Course Name: An Introduction to Climate Dynamics, Variability and Monitoring

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Lecture 34

TEMPERATURE DISTRIBUTION FOR A CONTINUOUSLY STRATIFIED ATMOSPHERE IN RADIATIVE EQUILIBRIUM - Part 2

Good morning, class. And in the previous lecture, we had started our discussion on the radiative equilibrium-based evaluation of the temperature profile of the atmosphere. We had assumed that the atmosphere is transparent to all shortwave radiation. And it is diffuse, that is the radiation intensity is independent of direction. And it is gray, that is the mass extinction coefficient is independent of the frequency nu in the infrared range. Which is a reasonable approximation especially for water vapor above the 14 micrometer wavelength band. And since water vapor is the strongest absorber of long wave radiation coming from the terrestrial radiation in the troposphere, this is a reasonable approximation for the tropospheric temperature profiles. We had evaluated two ODEs connecting the net upward radiation flux intensity and the net downward hemispherical radiation flux intensity in terms of the modified optical depth tau star which is 1.66 times the vertical optical depth tau. And the black body radiation value, black body irradiance sigma T to the power 4 which is the right hand side of this expression.

$$-\frac{dF^{\uparrow}(z)}{dz} + F^{\uparrow}(z) = \Pi B(T_z)$$

$$-\frac{dF^{\downarrow}(z)}{dz} + F^{\downarrow}(z) = \Pi B(T_z)$$

$$\Pi B(T_z) = \sigma T_z^4$$

So, we have the expression for the upward flux, the downward flux in terms of the black body radiation flux at the given temperature T at the Z. So, this is Tz.

Alright, next we will show that the net flux which is the upward going flux and the downward going flux must be a constant value and independent of Z because the volumetric heating rate is the gradient of the net flux density with Z and the volumetric heating rate must be 0 if the atmosphere has reached radiative equilibrium.

$$\rho(z)\dot{Q}_{\nu}^{\mathrm{LW}} = -\frac{dF_{z,\nu}}{dz}$$

$$F_{z,\nu} = F_{z,\nu}^{\uparrow} - F_{z,\nu}^{\downarrow}$$

So, the net flux F has to be 0. So, that subtraction of the net upward total upward hemispheric flux minus downward hemispheric flux must be a constant. Next, what we did is we looked at the situation at the top of the atmosphere where the optical path is 0.

$$\frac{dF_{z,\nu}}{dz} = 0$$

So, we are putting 0 at the brackets to signify it is the top of the atmosphere. There the downward hemispherical flux in the long wave spectra is 0 because no long wave IR radiation is coming from space. So, the net upward flux is just the total upward hemispherical flux at tau star equal to 0, that is at the top of the atmosphere, which is equal to the constant. However, if the earth as a whole is in the radiative equilibrium, then the net downward shortwave radiation, here because it is net, these two we can you know. Net long wave radiation must be equal and opposite to the net short wave radiation. So, the net short wave radiation is S0 by 4 1 minus alpha. So, the magnitude of the net upward long wave radiation is must be S0 by 4 1 minus alpha. The net solar radiation that is being absorbed by the world which is around 240 watt per meter square.

$$F_z^{\mathrm{LW}} = F_z^{\downarrow} = \frac{S_0}{4}(1-\alpha) \approx 240\,\mathrm{W/m^2}$$

So,

$$F_z(\tau^* = 0) = F_0 \approx \frac{S_0}{4}(1-\alpha) \approx 240\,\mathrm{W/m^2}$$

So, at any given z or τ^*:

$$F_z(\tau^*) = F^{\uparrow}(\tau^*) - F^{\downarrow}(\tau^*) = \frac{S_0}{4}(1-\alpha) \approx 240\,\mathrm{W/m^2}$$

However, we have noted that Fz is independent of z and is equals to a constant. So, Fz at any other location z with its own modified optical depth tau star will be equal to this value s0 by 4 1 minus alpha equals to 240 watt per meter square. So, this we have evaluated previously. Now, if you go back to the two ordinary differential equation sets that we have evaluated, these two, 3 and 4, we can add these two values. So, we were adding the first differential equation with the second differential, adding 3 and 4. Let us just do that. adding 3 and 4 we get minus d d tau star F z upward tau star minus F z downward at tau star plus F z upward at tau star plus F z downward at tau star is equals to 2 pi dt. This is just the plain addition of the two terms. F z tau star equals to f upward z tau star minus F downward z tau star equals to constant. If this is a constant then there is no the gradient of this Fz tau star with tau star is 0. So the first term goes to 0 entirely. Hence Fz sorry upward tau z star plus Fz downward tau z star equals to 2 pi dt. Let us call this equation 4. This is of course, Tz equation 4. This is the black body. radiation intensity and pi BT is sigma Tz to the power 4. Remember, pi BTz goes to sigma Tz to the power 4. So, this is the expression we are getting. Implies pi BTz is equals to sigma Tz to the power 4 equals to half F upward at tau star z plus F downward. So, let us because it is the same expression I am just calling this as equation 4. Now, this is when we are adding these two and we are getting this expression that pi into the blackbody radiation intensity which is basically the blackbody irradiance sigma Tz to the power 4 equals to half of upward radiation flux density plus downward radiation flux density.

$$-\frac{d}{d\tau^*}\left[F_z^{\uparrow}(\tau^*) - F_z^{\downarrow}(\tau^*)\right] + \left[F_z^{\uparrow}(\tau^*) + F_z^{\downarrow}(\tau^*)\right] = 2\pi B(T_z)$$

$$F_z(\tau^*) = F_z^{\uparrow}(\tau^*) - F_z^{\downarrow}(\tau^*) = \text{constant}$$

$$F_z^{\uparrow}(\tau_z) + F_z^{\downarrow}(\tau_z) = 2\pi B(T_z)$$

$$\Pi B(T_z) = \sigma T_z^4 = \frac{1}{2}\left[F_z^{\uparrow} + F_z^{\downarrow}\right]$$

So, this expression we are getting. Next we will do 3 minus 4. So, let us just go back. This expression minus this expression. So, the right hand side goes to 0, correct. Let us see what happens to the rest. If we subtract, so 3 minus 4, then we get d d tau z star equals to F upward tau star z plus F downward tau star z is equal to F upward tau star z minus F downward tau star z. So, this is the differential term and this is the constant term. So, this expression is equals to Fz tau star z equals to the constant term F0 which is equals to S0 by 4 1 minus alpha. So, then this constant term F0 which is the outward going radiation flux at the top of the atmosphere which is equals to the downward going shortwave radiation flux. So, this value we know is a constant.

$$\frac{d}{d\tau^*}\left[F_z^{\uparrow}(\tau_z^*) + F_z^{\downarrow}(\tau_z^*)\right] = F_z(\tau_z^*)$$

$$F_z = \frac{S_0}{4}(1-\alpha)$$

Upon integration:

$$F_z^{\uparrow} + F_z^{\downarrow} = F_z\tau_z^* + C_1$$

At the top of the atmosphere, $\tau_z^* = 0$ and $F_z^{\downarrow} = 0$, so:

$$C_1 = F_z^{\uparrow}(0) = F_z$$

So, now we can integrate this expression. On integration, then we have, we are now dropping this tau star z terms, they are all obviously there, F upward plus F downward equals to the constant F0 term into tau star z plus C1. This is when we are integrating this expression. At the top of atmosphere tau star z equals to 0 and F downward equal to 0, correct. So, this becomes C1 equals to F upward at 0 which is equals to F0. So, at the top of that downward long wave flux is 0, tau star z term is 0, optical depth is 0. So, these two terms cancels out. So, C1 is equals to a net upward total upward hemispherical flux which is F0, this value here.

$$F_z^{\uparrow}(\tau_z^*) + F_z^{\downarrow}(\tau_z^*) = F_0(1+\tau_z^*) \qquad \text{(v)}$$

$$F_z^{\uparrow} - F_z^{\downarrow} = F_0$$

$$F_z^{\uparrow} = \frac{1}{2}F_0(2+\tau_z^*) \qquad \text{(vi)}$$

$$F_z^{\downarrow} = \frac{1}{2}F_0\tau_z^* \qquad \text{(vii)}$$

So, the final expression then becomes F upward tau star z plus F downward tau star z plus equals to F0 which is that S0 by 4 1 minus alpha term into 1 plus tau star z. So, F upwards plus F downwards at any location z or any optical depth tau star z equals to the F0 term the constant 240 watt per meter square into 1 plus the optical depth term. So, this is expression 5. F upward minus F downward is also F0. This we have already evaluated. So, you can subtract these.

So, we get on adding F upward is equal to half of because when we add this, this becomes 2 f upwards and this becomes F0 2 plus tau z star. So, half of F 0 2 plus tau z star, where F 0 is 240 watt per meter square as we evaluated before. F downwards is equals to, so again here we just put this value here and we evaluate the F downward term is equals to half F0 tau z star. Remember that tau z star equals to 1.66 tau z equals to 1.66 integral z to infinity k absorption coefficient i which here is not dependent on z or anything into rho i which is a function of z into dz. We will use dummy variables here. This is the expression for tau z star. So, we are getting the expressions of both the upward hemispherical flux and the downward hemispherical flux in terms of the optical, modified optical depth tau z star.

$$F^{\uparrow}(\tau_z^*) - F^{\downarrow}(\tau_z^*) = F_0(1+\tau_z^*) \qquad \text{(V)}$$

$$F^{\uparrow} - F^{\downarrow} = F_0 \quad \text{where} \quad F_0 = 240\,\mu W/m^2$$

$$F^{\uparrow} = \frac{1}{2}F_0(2+\tau_z^*) \qquad \text{(VI')}$$

$$F^{\downarrow} = \frac{1}{2}F_0\tau_z^* \qquad \text{(VII)}$$

$$\tau_z^* = 1.66\tau_z = 1.66\int_0^{z_2} k_{\text{abs}}^i p_i(z)dz \qquad \text{(VIII)}$$

The other expression that is useful is we have evaluated that sigma Tz to the power 4 equals to pi BT is equals to half F plus plus F, F upwards plus F downwards, but F upwards plus F downwards is F0 into 1 plus tau star. So, this is F0 by 2, 1 plus tau z star. So, pi BT z equals to sigma Tz to the power 4 equals to F0 by 2 1 plus tau z star. This we are also getting. Why is this important? Because here we have the T as a function of Z. And this is what we finally want. We want the temperature of the atmospheric layer. Correct? That is what we are trying to get. So this expression helps us to get there finally. Okay. So then temperature of atmospheric layer under radiative equilibrium is T at z equals to F0 by 2 sigma 1 plus tau z star.

$$\sigma T_z^4 = \pi B(\tau) = \frac{1}{2}(F^{\uparrow} + F^{\downarrow}) = \frac{F_0}{2}(1+\tau_z^*)$$

So:

$$\pi B(\tau_z) = \sigma T_z^4 = \frac{F_0}{2}(1+\tau_z^*) \qquad \text{(IX)}$$

The temperature of the atmospheric layer under radiative equilibrium is:

$$T(z) = \left(\frac{F_0}{2\sigma}(1+\tau_z^*)\right)^{1/4} \qquad \text{(X)}$$

Now, some simplifications. Firstly, F0 is 240 watt per meter square equals to S0 by 4 into 1 minus alpha. But S0 by 4 into 1 minus alpha equals to sigma Te to the power 4 where Te is the black body emission temperature, correct. So, we can also write F0 equals to sigma Te to the power 4 implies F0 by sigma is Te to the power 4. This expression I am sorry is to the power see, this is Tz to the power 1 4. So, now in this expression F0 by 2 sigma 1 plus tau z star to the power 1 4, this entire thing is to the power 1 4. We can put F0 by 2 sigma, F0 by sigma is T to the power 4. So, we have T of z equals F0 by 2 sigma 1 plus tau star z to the power 1 fourth or if I take Te to the power 1 fourth here then T of z by Te which is the black body emission temperature equals to 1 plus tau z star by 2 d power 1. Clear? This is the expression that we are getting in terms of the blackbody emission temperature of the earth. several things. We will discuss the Tz expression a little bit later.

$$F_0 = 240\,\mu W/m^2 = \frac{S_0}{4}(1-\alpha)$$

But:

$$\frac{S_0}{4}(1-\alpha) = \sigma T_e^4$$

Where T_e is the black body emission temperature.

So:

$$F_0 = \sigma T_e^4 \quad \Rightarrow \quad F_0 = \frac{F_0}{2} = T_e^4$$

Thus:

$$T(z) = \left[\frac{F_0}{2\sigma}(1+\tau_z^*)\right]^{1/4} \quad \text{OR} \quad \frac{T(z)}{T_e} = \left[\frac{1+\tau_z^*}{2}\right]^{1/4} \qquad \text{(XI)}$$

First thing to note is what is the temperature of the top layer of the atmosphere? There Tz is 0, correct? So, at the top of atmosphere Tz star is equals to 0. So, here Tz by Te is 1 by 2 to the power 1. So, Tz equals to T at z tending to infinity is equals to 2 to the power minus 1 4 x 3. This is called the skin temperature of the earth, skin temperature of atmosphere. This is the temperature that the top of the atmosphere will have. Now, we need some expression to evaluate tau star z. Of course, we can evaluate tau star z if we have the optical path density values. But we can do a little bit better than that with some specific expressions. So, just remember where I put the tau star z, yes. Tau star z is 1.66 tau z, correct. So, in the tau z term you have the k absorption coefficient and the density rho i. We can model The density of an absorbing species rho i at a altitude z as rho i 0, let us not call it rho i 0, let us call it rho i at the surface, rho i at the surface into e to the power minus z by h i. We have used this model in the previous case, just in the previous class where H i is the scale height for absorbing species i. This scale height will be different for different absorbing species.

So, this is just a model for our cases. tau at any altitude z is equals to tau in the ground, we have a consistency here, let us call it rho i at the ground, tau at the ground. So, this is the optical depth at sea level at the ground equals to e to the power minus z by Hi. If this expression is valid and if k is independent of z, then if you integrate this term here, so we are taking k out and then rho i is rho i at the ground into d power minus z by h i, rho i at the ground you are also taking out, so you get the tau at the ground. Remember we have already evaluated what tau at the ground is, so tau at the ground. is equal to k rho i 0 and H i, k absorption coefficient rho i at the ground and the scale height H.

$$T_1 = 2^{1/4} T_e \quad \text{at the top of the atmosphere, } \tau_z^* = 0$$

$$p_i(z) = -p_i e^{-z/H_i} \qquad \text{(XIII)}$$

$$\tau_z = \tau_g e^{-z/H_i} \quad \text{and} \quad \tau_g = k_{\text{abs}}^i p_i^i H_i \qquad \text{(XIV)}$$

$$\tau_z^* = 1.66\tau_z(e^{-z/H_i}) \qquad \text{(XV)}$$

So, we have derived this, we have set tau 0 there. Since we are just putting 0 at the top of the atmosphere for many cases, I am just replacing the values of the ground with g. So this is 40, this is 50. So then we now know the optical depth and tau star is basically 1.66 tau z. So now we can evaluate these terms here. So tau z star is 1.66 tau z and tau z is tau at the ground into e to the power minus z by H i. So, putting tau z values we get thus hemispherical flux going upwards is half of F0 2 plus tau at the ground star e to the power minus z by H i and hemispherical flux going downwards is half F 0 tau at the ground e to the power minus z by H i. So, we can call these as 16 and 17, 16 and 17 where tau g star is 1.66 tau g and we have expressed tau g before.

So, these fluxes can be explicitly plotted for various values of h i which we will show in the next class. here Tz by Te is 1 plus tau star z by 2. So, we can also write then Tz by Te equals to half 1 plus tau g star e to the power minus z by hr. the power 1. So, Tz as a function Tz by Te can be plotted with respect to z by Hi to get the normalized expression of how the temperature is varying with altitude.

$$F^{\uparrow}(z) = \frac{1}{2}F_0\left(2 + \tau_g^* e^{-z/H_i}\right) \qquad \text{(XVI)}$$

$$F^{\downarrow}(z) = \frac{1}{2}F_0\tau_g^* e^{-z/H_i} \qquad \text{(XVII)}$$

$$\tau_g^* = 1.66\tau_g$$

$$\frac{T(z)}{T_e} = \left[\frac{1}{2}\left(1 + \tau_g^* e^{-z/H_i}\right)\right]^{1/4} \qquad \text{(XVIII)}$$

And then if you know the individual scale heights for the various absorbing species, you can put those. Remember, if you have multiple species, you have to sum up the absorption coefficients of all to get the total temperature or the total energy balance and then put it to get the final temperature. So, this is a simple case of a single absorbing species HI. So, the F upwards, F downwards and T are evaluated analytically using this simple set of expressions. We will stop here today. In the next class, we will just look at some of the plots, some of the physical principles and start from there. Thank you for listening and see you in the next class.