Finite Element Analysis and Constitutive Modelling in Geomechanics
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Lecture - 39
Cam Clay Models

Hello students, welcome back. In the previous class we had derived an equation for the elastoplastic constitutive matrix D ep and in terms of the dou F by dou sigma dou Q by dou sigma and so on.

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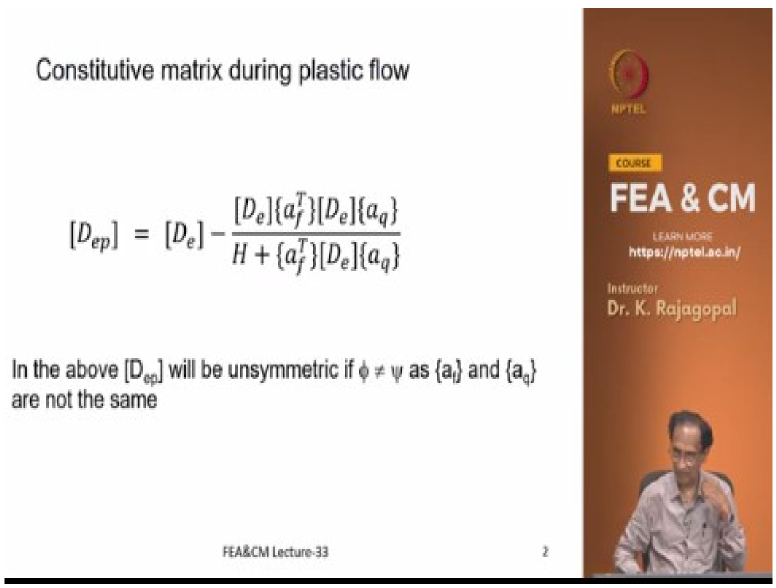

And we derived this equation D ep is D e - this whole thing and in general this D elastoplastic is unsymmetric when your dilation angle psi is not equal to friction angle phi.

$$[D_{ep}] = [D_e] - \frac{[D_e]\{a_f^T\}[D_e]\{a_q\}}{H + \{a_f^T\}[D_e]\{a_q\}}$$

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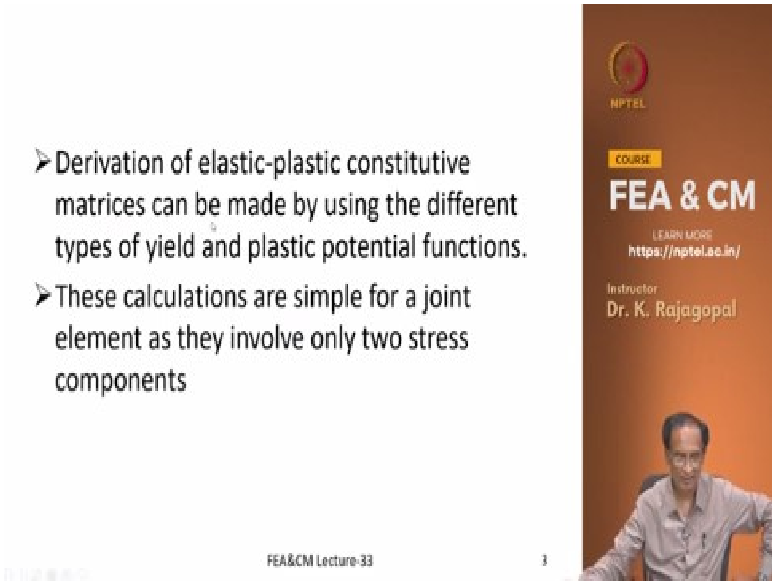

And that we will see in this class we will apply this for a very simple case of the joint element then we will derive all these quantities. So, that we can see what they are and by applying this f and q for different plasticity theories like for example for Tresca we had said that f is sigma 1 - sigma 3 - 2 c whereas for Mohr Coulomb relation it is a slightly longer equation and so on. We can generalize this for different plasticity theories.

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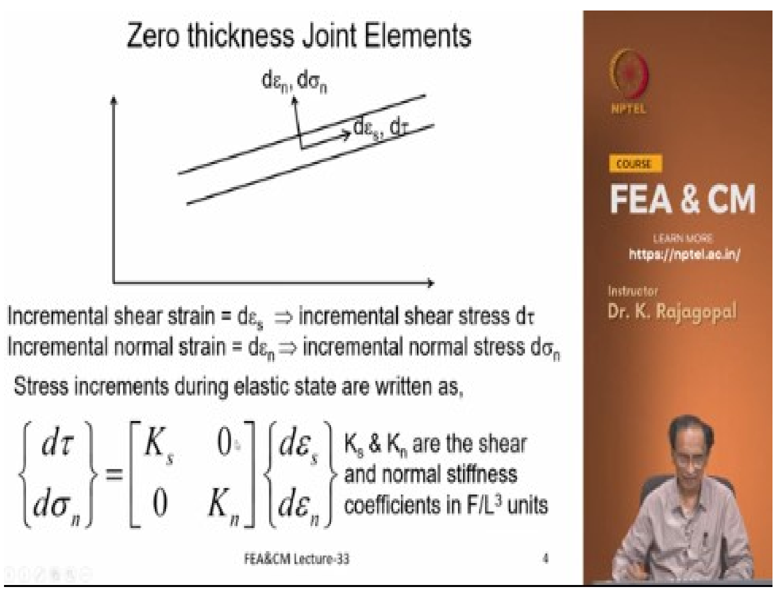
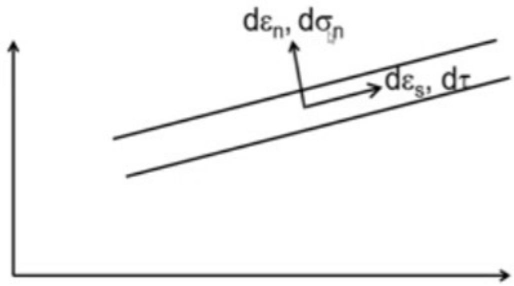


And in this class let us derive the elastoplastic constitutive matrix for the 0 thickness a joint element. This we had seen earlier long back. Let us say that we have a joint element like this that has a 0 thickness and there are only two strains and two stresses the shear strain d epsilon s that is the relative shear displacement between the two surfaces and the d epsilon n that is normal displacement either tensile or compressive.

Zero thickness Joint Elements



Incremental shear strain = $d\varepsilon_s \Rightarrow$ incremental shear stress $d\tau$
Incremental normal strain = $d\varepsilon_n \Rightarrow$ incremental normal stress $d\sigma_n$

And then the corresponding stresses d tau and the d sigma n data with the shear stress and d sigma n is the normal stress and in the elastic state these two stresses and strains are decoupled. So, if you apply shear strain, we will only produce shear stress and if you apply normal stresses, we will produce only normal strains. And the stress strain relation during the elastic state or like this d tau and d sigma n is K s 0 0 K n times d epsilon s d epsilon n.

Stress increments during elastic state are written as,

$$\begin{Bmatrix} d\tau \\ d\sigma_n \end{Bmatrix} = \begin{bmatrix} K_s & 0 \\ 0 & K_n \end{bmatrix} \begin{Bmatrix} d\varepsilon_s \\ d\varepsilon_n \end{Bmatrix}$$

K_s & K_n are the shear and normal stiffness coefficients in F/L^3 units

That is d tau is K s times epsilon s and d sigma n is k n times d epsilon n and the cross terms are 0. And our K s and K n are the shear and normal stiffness coefficients in the units of F by L cube units because the strains for the joint elements we are defined them in terms of the relative deformations. These are not the gradient of the displacement but it is basically at the interface what is the relative displacement between the two surfaces.

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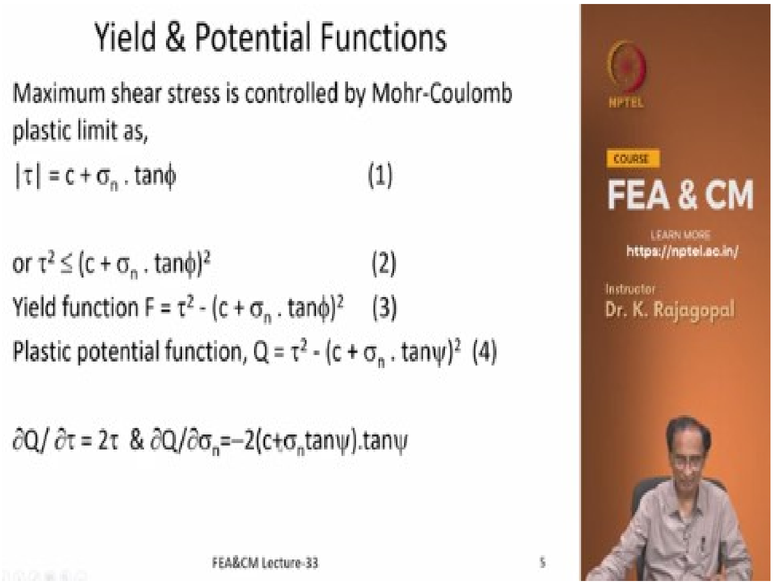

And let us say that we are using a Mohr Coulomb relation for the limiting shear stress and because the shear stress could be that positive or negative with the same consequence, we can write the tau as c + sigma and tan phi where our compressive stress is taken as positive. So, we have the c + sigma and tan phi and in the elasticity convention the compressive stresses are negative. So, if you look at any finite element program you will see this as c - sigma and tan phi.

Maximum shear stress is controlled by Mohr-Coulomb plastic limit as,

$|\tau| = c + \sigma_n . \tan\phi$ (1)

or $\tau^2 \leq (c + \sigma_n . \tan\phi)^2$ (2)

Yield function $F = \tau^2 - (c + \sigma_n . \tan\phi)^2$ (3)

Plastic potential function, $Q = \tau^2 - (c + \sigma_n . \tan\psi)^2$ (4)

$\partial Q/ \partial \tau = 2\tau$ & $\partial Q/\partial \sigma_n = -2(c+\sigma_n \tan\psi).\tan\psi$

Or by squaring on both the sides tau square is less than or equal to c + sigma and tan phi whole square this is your yield limit. The tau max should be less than this quantity c + sigma and tan phi the yield function F can be written as F is a tau square - c + sigma and tan phi whole square. And if F is less than 0 we are in the elastic state and when F is greater than 0 we have exceeded the yield limit and so we have to come back to the yield surface.

And the plastic potential function Q can be written in terms of psi like this Q is tau square - c + sigma and tan psi whole square and dou Q by dou tau is just simply 2 tau and dou Q by dou sigma n is - 2 times c + sigma sin tan psi.

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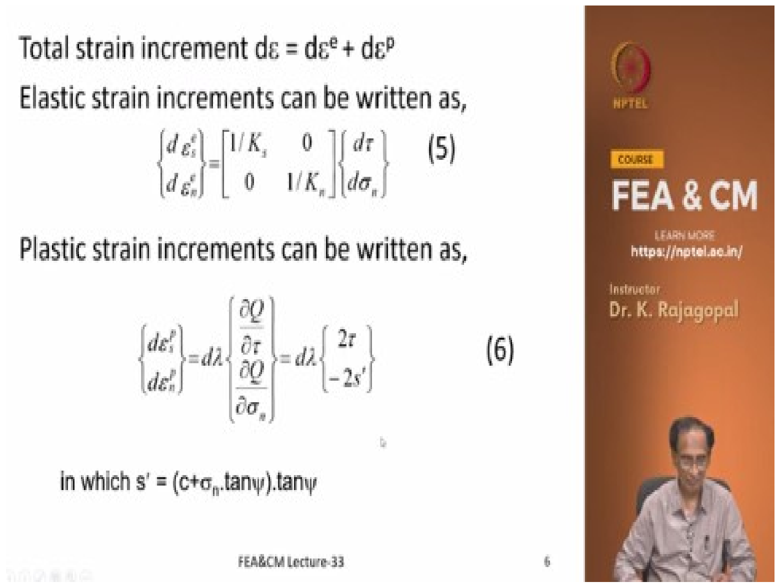
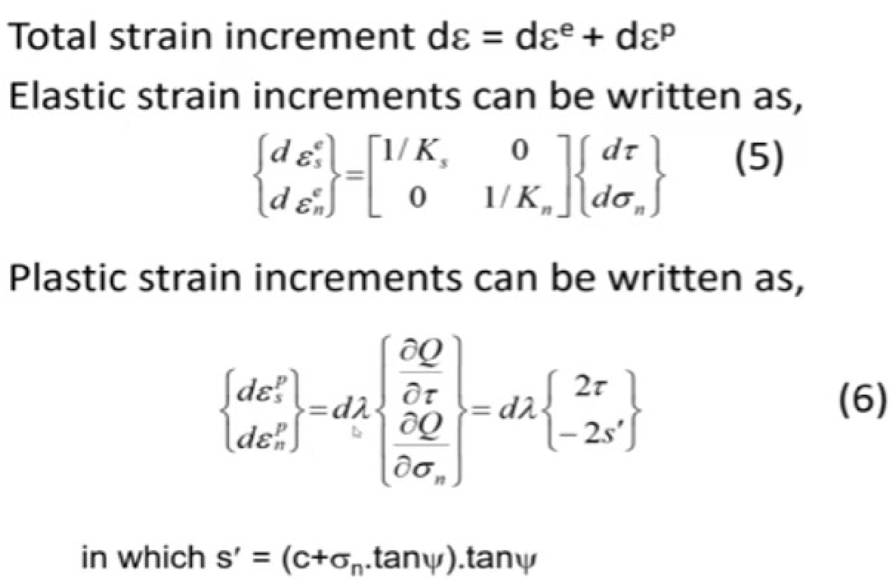


And our in the elastic state our strain increments d epsilon e and d epsilon n e is just simply 1 by K s times d tau 1 by K n times d sigma n and then during the plastic state the plastic strain increments d epsilon s p and d epsilon n p is d lambda dou Q by dou sigma and dou Q by dou sigma n. So, because our tau is in the shear direction epsilon s and the sigma n is in the normal direction.



And this we can write as d lambda times 2 tau by - 2 s prime where s prime is c + sigma and tan psi is actually our dou Q by dou tau and dou Q by dou sigma n are written like this. So, by using this s prime as c + sigma and tan psi tan psi we can write like this.

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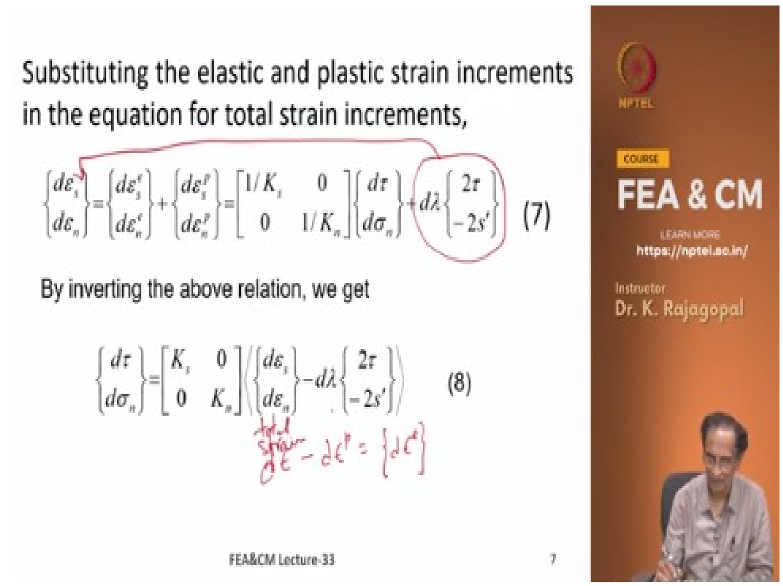

And so, substituting the elastic and the plastic strain increments in the equation for total strain we can write the total strain as d epsilon s and d epsilon n is d epsilon elastic d epsilon n normal elastic plus the plastic strain increments and in terms of the stress increment d tau and d sigma n, we can write the elastic strain increments like this. And then these are the plastic strain increments d lambda times dou Q by dou sigma dou tau and dou Q by dou sigma n.

$$\begin{Bmatrix} d\varepsilon_s \\ d\varepsilon_n \end{Bmatrix} = \begin{Bmatrix} d\varepsilon_s^e \\ d\varepsilon_n^e \end{Bmatrix} + \begin{Bmatrix} d\varepsilon_s^p \\ d\varepsilon_n^p \end{Bmatrix} = \begin{bmatrix} 1/K_s & 0 \\ 0 & 1/K_n \end{bmatrix} \begin{Bmatrix} d\tau \\ d\sigma_n \end{Bmatrix} + d\lambda \begin{Bmatrix} 2\tau \\ -2s' \end{Bmatrix} \quad (7)$$

By inverting the above relation, we get

$$\begin{Bmatrix} d\tau \\ d\sigma_n \end{Bmatrix} = \begin{bmatrix} K_s & 0 \\ 0 & K_n \end{bmatrix} \left\langle \begin{Bmatrix} d\varepsilon_s \\ d\varepsilon_n \end{Bmatrix} - d\lambda \begin{Bmatrix} 2\tau \\ -2s' \end{Bmatrix} \right\rangle \quad (8)$$

And so, by inverting this relation by taking this d lambda times 2 tau and this quantity let me just show this. See by taking this quantity to the other side and we can write d tau and d sigma n in terms of the other quantities. So, this K s 0 0 K n times d epsilon s and d epsilon n are the this is the elastic stress increment minus this d lambda 2 tau 2 s prime. This is the plastic strain part. Basically, this is sorry I think so we have seen that the stress increment during the plastic strains is only because of the elastic part.

So, that is some constitutive matrix multiplied by total strain minus elastic strains or sorry the total strains minus the plastic strains that is what the d s d epsilon s d epsilon n is the total strain and this is the plastic strain.

$$d\epsilon - d\epsilon^p = \{d\epsilon^e\}$$

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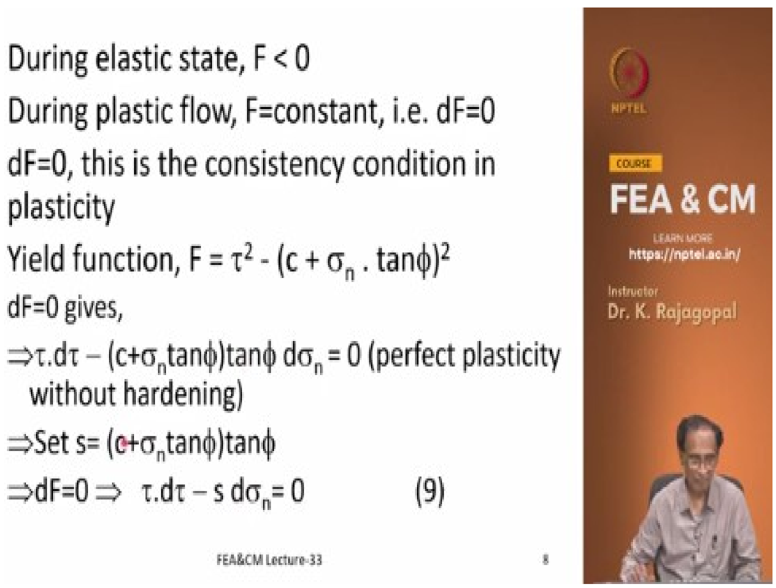
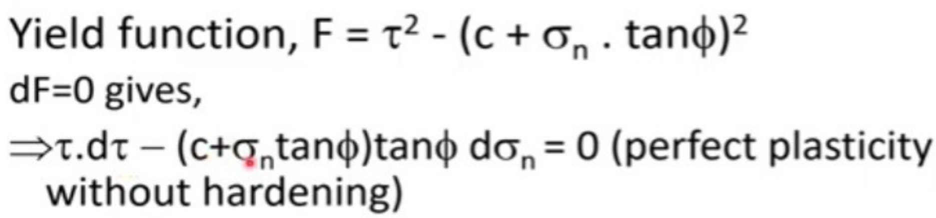


So, we get the stress increment. So, during the plastic flow the change in the yield function should be 0 dF is 0. And our F is written as tau square - c + sigma and tan phi whole square and the dF of 0 means tau times d tau - c + sigma and tan phi times phi times d sigma and this is 0. This is for perfect plasticity where your yield limit is not increasing with the plastic strains. And we can set the c + sigma and tan phi times tan phi 2 s and the dF of 0 means tau times d tau - s times d sigma n is 0.



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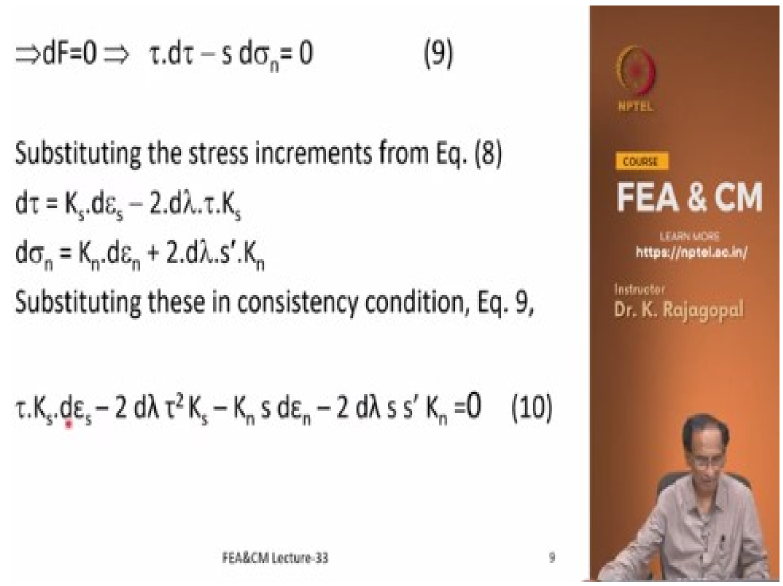

And our d tau and d sigma n we can get from equation 8. So, if you look at our previous equation 8 we have d tau and d sigma n in terms of the other quantities and we can substitute them in this equation 9 and d tau and the d sigma n from equation 8 or this and by substituting this in the consistency equation that is equation 9 the tau K s d epsilon s - 2 d tau d lambda tau square K s and so on and so from here we can get d lambda because all the other quantities are known except d lambda.

$$\Rightarrow dF=0 \Rightarrow \quad \tau.d\tau - s\, d\sigma_n = 0 \qquad (9)$$

Substituting the stress increments from Eq. (8)

$d\tau = K_s.d\varepsilon_s - 2.d\lambda.\tau.K_s$

$d\sigma_n = K_n.d\varepsilon_n + 2.d\lambda.s'.K_n$

Substituting these in consistency condition, Eq. 9,

$$\tau.K_s.d\varepsilon_s - 2\, d\lambda\, \tau^2 K_s - K_n\, s\, d\varepsilon_n - 2\, d\lambda\, s\, s'\, K_n = 0 \qquad (10)$$

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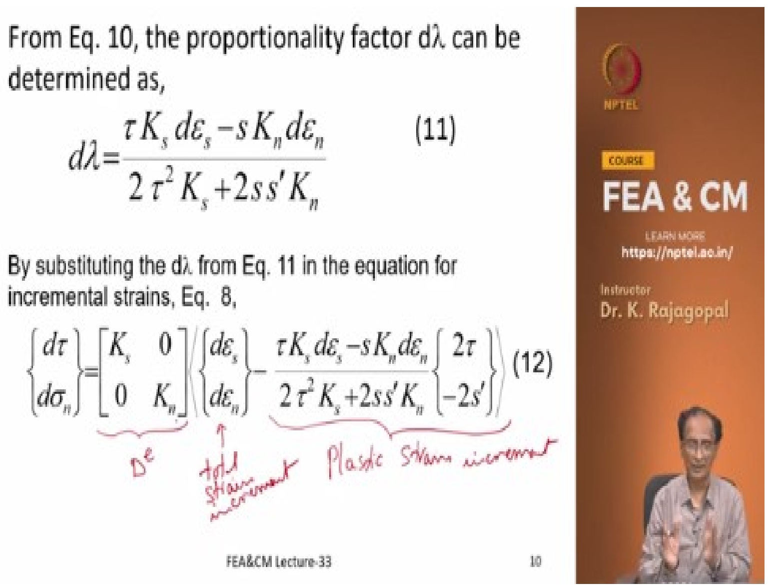

And so, d lambda can be determined like this and now the stress increment during the elastic plastic state is detail d sigma n is elastic constitutive a matrix multiplied by d epsilon - the plastic strains. See basically the plastic strain is d lambda times dou Q by dou tau and dou Q by dou sigma n. So, this is our plastic strain increment and this is our elastic D elastic, this is our total strain.

From Eq. 10, the proportionality factor $d\lambda$ can be determined as,

$$d\lambda = \frac{\tau K_s\, d\varepsilon_s - s\, K_n d\varepsilon_n}{2\,\tau^2 K_s + 2 s s' K_n} \qquad (11)$$

By substituting the $d\lambda$ from Eq. 11 in the equation for incremental strains, Eq. 8,

$$\begin{Bmatrix} d\tau \\ d\sigma_n \end{Bmatrix} = \begin{bmatrix} K_s & 0 \\ 0 & K_n \end{bmatrix} \left\langle \begin{Bmatrix} d\varepsilon_s \\ d\varepsilon_n \end{Bmatrix} - \frac{\tau K_s\, d\varepsilon_s - s K_n d\varepsilon_n}{2\,\tau^2 K_s + 2 s s' K_n} \begin{Bmatrix} 2\tau \\ -2s' \end{Bmatrix} \right\rangle \qquad (12)$$

So, in the plastic part D elastic times the elastic strain increment that is the total strain increment minus the plastic strain increment.

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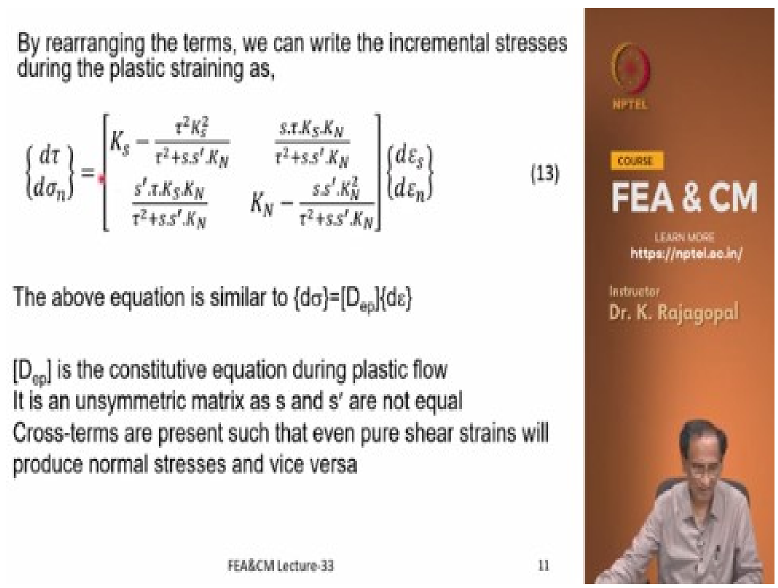

So, by rearranging all the terms, we get this equation d tau d sigma n is this multiplied by d epsilon s and d epsilon n and this bracket these the terms the square bracket they correspond to our constitutive matrix in the elastic elastoplastic part D ep. So, this is actually it is if you look at this is very interesting to see that during the plastic part our two diagonal terms are lesser than the elastic values.

By rearranging the terms, we can write the incremental stresses during the plastic straining as,

$$\begin{Bmatrix} d\tau \\ d\sigma_n \end{Bmatrix} = \begin{bmatrix} K_S - \frac{\tau^2 K_S^2}{\tau^2 + s.s'.K_N} & \frac{s.\tau.K_S.K_N}{\tau^2 + s.s'.K_N} \\ \frac{s'.\tau.K_S.K_N}{\tau^2 + s.s'.K_N} & K_N - \frac{s.s'.K_N^2}{\tau^2 + s.s'.K_N} \end{bmatrix} \begin{Bmatrix} d\varepsilon_s \\ d\varepsilon_n \end{Bmatrix} \quad (13)$$

The above equation is similar to $\{d\sigma\}=[D_{ep}]\{d\varepsilon\}$

$[D_{ep}]$ is the constitutive equation during plastic flow
It is an unsymmetric matrix as s and s′ are not equal
Cross-terms are present such that even pure shear strains will produce normal stresses and vice versa

Because you have K s minus something and K s minus something then if you look at this half diagonal terms, we have s and tau and here, s prime tau and s and s prime are not the same. Because one is related to friction angle phi and the other is related to dilation angle psi. So, we see that the half diagonal terms are not the same so we see that our constitutive matrix zealous is unsymmetric and because of the presence of this cross terms even if you apply a pure shear strain d epsilon s.

There will be some normal stress d sigma n and vice versa like if you apply a pure normal stress so we can also produce some shear stress.

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Numerical Example for stiffness coefficients

Let, $K_s=10^6$ & $K_N=10^8$
Let c=10, $\phi=30^\circ$ and $\psi=10^\circ$ & $\sigma_n=50$
$s=(c+\sigma_n.\tan\phi).\tan\phi=22.44$
$s'=(c+\sigma_n.\tan\psi).\tan\psi=3.32$
$\tau=c+\sigma_n.\tan\phi = 38.86$
Denominator = $\tau^2+s.s'.K_n = 7.45\times10^9$

$s.\tau.K_s.K_n = 8.720\times10^{16}$
$s'.\tau.K_s.K_n = 1.290\times10^{16}$
$K_{11} = 797302.07 = 7.97\times10^5$
$K_{12} = 1.170\times10^7$
$K_{21} = 1.731\times10^6$
$K_{22} = -1073.8$

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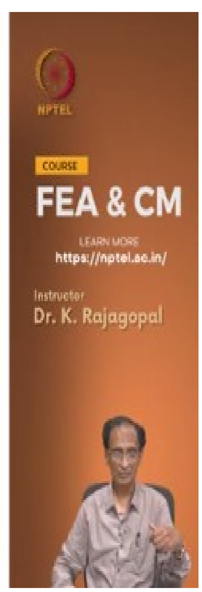

Let us look at some numerical values, let us give some arbitrary values K s is 10 to the power of 6, K N is 10 to the power of 8, c and phi are given like this and phi is 30 degrees and the dilation angle is 10 degrees that means that we are dealing with non-associated flow rule and let sigma n be 50 and this quantity s is a c + sigma and tan phi times tan phi that is 22.44 and s prime is c + sigma and tan psi times tan psi that is 3.32 and then the tau during the plastic strain is at the limit that is c + sigma and tan phi that is 38.86.

Let, $K_s=10^6$ & $K_N=10^8$

Let $c=10$, $\phi=30^\circ$ and $\psi=10^\circ$ & $\sigma_n=50$

$s=(c+\sigma_n.\tan\phi).\tan\phi=22.44$

$s'=(c+\sigma_n.\tan\psi).\tan\psi=3.32$

$\tau=c+\sigma_n.\tan\phi = 38.86$

Denominator = $\tau^2+s.s'.K_n = 7.45\times10^9$

$s.\tau.K_s.K_n = 8.720\times10^{16}$

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$K_{11} = 797302.07 = 7.97\times10^5$

$K_{12} = 1.170\times10^7$

$K_{21} = 1.731\times10^6$

$K_{22} = -1073.8$

The denominator in our stiffness matrix or the constitute matrix terms is tau square + s times s prime K N that is this and then our s 2 times K s and these are the numerator terms in the off-diagonal terms. And the K 11 is so we can actually do this calculation K 11 is K s - tau square K square divided by this denominator that is tau square + s times s prime K N. Because if you see all the denominators are the same so that is why I just calculated one by once denominator.

And K 11 is approximately 8 times 10 to the power of 5 and the elastic part it is 10 to the power of 6. So, K 11 is a smaller than elastic part K 12 is if you do the calculation it comes to 1.17 times 10 to the power of 7 and K 21 is 1.73 times 10 to the power of 6 it is not the same. Then if you look at K 22 it is negative it is actually K N minus all this quantity and it just so happens that for this particular stress condition one of the diagonal terms is negative.

And if too many diagonal terms become negative in a stiffness matrix, then we will have a numerical difficulty and we will be getting some ridiculous results like our displacements will be suddenly they will increase the 10 power of 20 10 power of 30 and so on and even the stresses that you compute will be meaningless. So, maybe a couple of terms within a stiffness matrix like negative diagonal terms is but if too many of them become negative your computations in fact the programs will simply stop.

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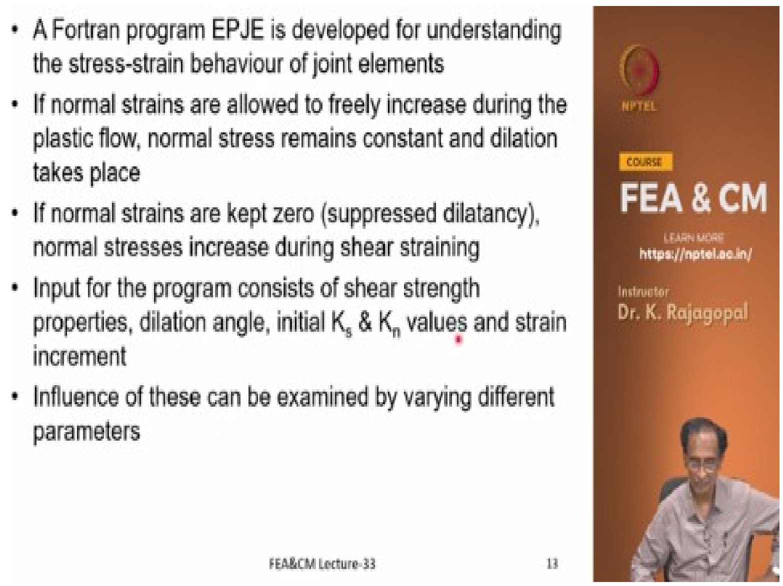

So, I am going to give you one Fortran program EPJE elastoplastic joint element and for understanding the stress strain behaviour of joint elements. The actual it is a very simple program it is not it does not do much, it will do simple calculations. It will allow you to a place shear strain and then calculate the shear stress and normal stress and then the normal strain. If normal strains are allowed to freely increase during the plastic flow.

Normal stress will remain constant and dilation takes place. But if the normal stresses are kept 0 that is suppressed dilatancy we will see these normal stresses will increase and then the normal strain will become will remain 0 and the input for the program is very simple. The strength properties c and phi and that the dilation angle and then the initial K s and the K n values and then the shear strain increment.

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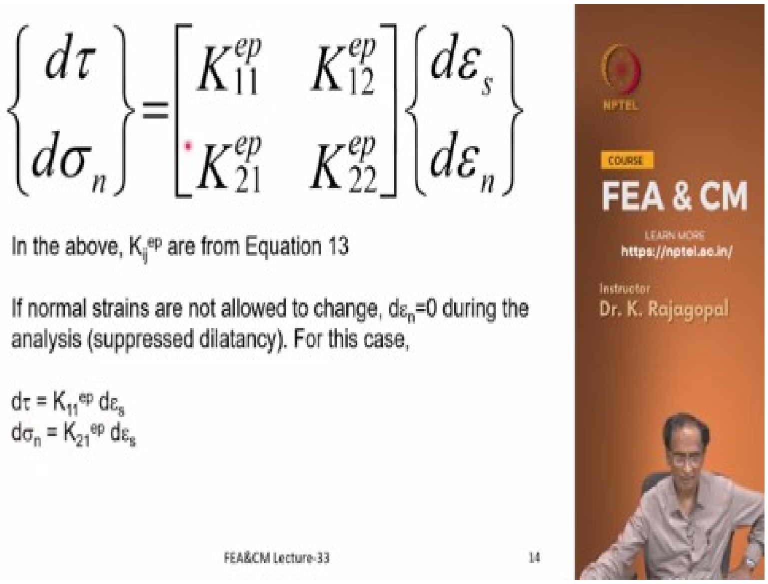
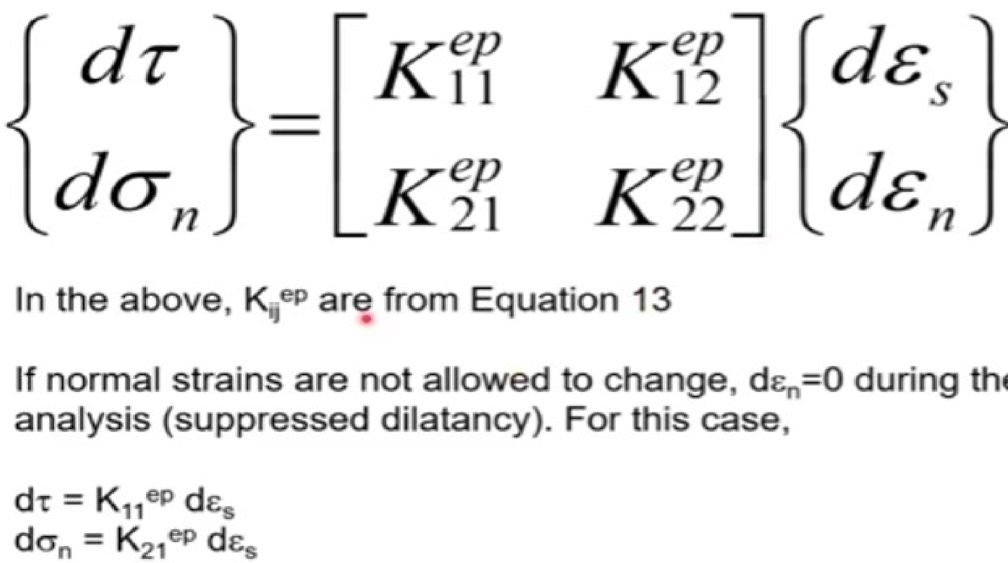


So, the equations implemented are d tau and d sigma n is K 11 elastoplastic, K 12 elastoplastic, K 21, K 22 elastoplastic multiplied by d epsilon s and d epsilon n and during the elastic part it is just simply these two cross terms are 0. Whereas in the plastic part we have this influence of the cross terms. So, the when d epsilon n is 0 d tau is K 11 d epsilon s and d sigma n is K 21 times d epsilon s.



So, we see that because of this cross term we will get an increase in the normal stress. Although we are applying pure shear strain, we get an increase in the normal stress.

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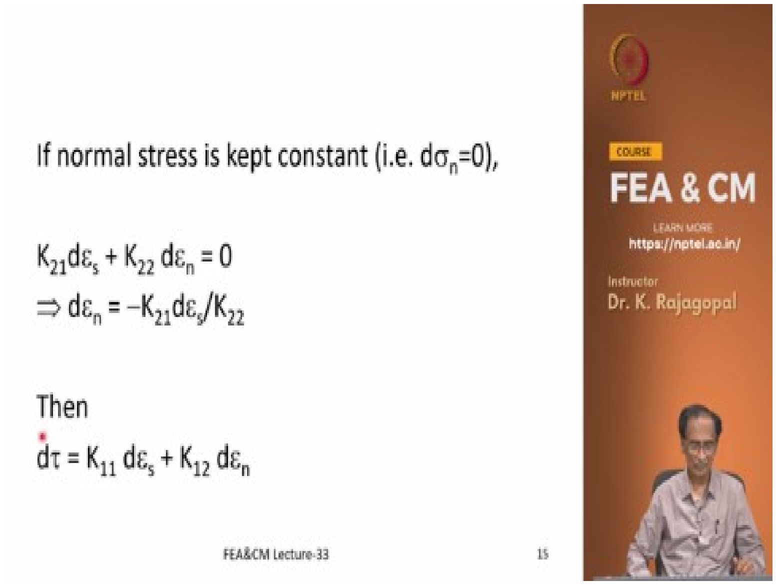

And if normal stress is kept constant that is d sigma n we will get some dilation.

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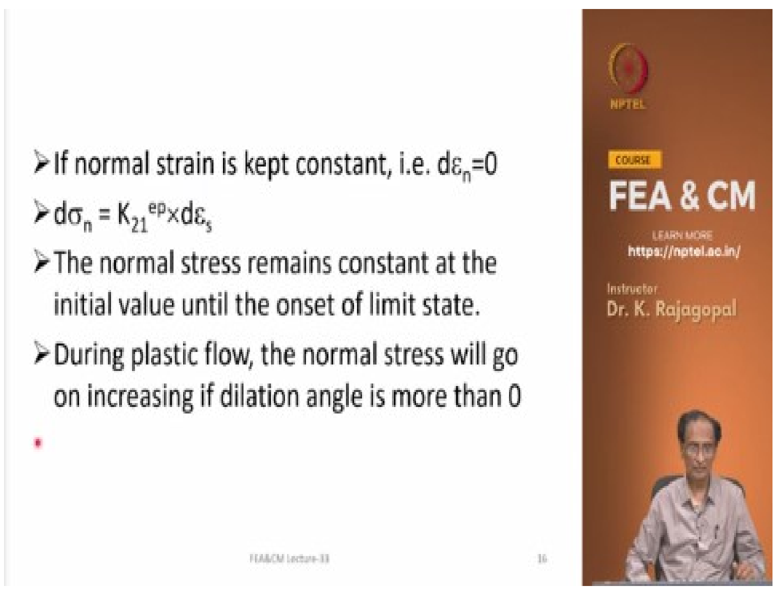

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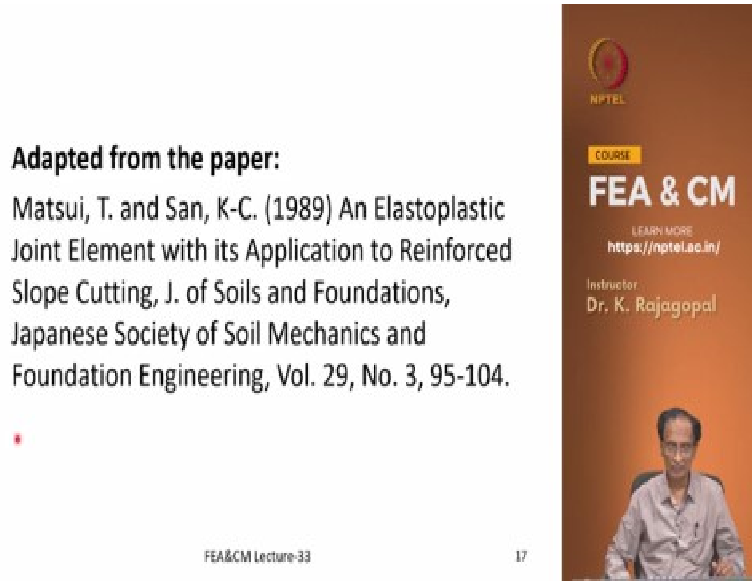

And this particular lecture is based on this paper by Matsui and San elastoplastic joint element with its application to reinforce slope cutting where they considered only associated flow rule. Here I generalize this for non-associated flow rule with F and Q. So, let us look at this program let me keep the annotations.

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So, this is the program and whenever you run the program it will prompt for a file name to store the results. Let us say elastoplastic it will be a text file so that you can open it within any editor and go through it. So, I am calling this as elastoplastic dot out and K s and the K n say 1.086. You can separate out the values either with the space or a common and c and phi and the psi. Let us say c of 10 and the friction angle of 30 degrees and dilation angle of 10 degrees let us say.

And normal stress let us say 100 is the normal stress and then the shear strain increment let us say we apply in increments of 0.001. And what is the maximum strain let us say 0.06 and then do you want to keep normal strain or normal stress constant. See one for constant normal strain or two for constant normal stress that is if you use two that means the you know the soil is free to dilate. So, that our normal stresses remain constant.

So, I am using two and it will the program will run and then let me see where is the result and the shear stress is I think it supplied the loading is applied too fast and right from beginning the shear stress is reach the yield limit and the normal stress is remaining constant at 100. Let me run this program once again. I think I have not paid much attention K s is 1.08 power 6. Now I am using a comma for separating out these two values and the c and phi 10, 30 and 10.

And the normal stress is 100 and the shear strain increment. I am using very small increment of 0.0001 and let us say maximum shear strain is 0.05 and let us take two for keeping the normal stress constant. Even with this it is what is happening. Actually, I think I have to decrease the shear strain increment much further or let me just control the shear modulus. Let us take K s s 10000 and the K n has a very large value 10, 30, 10 and let us say 100.

And shear strain increment of 0.001, 0.001 means it will be 1000 ten that is fine maximum strain and let us say two, I think. So, now we are getting good results. See the shear stress increment is 10 in the elastic path so initially it was let me zoom it a bit. So, with every shear strain increment the shear stress increases by ten in the elastic part. In the elastic part you see that there is no normal string it is 0 10 has become 20, 30, 40, 50, 60.

And next it has it should become 70 but then it has reached the plasticity. So, it has reached the yield limit of c + sigma and tan phi that is 67.735 normal stress is remaining constant at 100. But then your normal strain has started increasing because gradually the normal strain will increase because of your cross terms the interaction terms. And this normal strain increment and normal strains increase will be higher for higher dilation angle.

Let me compare this I will use some other dilation angle for the same soil properties. I am calling this result as plastic two dot out 10000. So, next time previously we had done the analysis with the dilation angle of 10. Now I am doing with the dilation angle of 25 and normal stress is 100 and the shear strain increment is 0.001, maximum shear strain is 0.05 and then I want to keep the normal stress constant.

So, I am using a 2 and let us see what happens with the plastic two and so this is the previous result. Let me zoom it a bit. See this previous one is for a dilation angle of 10 degrees and then this is with the dilation angle of 25 degrees and you see here. See this at shear strain of 8 times 10 to the power of - 3 your normal strain is 2.46 times 10 to the power of - 8. Whereas at the same shear strain with a higher dilation angle we get a larger dilation because our dilation angle is larger.

So, we see that with the higher dilation angle we get more dilation more volume increase and this is this volume increase is continuing. Because we have not put any limit or we have not used the strains offering type model where beyond certain yielding the soil reaches a constant volume state that we have not enforced. So, we see a continuous increase in the strain. Now let us run the same program with the suppressed dilatancy I am just calling it as SUPP.

And our K s is 10000 and then our K n stands for of eight c of 10 phi of 30 degrees and dilation angle of 20 and the normal stress is 100 and the shear strain increment is 0.001 and now I am using constant strain so that we get a stress increase. So, now we see during the elastic state our normal stress is remaining constant at 100. But then in the plastic state the normal stress will go on increasing because of suppressed dilatancy.

But once the normal stress increases the allowable shear stress also is increasing. So, at any stage your shear stress is exactly equal to c + sigma and tan phi that is what we see here at a shear strain of 0.002 the normal stress is increased to 290 and so our shear stress is also increased 277.65 that is c + sigma and tan phi. So, let me run the same program but with suppressed dilatancy but a dilation angle of 0 and let us see K s is a 10000, K n is a 10 to power of 8, c of 10, phi of 30, dilation angle is 0 and normal stress is 100.

So, here in this case I used a dilation angle of 0 and then I did not allow the soil to expand our normal strain is remaining constant. But then our normal stress is also remaining constant at 100 because there is no dilation induced by the shear strain. So, our normal stress is constant and our shear stress is also constant and the same value of 67.735. So, with this type of model we can simulate the volume expansion.

And if you prevent the volume expansion you get your higher stresses like similar to your swelling of the soil. If you allow the soil to swell freely, they will not be any soil pressure. But if you prevent the swelling from happening then you get the swell pressure. The same thing happens even with the dilation and this suppressed dilatancy can increase our shear strength that is what we have seen here.

See with increase in the normal stresses our corresponding shear stress will also increase that is what we have seen in the plastic part. Our normal stresses are increasing and the shear stresses are also increasing or the allowable shear stress. So, that leads to higher shear strength of the soil.
(Video Ends: 35:47)
So, that is a brief introduction to our elastoplastic a joint element and in next class we will examine the same equation the elastoplastic for the Prandtl-Lewis material. We will do step by step calculation so that we understand how to do these computations. I will also try to give some numerical example. So, that you appreciate what we have gone through. So, thank you very much, we will meet next time.