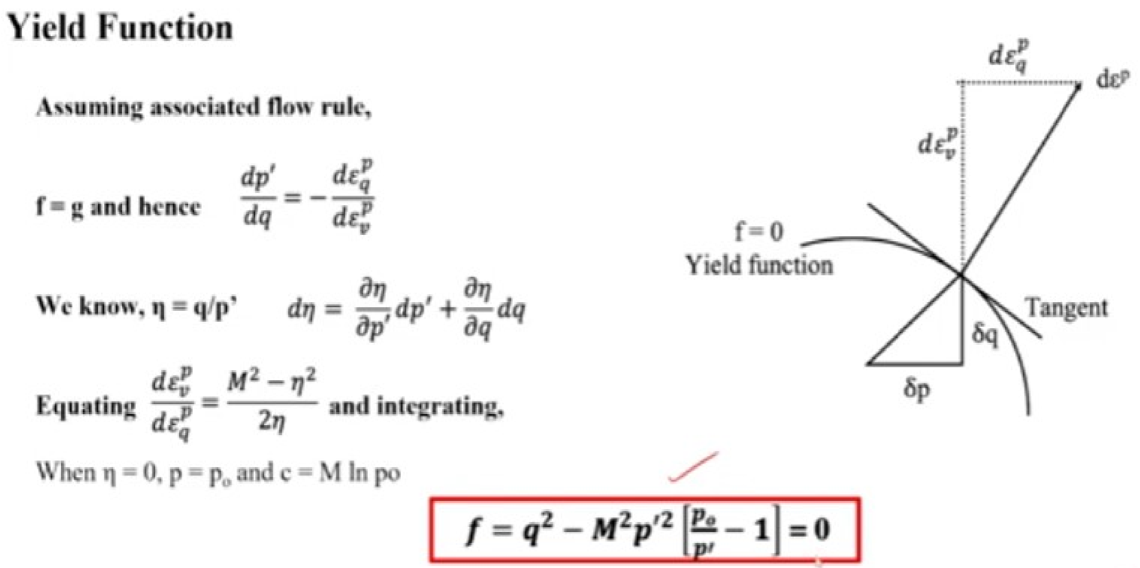


So, again as I said this has an advantage. So, it is a unified framework which predicts both the shear and volume change behaviour fairly accurately. And the input parameters are few so you can actually perform some triaxial compression test and obtain the model parameters, triaxial compression and odometer test. You can obtain the model parameters I am just saying few compared to other complex models out there which has extensive parameters.



For example, some models will have 15 or 21 parameters. So, however this MCC model has a few number of parameters which we can obtain experimentally. You have M, you have kappa, you have lambda you have n and you also need to know the elastic parameters K and mu. If you know these parameters you can actually model the material response you need to determine these number of parameters.

Of course, this is higher than your traditional Mohr Coulomb model where you need to know C phi and elastic parameters. So, you need to have fewer number of parameters in modular model but that does not predict your material response better. So, that is why you are going with a modified cam clay model. Also, this has some limitations. For example, it overestimates when you are performing and K 0 loading and the elastic response is not predicted accurately.

So, we will see in the later slides that the elastic response is over predicted. So, you have so that is not predicted accurately, it over predicts the elastic region significantly especially under extension conditions. Again, we are using an associated flow rule and that is the one of the reasons for not predicting the post peak response fairly accurately. Like original cam clay this also does not capture the anisotropy and the red dependent behaviour cannot be modelled.

So, the three dimensional yield surface is also not unique and this is a function of stress history. These are some of the limitations. So, we will see how to model this.

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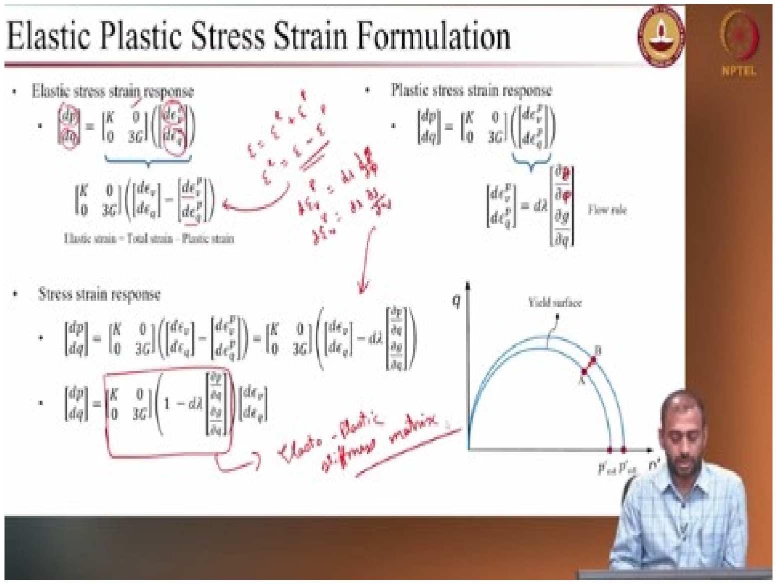
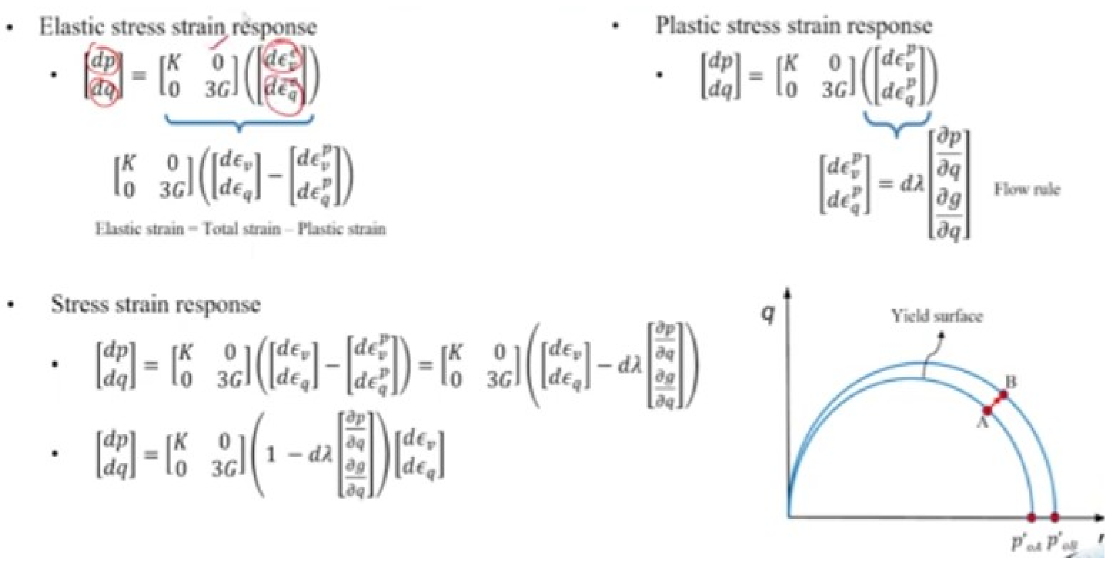


So, this is your basic yield function. Using this yield equation and the plasticity framework which we discussed how do we model the material response. We will see with some examples we will see how using this model we can predict both trained and undrained response. So, as we know this is an elastic stiffness matrix. So, the volumetric part the mean effective stress and the deviatoric stress are related to the volumetric strain and shear strain using the stiffness matrix.



So, we all know total strain is equal to elastic strain plus plastic strain. So, if you want to know the elastic strain is equal to total strain minus plastic strain. So, if you just write this in this equation and we have the relation for the plastic strain. Plastic strain d epsilon v p is nothing but d lambda into dp by dq sorry I think this is there is a error here. So, this is dg by dp. Similarly, d epsilon qp = d lambda into f by dot q.

So, if you have this you just try to put this in this equation and try to form this elastoplastic stiffness matrix. So, you need to determine this elastoplastic stiffness matrix.

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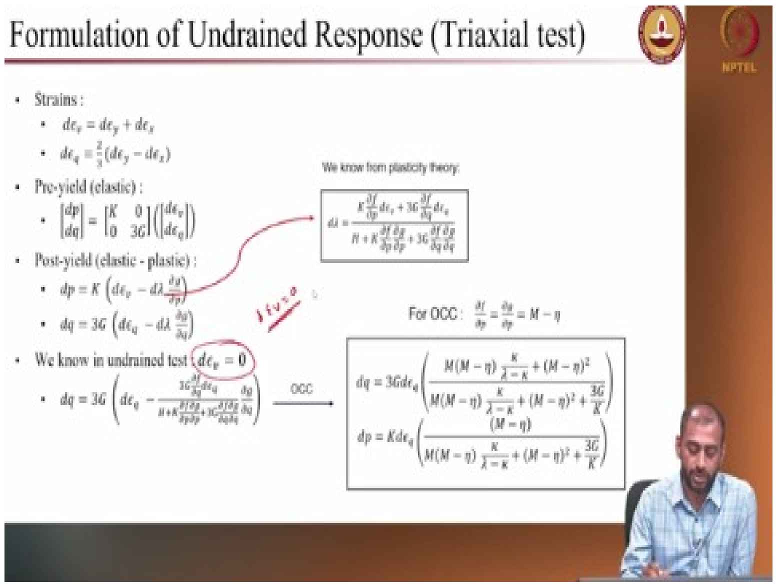

So, for that again as I said we know the volumetric strain, we know the deviatoric strain, pre-yield you have only the elastic stress strained relations. So, this K and g is just constant but you can make it as a function of sigma 3. We can use pressure dependent bulk modulus and shear modulus and post yield you can also you will also have you will follow the elastic plastic stress strain relations.

- Strains :
  - $d\epsilon_v = d\epsilon_y + d\epsilon_x$
  - $d\epsilon_q = \frac{2}{3}(d\epsilon_y - d\epsilon_x)$
- Pre-yield (elastic zone) :
  - $\begin{bmatrix} dp \\ dq \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \left(\begin{bmatrix} d\epsilon_v \\ d\epsilon_q \end{bmatrix} \right)$
- Post-yield (elastic – plastic zone) :
  - $dp = K \left(d\epsilon_v - d\lambda \frac{\partial g}{\partial p} \right)$
  - $dq = 3G \left(d\epsilon_q - d\lambda \frac{\partial g}{\partial q} \right)$
  - $\eta = \frac{dq}{dp} = \frac{3G}{K} \left(\frac{d\epsilon_q - d\lambda \frac{\partial g}{\partial q}}{d\epsilon_v - d\lambda \frac{\partial g}{\partial p}} \right)$
- We know in drained test: $\eta = \frac{dp}{dq} = 3$
  - $d\epsilon_v = \frac{G}{K}\left(d\epsilon_q - d\lambda \frac{\partial g}{\partial q} \right) + d\lambda \frac{\partial g}{\partial p}$

We know from plasticity theory:

$$d\lambda = \frac{K \frac{\partial f}{\partial p} d\epsilon_v + 3G \frac{\partial f}{\partial q} d\epsilon_q}{H + K \frac{\partial f}{\partial p}\frac{\partial g}{\partial p} + 3G \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}}$$

For OCC : $\frac{\partial f}{\partial p} = \frac{\partial g}{\partial p} = M - \eta$

OCC →

$$\left(\frac{d\epsilon_q}{d\epsilon_p} \right) = \frac{\eta (M - \eta)\kappa + \frac{3G}{K}(\lambda - \kappa)}{\frac{3G}{K}(M - \eta)\lambda}$$

So, you have these two equations you have the stress ratio which is nothing but dq by dp prime. So, in case of drained test so your eta = 3 this is always the stress path is always taking in a slope of 3. So, you can use this condition to derive your volumetric strain response. So, again we use an associated flow rule so you use M - eta and then try to form the equations. This is the formulation for drained response.

Similarly, you can also formulate for undrained response, you follow the same procedure but you have a condition where your volumetric strain is 0 d epsilon the volumetric strain increment d epsilon v = 0. So, you have used this condition and you can find a relation between volumetric or mean effective stress and volumetric strain and the deviated stress and shear strain.

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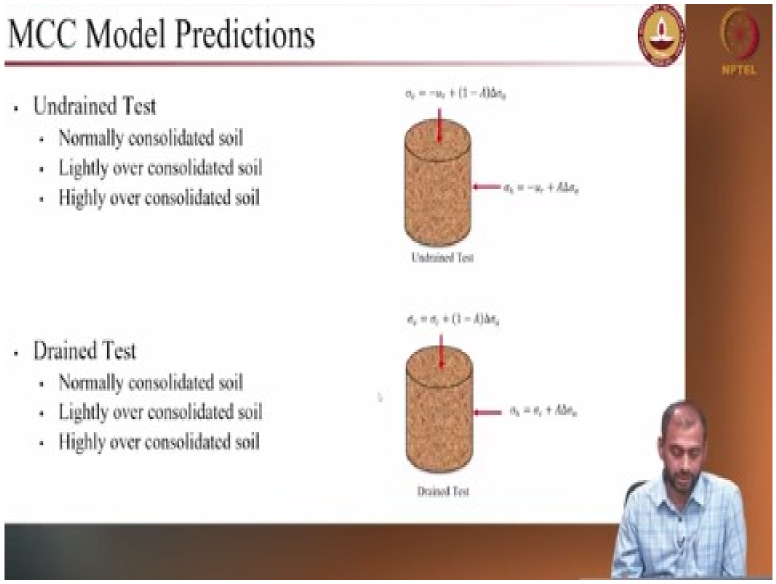

Once you have all these things we can actually model and try to predict the material response. So, this is a typical response what you observe in an undrained and drained condition. When you are performing a normally consolidated soil and when you are having a lightly over consolidated and highly over consolidated soil.

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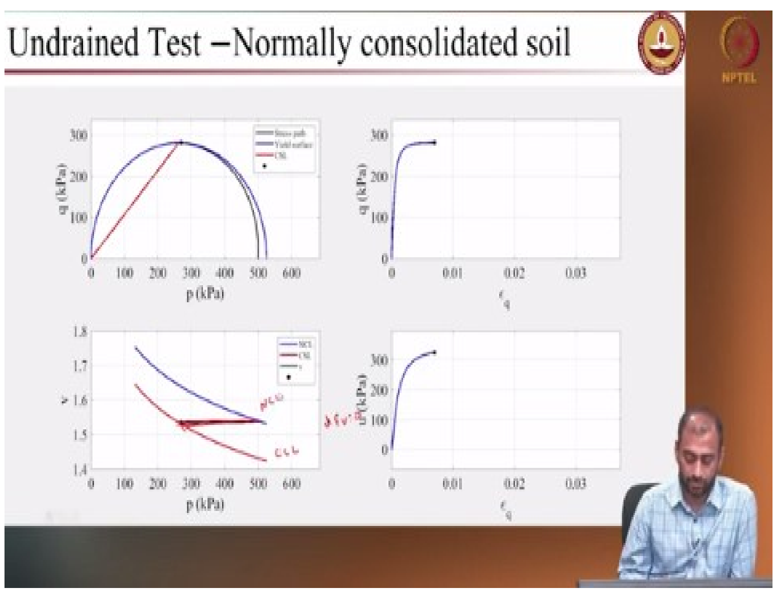

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So, on the screen what you see is an undrained response for a normally consolidated soil. So, as you can see since it is a undrained response the volume change is not happening. So, your volumetric strain is d epsilon v = 0. So, your curve is on the specific volume p prime space, your curve is actually moving horizontally and reaches the critical string. So, this is your critical straight line this is your NCL.

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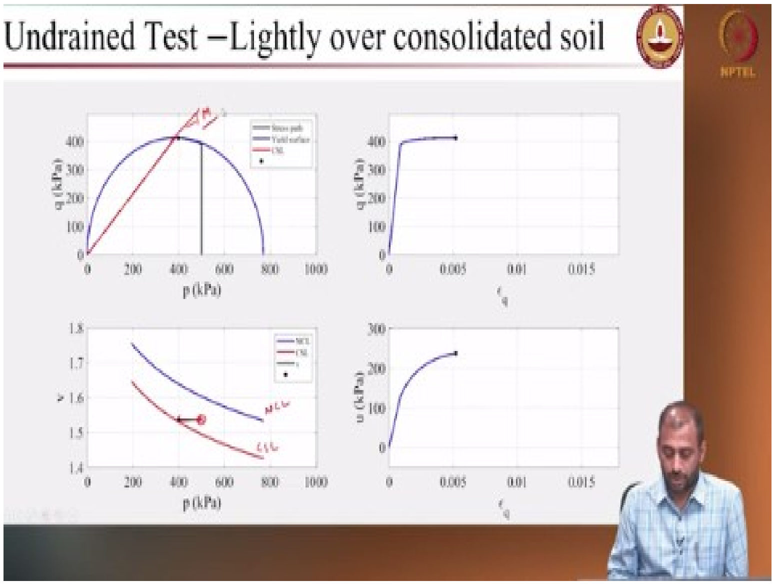

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So, your stresses it keeps since it is an undrained it actually continuously increases and then reaches your critical state line. And you can also see the stress strain response and the pore pressure response. The pore pressure keeps increasing because since it is a normally consolidated soil you continuously see the contraction and your pore pressure keeps on increasing. So, this is a typical again this is a predictions prediction that is obtained using modified cam clay mode.

So, you have q versus p prime, specific volume versus p prime and stress versus strain and pore pressure was a strain so you have all these four plots. Now when you have a lightly over consolidated soil. For example, as you are moving in you have a slightly higher pre consolidation pressure. So, once you start at a pressure which is slightly lower than the pre consolidation pressure if you see carefully until it reaches the failure locus when the material is deforming elastically there is not much of a volume change that is happening.

But after it reaches the yield surface since it is an undrained condition it travels horizontally and reaches the critical straight line. So, this is how the material reaches its critical straight line of M.

So, here also you can see it is not dilating much. So, the pore pressure drastic increases and then slowly starts reaching a constant value. So, this is for a lightly over consolidated soil under a drained condition.

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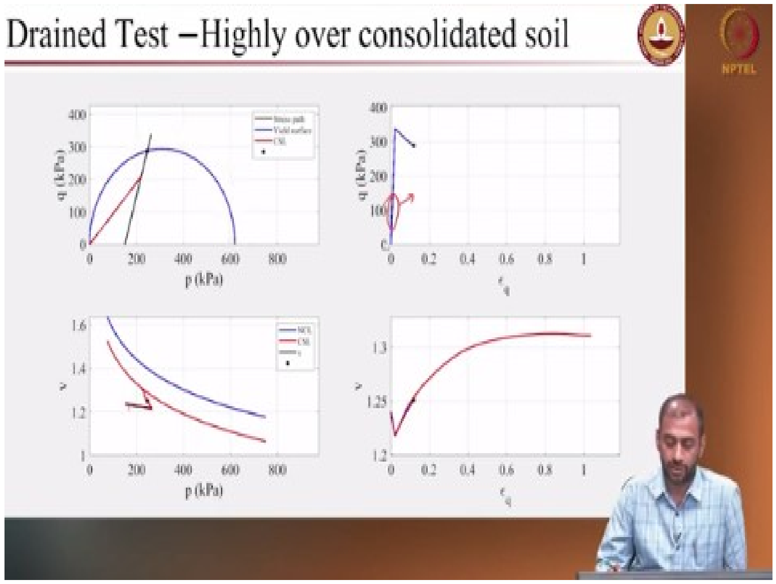

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So, now you are seeing a highly over consolidated soil. Your pre-consolidation pressure is very high compared to the initial confining pressure. So, you actually cross the yield surface and then across the critical straight line and touch the failure locus. Once the it reaches the failure locus the volume change happens and the specific volume is constant and reaches the critical straight line and also here you can see carefully the material behaviour.

Since it is over consolidated soil the material initially contracts and then it dilates so it is contracting and then it is dilating to reach the critical state. So, when it is contracting it is having a higher pore pressure and then as it dilates the pore pressure reduces and then reaches a constant value. So, you can clearly see a peak strength that is observed in case of a highly over consolidated soil. So, this is about the undrained response.

Similarly, we will also see the drained response prediction. As you can see this is again drain response normally consolidated soil. So, it reaches a hydrostatic stress and then after that it starts shearing at a slope of three. And here since it is a drain condition you can see the specific volume keeps on decreasing. So, the material is actually contracting and reaches this critical straight line. So, you can see a nice contractile response the volume change actually reduces the specific volume reduces.

And reaches a constant value on the stresses are like a hyperbolic in shape and it reaches a constant one. So, this is about drained response under normally consolidated soil. Similarly, if you have a over consolidated soil, you can see if the stress state is in the elastic regime the change in specific volume is not that much. But once it reaches yield the specific volume decreases rapidly and then reaches this critical straight line.

And you can hear also you can see a continuous contraction in the specific volume and reaches a constant value. In case of heavily over consolidated soil you can see the specific volume contracts after that it dilates. You can see a clear increase in void ratio before it reaches the critical state and the yield locus also contracts since it is experiencing some sort of dilation. So, you can clearly see this contraction and then the dilation response in case of over consolidated soil.

However, as I as we discussed earlier one of the limitations of mercury model the initial non-linear initial elastic region seems to be too steep. It is over predicting the elastic part of the material response. So, we need to bring in a better pressure dependent model to clearly model the elastic regime in modified cam clay model.

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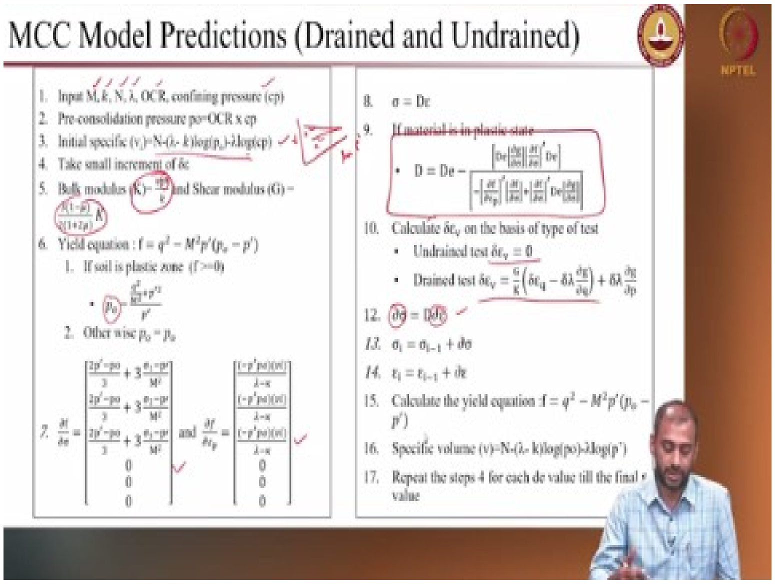
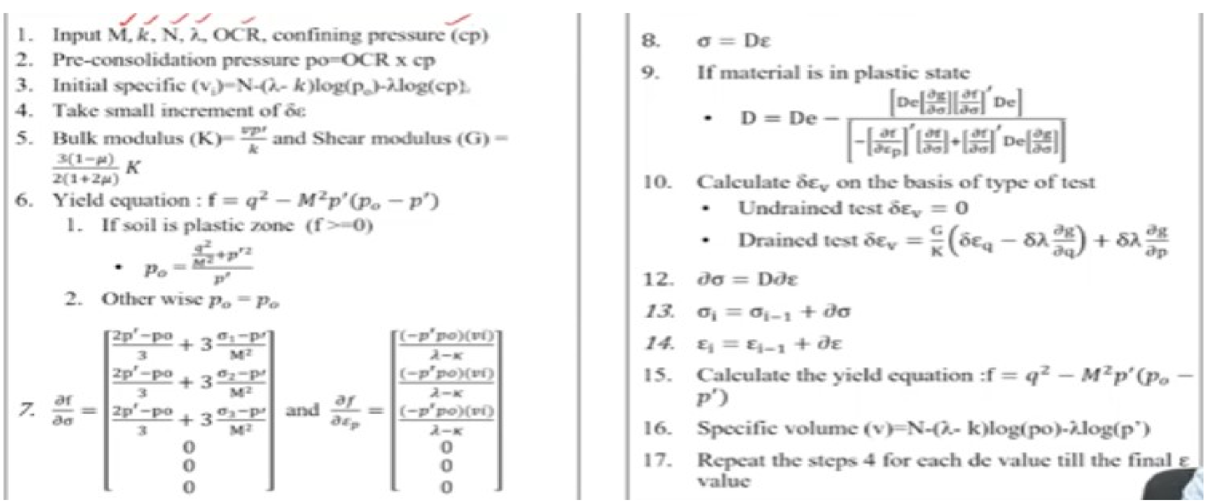


So, here on the screen what you are seeing is how you perform a drained or undrained respond. These are the inputs so you need to know the credit so these are the input parameters you need to know this critical stress ratio, kappa parameter n, the lambda parameter. We also need to know whether it is a what is the over consolidation ratio and then what confining pressure you are sharing this specimen.

So, if you know all these things you can find out the specific volume using this relation from the odometer test you will have ln P prime which is new. You have this equation you can obtain this particular equation from this is your lambda, this is your kappa. So, you can get this equation specific volume from these experimental results or these experimental plots. So, you have this equation if you know the parameters you can get the initial specified.



For a small increment of strains, you can use a bulk modulus so here you can see the bulk modulus is dependent on P prime instead of keeping it constant we can make it dependent on P prime and also shear modulus. So, after that every time you need to check the yield condition or yield equation. So, if it is less than f is less than 0 then I think it is under elastic condition, if f = 0 then the material has reached yield.

Once it has reached yield you need to calculate the pre-consolidation pressure. Again, you need to determine this elastoplastic stiffness matrix. What you are showing here once we know the elastoplastic stiffness matrix and formulate the relation, we have conditions like for in case of untrained condition your volumetric strain is 0. So, use the condition and simplify the equation bit further and then solve the problem.

If it is drained condition use this relation d epsilon v = g by k. This relation and then you can like solve the problem. So, once you have this for a particular increment you will get the for a particular strain increment you will get the stress. So, again you have to repeat this process for the next strain increment continue the steps change the specific volume and finally you will update the stresses and strains at each and every step.

So, this using this particular method you can predict this stress strain response and the previous animations what you were seeing is just based on following this particular procedure and using the MCC yield equation we predicted this stress strain response.

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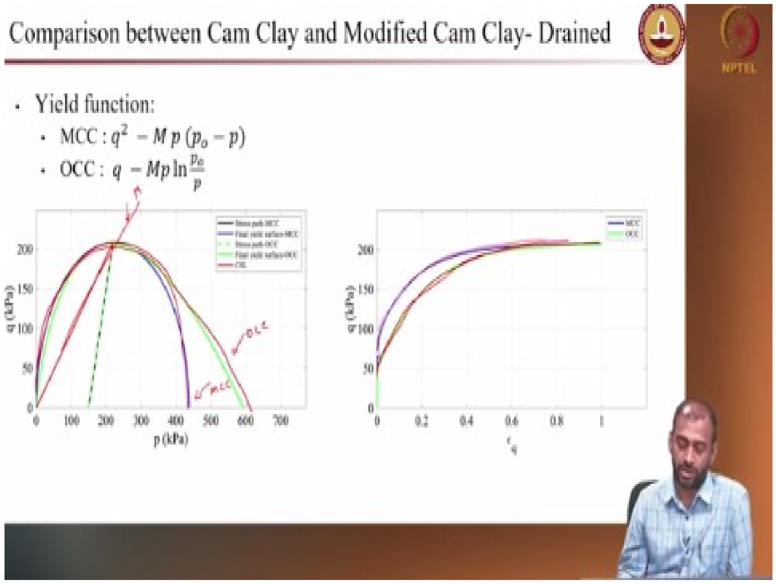

So, here I am trying to compare the stress strain and volume change response between the original cam clay model and the modified cam clay model. As you can see the failure locus somewhat leaf-like because of the presence of logarithmic term in case of original cam clay model and this is an elliptical failure locus in case of an modified cam clay model. This is your failure line failure and M.

So, for the same set of parameters you can see the modified the stresses are slightly under predicted in case of original cam clay and it is over predicted in case of modified cam clay. It is not over predicted I think the predictions for from original cam clay is somewhat underpredicting whereas modified cam clay is slightly better in predicting the material response again. So, these two models works very well in case of a normally consolidated soil.

If this material response if the material is highly over consulted then the response is not close to the actual response that you observe from your experimental results.

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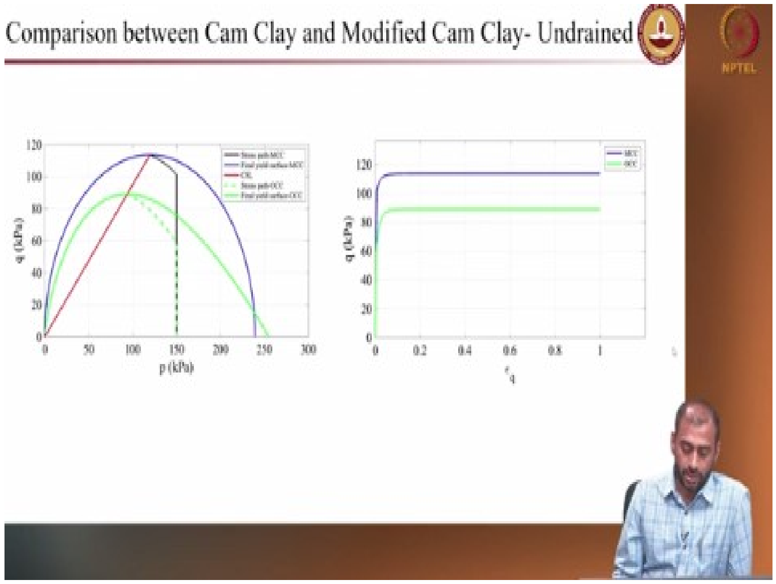

Similarly, you can see the volume change. So, this is your NCL but you can see the critical state line for your original cam clay and this is your critical straight line for modified cam clay. So, your specific volume actually decreases and reaches your critical straight line. You can see a difference in the specific volume predictions based this is specifically because of the yield equation that you are using for original cam clay and modified cam clay model.

The modified cam clay is predicting much better actually the original cam clay the specific volume is all predicted. As you can also see here the specific volume is over predicted in case of OCC original cam clay. So, here also we are trying to compare the untrained response, we can see the OCC the strength is under predicted.

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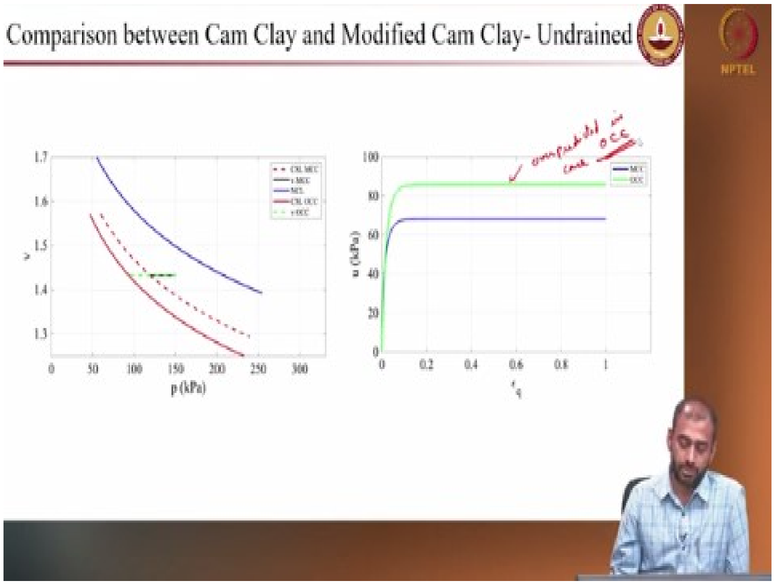

And the pore pressure is sort of overpriced in case of OCC. So, as we saw the limitations we actually try to predict and then we showed that the stress strain response between OCC and MCC.

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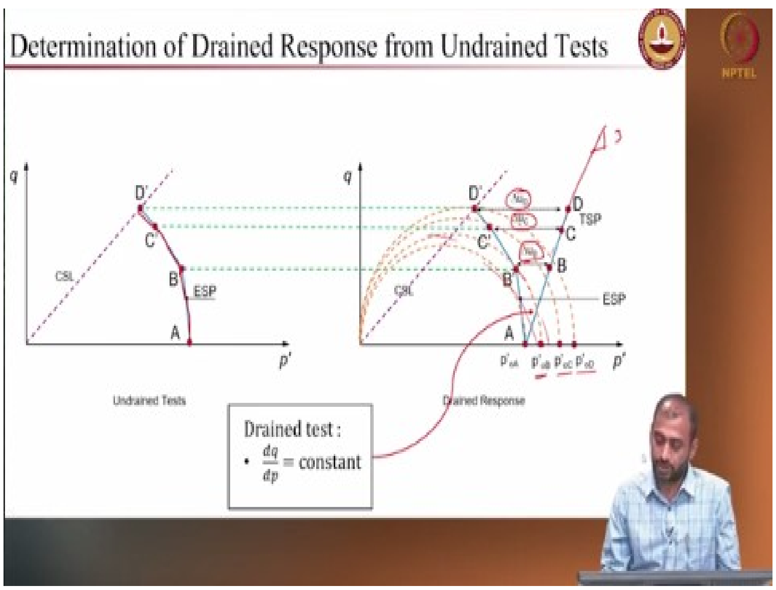

So, one of the advantages of using these models is that you can use the critical state framework, you can use this model and if you just have a drained response or if you have an untrained test but if you want to quantify the drained response you can use this framework and quantify the drained response from the undrained test. So, for example this is your undrained test so this is between p prime q space you are having in this sort of an undrained response.

But I want to know the drain response of the material. So, what you need to do is so the drained response it will you have to draw the stress path which is at a slope of three. So, you have this slope, you have this Mohr Coulomb modified cam clay failure envelope using at different points you can draw the envelope. So, that will give you your new pre-consolidation pressure p 0 b, p 0 c, p 0 d.

And the difference between that current the stresses on the untrained state and the total stress will give you the excess pore pressure. So, you will know the pore pressure from so you have the total stress path you have the effective stress path and the difference between these two will you will get the pore water pressure and you will also get the new pre-consolidation pressure.

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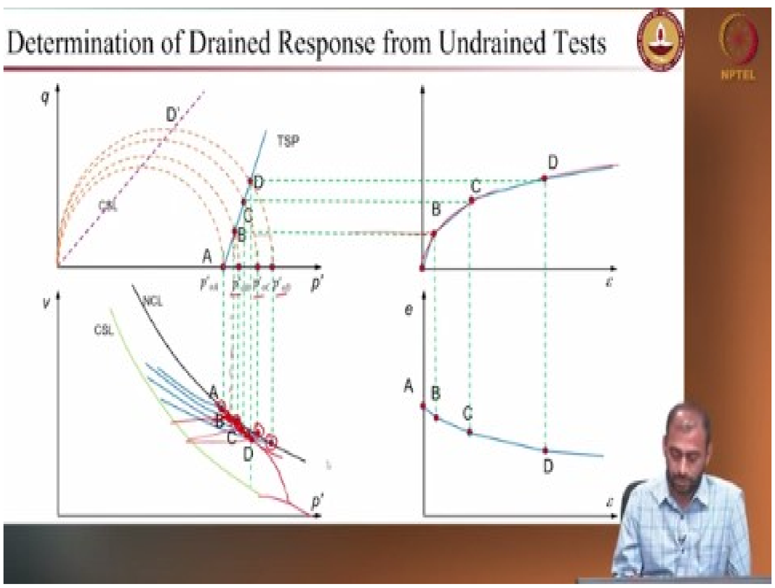

So, you have this new pre-consolidation pressures. From this you can actually draw the stress profile for the drain test. So, you already know the undrained behaviour but you are trying to predict the drained response. Also, the challenges since you it is undrained interest you have only the pore pressure response but you want to know the volume change response if there is an undrained test that has been performed.

You do not have the undrained sorry the drain test but you are predicting from your using your MCC model. So, from here if you just project the PC the pre-consolidation pressure onto your NCL line you will get these points you get these points on your normally consolidated line.

Similarly, from the intersection of the total stress path you can draw a line on the specific volume p prime space and it intercepts at a particular point.

Because you know in case of undrained test you have a constant specific volume. So, you can have this intersection point and then you can connect and draw your undrained response sorry you can draw your drained volume change from your undrained response. This is how you predict the drained response from your undrained tests.

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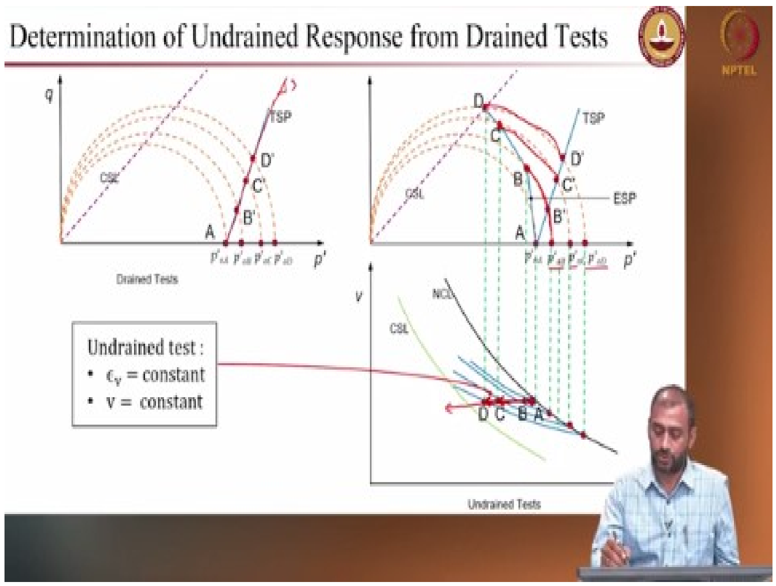

Similarly, if you have drain test and if you want to determine the undrained response. So, if you have a drained test, you will have this particular total stress path of a slope three. So, once you know this how to determine the untrained response. So, you know the pre-consolidation so from the Mohr Coulomb failure envelope you will determine PC POB which is the pre-consolidation pressure at the point B, point C and point D.

And then you can extend it on the normally consolidated line. And also, since this is you are trying to predict the undrained response, your specific volume is constant so it will move along this line. So, you know this point from your modified cam clay you can get this point on this particular failure locus. The point B on the failure locus can be determined similarly the point C and the failure locus and the point D on the failure locus is determined.

So, if you locate these two then you can find the difference between the pore water pressure or difference between the total stress path and effective stress path to quantify the pore water pressure. So, that is what I have done here.

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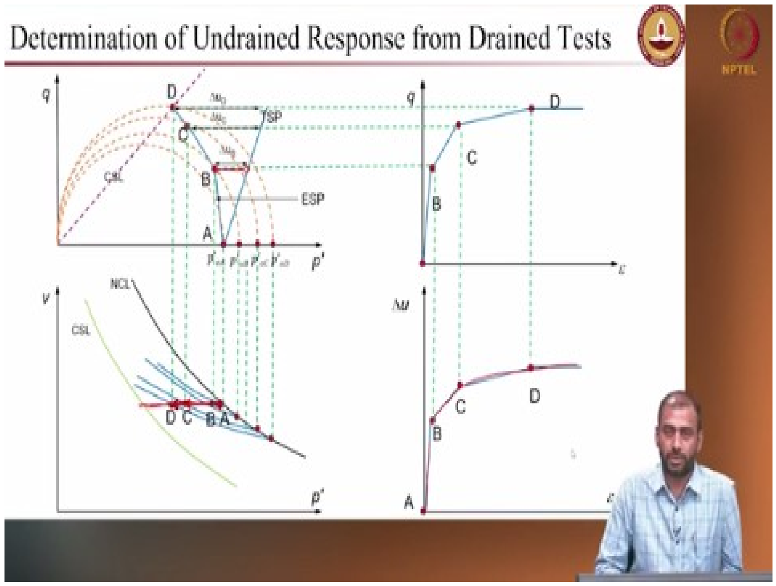

So, you can find out the difference between the total stress path and effective stress path to find to plot the pore pressure curve. Similarly, you can also apply identify the specific volume in this particular space. So, you have a drained test and you are trying to predict the undrained response. So, this model this original cam clay and modified cam clay model predicts the material response fairly accurately.

And also, this is capable of determining the drained response from an undrained behaviour or untrained response from a drained behaviour. So, it is an useful model to predict the material and this is one of a commonly employed models in any finite element framework which you can use these models and use input the appropriate parameters and then solve some complex boundary value problems.

So, that is about the discussion on original cam clay model, modified cam clay model and how it is used to predict the material response. And these needs to be this is whatever I have showed is a single point integration to determine the material response. But you have to do this on a finite element implement in a finite element framework to solve for multiple nodes and elements. So, anyway thank you.