Course Name: An Introduction to Climate Dynamics, Variability and Monitoring

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INTRODUCTION TO PRINCIPLES OF MEASUREMENT AND INSTRUMENTATION - PART 2

Good morning class and welcome to our continuing lectures on climate modeling, climate variability, climate dynamics and climate monitoring. In the previous lecture, we started our discussion on climate instrumentation and measurements, and we were discussing the general principles measurements. We discussed some of the ideas about instrument response, instrument sensitivity, instrument resolution, instrument calibration, instrument dynamic range. Today, we will continue our discussion on understanding the measurement quality of an instrument. It is how reliable the instrument is in measuring the physical parameter that it is measuring. Every measuring instrument has some uncertainty, which is previously called errors.

Uncertainty is defined as the variability in output of an instrument on repeated measurement of the same physical parameter value. So, for example, suppose you are measuring the temperature of boiling water at one atmosphere. We know that its temperature will be 100 degrees centigrade exactly. What is the instrument measuring if it takes a 100 repeated temperature samples of boiling water? We know the fixed input; we look at the output and the spread will be the uncertainty.

So, this is one way, it is called physical standardization of the measurement. Another way is to compare the instrument's output with respect to a better instrument, a calibration standard whose uncertainties are low and about which we have very high confidence that its values are accurate. When we do this, we see two types of uncertainties. The first type is systemic uncertainty, which is a consistent repeated offset in the measurement as a result of a fixed and regular discrepancy in the instrument response. So, the values that we are getting are consistently of corresponding to the real physical value by a certain amount and this amount is not changing from one measurement to another measurement.

The calibration process can help quantify such systemic uncertainties and they can be offset in further post-processing. The other type of uncertainty is random uncertainty. These are variations in measurements due to statistical fluctuations in either the quantity being sensed or the internal operations of the instrument. So, it can either be small random fluctuations in the actual physical parameter that is being measured or the internal operations that are statistical fluctuations within the internal operations of the instrument. These are random in nature and hence they are called random uncertainties.

Random uncertainties can be quantified and you will see the quantification process in this class and reduced by averaging procedures. So, in this context then we can define something called instrument precision and instrument accuracy. Precision is defined as the ability of the instrument to give the same output value for a given input parameter. So, an instrument is precise if on repeated trials it is able to give the same output value for the same value of input parameter. So, if we are keeping the input parameter constant, The instrument is giving a constant output value.

Then the instrument is precise. More practically, the output is within a narrow band of statistical variables. Usually, the same output value will not be achieved. However, it will be within a very narrow band of statistical variables. So, precision requires that the systemic uncertainty remain constant during repeated measurement processes.

So, if the systemic uncertainty remains constant, then if the physical parameter is not changing, then the actual output value will not vary as well. So, that makes the instrument precise. Okay. However, note that a precise instrument may not give the physical variable value correctly. Suppose you have a systemic uncertainty.

Suppose your temperature measurement has a systemic error or systemic uncertainty of 2 kelvins. Then, even though you are measuring water boiling at 1 atmosphere, whose temperature is 100 degrees centigrade, it is consistently giving you 102 degrees centigrade. It is precise because it is giving the same temperature value in say 50 to 100 rounds, but it is not accurate. So, how do we quantify accuracy? Accuracy is the measure of the overall deviation or uncertainty in the output value of the parameter measured compared to the actual value. So, accuracy is the overall measure of the deviation or uncertainty that the actual that the output value has compared to the actual value of the parameter that is being measured.

Measurements by a calibration standard is used as a stand-in for the actual parameter value during accuracy determination. So, you have a good calibration standard whose value we already know matches well with the actual physical parameter value. Then those values and the values you are getting from your instrument, become the two series that are compared to evaluate the overall accuracy of your instrument. It is determined by a combination of systemic and random uncertainties in the instrument. We have already seen that if you have a systemic uncertainty you have to identify that and there are statistical means by which you can also evaluate the extent of random uncertainty, both should be low in an accurate instrument.

So, an instrument is accurate if its systemic uncertainty is low and random uncertainty is low. Accuracy is reported either as an absolute accuracy, that is say for example in a pressure measuring device, it can be reported as ±0.1 kilopascals. So, whatever it is measuring, it should be accurate by ±0.1 kPa or as relative accuracy, ±0.1 percent of the measured value. So, depending on the instrument type, one measure is used in some instrument, another measure is used in other instruments. So, it can be 0.1 percent of the value being measured, plus minus or an absolute value. Note that an accurate instrument must be precise, its systemic uncertainty will be low or should be constant, but a precise instrument need not be accurate.

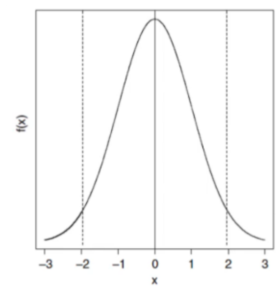
Now, how do we quantify random uncertainty? That is the next question. Let us assume that we have already evaluated the systemic uncertainty and we have taken that out of the picture. So, now we are left with only random uncertainty. These uncertainties are caused by stochastic fluctuations in the quantity itself or noise in the measuring instrument. So stochastic fluctuations in the quantity can be causing it or it can be caused by random noise that is generated within the instrument processes.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right)$$

Then the measured output variable values x will have a normal probability distribution around the parameter mean μ which is the actual value that the parameter has, but the measured values will be normally distributed around this μ with a standard deviation of sigma. So this gives you the probability density function $f(x)$. What $f(x)$ is giving is the probability of measuring the variable as having a certain value x, okay, where the mean value or the expected value is μ. What is the probability of finding the value to be x? All right. And that is given by $\frac{1}{\sigma\sqrt{2\pi}}$, where sigma is a standard deviation of this PDF, $exp\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right)$, and this is the probability distribution in such cases.

Here x is basically you can think of is a normal distribution product for mean x of 0. Suppose we are measuring say melting ice. Ice melting at 1 atmosphere is 0 degree centigrade. The mean mu is 0. And what is the probability of finding its temperature to be 1 degree centigrade, 2 degrees centigrade or -1 degree centigrade, -2 degree centigrade, something like that.

This will be given by a normal distribution. So, finding it as -2 degree centigrade, the probability is this much. whether this is the probability of finding it as exactly 0 degree centigrade is of course much higher. If the values of the mean are not 0 as is the case, this x can be μ, then it will be $\mu - 1$, $\mu + 1$, etcetera. Here, remember that the standard deviation is 1.



Normal distribution function $f(X)$, plotted for a mean X of zero and unit standard deviation σ. The dashed vertical lines enclose 95% of the values $\pm 1.96\sigma$

So, we have basically normalized this solution. We have taken the mean to be 0 and the standard deviation is 1. Of course, actual variable will have a certain non-zero mean and a certain non-unit sigma. Then the corresponding values will be plotted, the dashed vertical lines enclose 95 percent of the probability.

So, ±1.96 sigma is the range within which, there is 95 percent chance that the instrument will be measured. As long as we have eliminated all systemic uncertainties. Given that the stochastic PDF follows a standard deviation sigma, then ±2 sigma is the range within which it is probable that 95 percent of the measured values will lie, if it is a normal distribution. Suppose the instrument takes N measurements. Then the mean of these measurements called the sample mean, we can call as $\bar{X}$ is 1 by $\frac{1}{N}\sum_{i=1}^{N} X_i$.

$$\bar{X} = \frac{1}{N}(X_1 + X_2 + \cdots + X_N) = \frac{1}{N}\sum_{i=1}^{N} X_I$$

Suppose we take a 100 measurements of this melting ice. Then the sample mean, the mean temperature as obtained from these 100 measurements is 1 by 100 summation of all the temperatures. So the question that is asked is how far away is the sample mean x bar from the true mean μ? That is the question. Is the sample mean also 0 degree centigrade or how far away is the expected value of the sample mean from the true mean μ? Alright, that is the question. How do we go about evaluating this? Let us take M sets of repeated N samples using the instrument.

What does this mean? Suppose again we are measuring boiling water. 100 degrees centigrade is the expected value. So, μ is 100 degrees centigrade. We take 100 temperature measurements of this boiling water and we take 50 such sets of 100 temperature measurements. So, 15 to 100, 5000, 5000 measurements in sets of n.

Each individual 100 set, each individual set contains 100 samples of temperature data and you get a corresponding sample mean $\overline{X_J}$. where j is varying from 1 to m. So, each set will have its own sample mean. There will be 50 such sample mean temperatures, X_1, X_2, X_3 up to X_{50}. Since the error is stochastic and the measurements are independent, the distribution of the sample means about the true mean will also follow a normal distribution.

This is the key point. Since the error in measurement is stochastic and the measurements are independent of each other, the sample mean probability distribution will also follow a normal distribution with the true mean being μ. The standard deviation of the sample means about the true mean is given by σ_m. So, the actual case the standard deviation was sigma. The standard deviation will be different. It will be the standard deviation of the sample mean PDF about the true mean.

And this σ_m is given by $\frac{\sigma}{\sqrt{N}}$. So, note this again is a theory we will not derive here. The standard deviation of the PDF of the sample means is sigma by root n where n is the number of samples taken in each set. So, if there is 100, Then σ_m is $\frac{\sigma}{\sqrt{100}}$, so sigma by 10.

Notice what is happening. The standard deviation has reduced significantly. If the original standard deviation was ±1 degree centigrade, because you have taken 100 samples, the standard deviation for the sample means is ±0.1 degree centigrade. Here sigma is the standard deviation of the variable period. Now, the question is how do we know the standard deviation of the variable period? Theoretically, we can show that it can be modeled as being equal to the standard deviation of that sample s.

So, we have a sample of 100 temperature points. The standard deviation of this sample which we call s is $\left\{\frac{1}{(N-1)}\sum_{I=1}^{N}(X_I-\bar{X})^2\right\}^{\frac{1}{2}}$. So, we can find the standard deviation of our measured sample. That standard deviation s can be said to be equal to the standard deviation of the random error sigma. As a result, we can then write that σ_m, the standard deviation of the sample mean period is almost equal to the sample standard deviation by root n.

Now that we know this, because it's a PDF, the true mean has a 95% chance of lying within x bar ±1.96 σ_m. correct. So, the true mean mu has 95 percent probability because it is a normal deviation distribution applying between ±1.96 σ_m. So, if the original center deviation was 1 degree centigrade in the sample μ, in the and you have taken 100 samples, then the σ_m is 1 by root 100 or 1 by 10 or 0.1 degree centigrade. So, your true mean should lie between x bar ±1.96 into 0.1 degree centigrade or approximately 0.2 degree centigrade. This is conventionally adopted as the random uncertainty estimate of the reported results. So, when we say that the random uncertainty of a measuring instrument is say ±0.01, that 0.01 is basically 1.96 into this sigma n value. So, that way then we can evaluate the measurements. It is important to note here that if you increase the number of samples, this expression decreases. And the other point to note is the sample standard deviation is kind of the intrinsic measure of the accuracy of your instrument. Because if the original sample standard deviation is low, then your sigma m will be low even for a small number of samples taken.

Otherwise, it will be almost equals to S. So, if you take one measurement, then n is 1. So, sigma m is equal to S. Clear? All right. Now, the point is, you have to combine these uncertainties together. So, for example, you can have another parameter Z, which is a combination, either some product, a division or a power of the actual measured samples.

So, for example, if you think of the ideal gas law, T equals to rho RT, and you are trying to find the density rho while you have measured pressure and temperature. So, rho is p by RT. You are measuring pressure and you are measuring temperature. So, rho z is equals to its proportional constant is of course constant is basically pressure measurement value by temperature. Now the question is, how do the uncertainties in the pressure measurement value and the temperature measurement value be combined to give you the estimate of the uncertainty of the density that you are calculating out of it? That is one thing.

Similarly, many instruments do it directly inside of them, that they measure two or three physical parameters, then combine them to give us the physical parameter we want. In that case also, the uncertainties in the measured parameters have to be combined appropriately for the instrument to tell us what is the uncertainty it is actually giving us. So, here I have not derived this. This is not the class to do that. If z is a plus minus b, then the uncertainty in z is equals to root over of the square of the uncertainty in A plus square of the uncertainty in B.

Summary of methods for estimating the resultant uncertainty for common combinations of variables *A* and *B*, each with individual uncertainties ΔA and ΔB

Functional relationship	Resultant uncertainty or fractional uncertainty in *Z*
$Z = A \pm B$	$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$
$Z = AB$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
$Z = \frac{A}{B}$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
$Z = A^n$	$\left(\frac{\Delta Z}{Z}\right) = n\left(\frac{\Delta A}{A}\right)$

Remember, delta A and delta B are basically this one, 1.96 σ_m for each of those individual measurements. So, it is the sum root, sum of the squares root of that. If z is a by b, then the fractional uncertainty becomes important, del z by z square is equals to sum of the fractional square of the fractional uncertainty between a and b, delta a by a square plus delta b by b square. This is also true when z equals to a by b.

So, that is your answer here. And when z equals to a to the power n, then delta z by z equals to n times delta a by a So, the power is particularly vulnerable of being increasing the uncertainty values. So, this kind of gives a snapshot of the uncertainties. We will start the instrument response times today as well and then we will continue in the next class. Now, what is instrument response time? It is the time it takes for the instrument to respond to a step change in the physical parameter. So, this you can see here that the physical parameter suddenly changed from 0 to 1 value at time t equal to 0.

Now we have three instruments, a fast responding instrument that rapidly rose as a response to this physical parameter value and went up to near the value of 1 by time 1 as moderately responding instrument takes more time, around time equals to 4, while a slow responding instrument may take up to 10 time, 10 seconds. So, this is a unit, 10 seconds to go to the instrument, the new physical parameter value. So, signal abruptly changes from x_0 to x_a at time t equal to 0. The rate of the change of the instrument output x, $\frac{dx}{dt}$, how rapidly the instrument output is changing with time is given as $-\frac{x-x_0}{\tau}$. the response at the given time minus the original response x_0 by τ where, τ is the exponential response time.

This dx dt is negative because as you can see the slope slowly decreases with time. The rate of change of the output value with time slowly decays as it reaches steady state. If we integrate this, we get the time varying instrument response as x which is a function of time equals the actual new parameter value x_a plus the difference between the old parameter value and the new parameter value $(x_0 - x_a)\exp\left(-\frac{t}{\tau}\right)$, exponentially decaying at t equals to tau, so x at $\tau{-}x_0$. So, this is the change in the actual instrument response from its time t equal to 0 value is given as $(x_a - x_0)(1 - e^{-1})$

$$x(t) = x_a + (x_0 - x_a)\exp\left(-\frac{t}{\tau}\right)$$

You can evaluate this directly also. So, this is equals to 0.63 into $(x_a - x_0)$. So, at the instrument response time value tau, the instrument has reached 63 percent of the total change it has to get to properly record the actual change in the physical parameter. And at t equals to 3τ, $x(3\tau) - x_0$ is of the order of 0.95. So, 95 percent of the change has already been registered. So, if the response time is tau, it is actually takes around 3τ seconds for the instrument to actually go up to the new parameter value. If tau is small, the device responds fast to step changes in the parameter value and vice versa. Hence, response time is an important parameter when it comes to various instruments. We will look at those things later in the class.

So, we will stop here today. We will continue the instrument response time in the next class and also start discussing how to record the physical parameter values. Thank you for listening.