Underground Space Technology
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Module No # 04
Lecture No # 17
Elastic Stress Distribution around Circular Tunnels – 02

Hello everyone, in the previous class, we started our discussion on elastic stress distribution around circular tunnels. And I told you that what is the geometry of the problem? What are the loading conditions, material properties, and the boundary conditions? And then we started deriving using the Airy's stress function approach and we were in the middle of the application of the boundary conditions. We already applied the boundary conditions at r tending to infinity.

So, let us continue from the previous class, we have 2 more boundary conditions which is at r = a, that is the radius of the tunnel. So, what happens when we excavate? There is a stress-free boundary. So, at the periphery of the tunnel that is at r = a, there will not be any stresses. So, therefore, the radial stress as well as the shear stress, is going to be equal to 0. So, let us apply that and then try to get the expression for various stress components.

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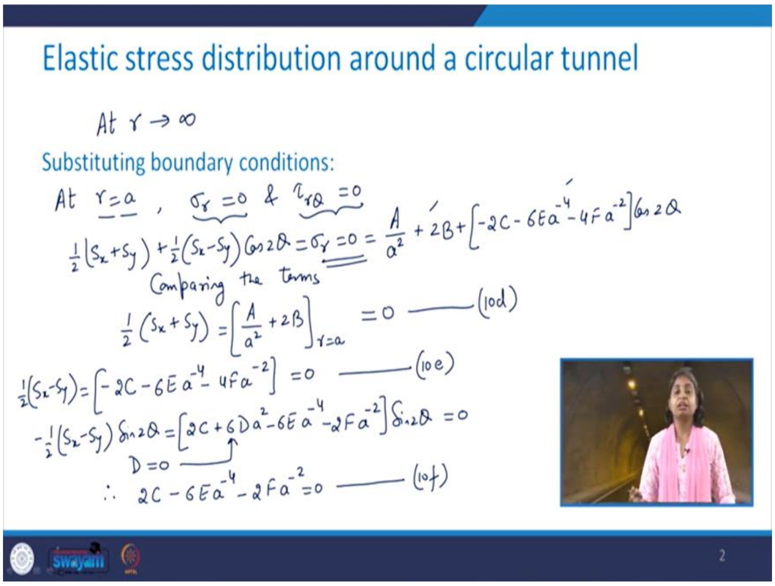

So here in the previous class, we applied the boundary conditions at r when it was tending to infinity. So, we have now that is at r = a; we have the stress-free boundary that is $\sigma_r = 0$ and $\tau_{r\theta}$ is also equal to 0. Let us apply to this that is σ_r to be equal to 0. We have:

$$\frac{1}{2}(s_x + s_y) + \frac{1}{2}(s_x - s_y)cos2\theta = \sigma_r = 0 = \frac{A}{a^2} + 2B + [-2C - 6Ea^{-4} - 4Fa^{-2}]cos2\theta$$

Now, if you are comparing the terms on both side of the equation, so what we will get is:

$$\frac{1}{2}(s_x + s_y) = \left[\frac{A}{a^2} + 2B\right]_{r=a} = 0 \qquad 10(d)$$

This is going to be equal to this is going to be equation 10d. So, these equation numbers are in continuation with the previous class. So, then the other term is going to be:

$$\frac{1}{2}(s_x - s_y) = [-2C - 6Ea^{-4} - 4Fa^{-2}] = 0 \qquad 10(e)$$

So individually, both the terms have to be equal to 0 if this equation is to be satisfied. So, I will write this as equation 10e.

So, we have another condition that is $\tau_{r\theta}$ to be equal to 0. So, we do the similar exercise with reference to that, so what we are going to get is:

$$-\frac{1}{2}(s_x - s_y)sin2\theta = [2C + 6Da^2 - 6Ea^{-4} - 2Fa^{-2}]sin2\theta = 0$$

Now, we have already seen that this D to be equal to 0. In the previous class, so if we just substitute it here. what we get is:

$$2C - 6Ea^{-4} - 2Fa^{-2} = 0 \qquad 10(f)$$

I mark this equation as 10f. So here we have got the equations from 10a to 10f. So, let us try to see that how we can determine the different boundary conditions here.

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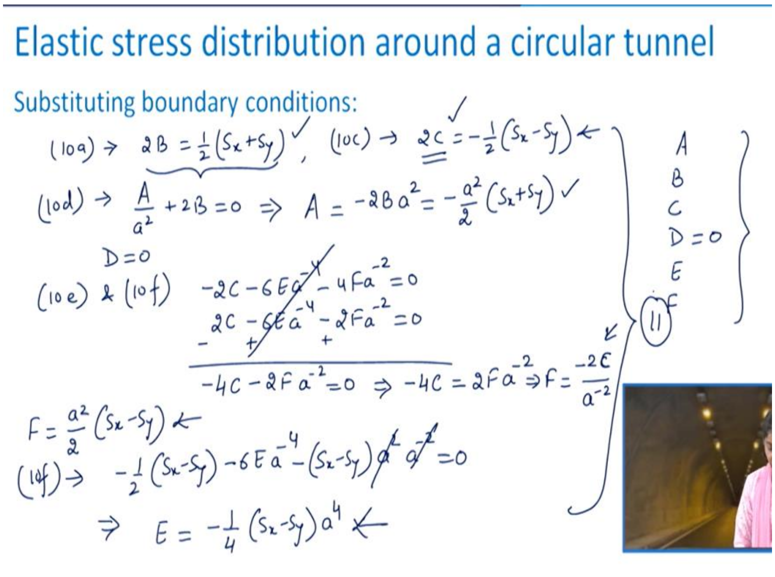

So, this equation 10a gave us that 2B = half (s_x + s_y), then equation 10c gave us 2C equal to - half (s_x + s_y). Then 10d was (A upon a^2) + 2B to be equal to 0, and from here, you can determine the constant A in terms of the constant B, which you can obtain from here. So, this is going to be – 2B into a^2, and then 2B here you just substitute this so $-a^2$ upon 2 $(s_x + s_y)$.

Then, we have D equal to 0 already. Now, if you add equation 10e and 10f together or if you combine these together, see what will happen. So 10e was - 2C– $6Ea^{-4}$ $-4Fa^{-2}$ equal to 0. And then you had 2C + $6Ea^{-4}$ $-2Fa^{-2}$ to be equal to 0, just subtract it so this will become minus, this is plus, and this will get cancelled. So, this is going to be – 4C – $2Fa^{-2}$ to be equal to 0.

And from here, what we can get is 4C = $2Fa^{-2}$, or you can write as – 4C = $2Fa^{-2}$. Or in other words, we can write it as F = -2C upon a^{-2}, or we can write as 2C is this. So, we can write this F to be equal to a^2 upon 2 $(s_x + s_y)$ just substitute 2C to be equal to minus half $(s_x + s_y)$. Here, in this expression and this is what that you are going to get now from equation 10f.

Now, what we have is minus half $(s_x + s_y)$ - $6Ea^{-4}$ and then $-2Fa^{-2}$. So, this is going to be – $(s_x + s_y)$ a^2 into a^{-2}, this is going to be equal to 0. So, this and this will get cancel out, and from here, what you are going to get as the expression for E as -1 upon 4 $(s_x + s_y)$ a^4. So, if you just take a look at this slide, you will get to know that we already have got A by this expression, then B by this expression, C by this expression, D is already equal to 0.

Then, we got E by this expression, and F using this expression. So, this is how using the boundary conditions, we could find all the arbitrary constant. So now, once I have the expression for all of these, substitute it back and then try to get the expression for stresses. So, I just what I will do is will mark this complete set as equation number 11.

$$\left.\begin{aligned} 2B &= \frac{1}{2}(s_x + s_y) \\ 2C &= -\frac{1}{2}(s_x - s_y) \\ D &= 0 \\ -4C - 2Fa^{-2} = 0 \Rightarrow\ -4C &= 2Fa^{-2}\ \Rightarrow F = \frac{-2C}{a^{-2}} \\ F &= \frac{a^2}{2}(s_x - s_y) \\ -\frac{1}{2}(s_x - s_y) - 6Ea^{-4} - (s_x - s_y) &= 0 \\ E &= -\frac{1}{4}(s_x - s_y)a^4 \end{aligned}\right\} \quad 11$$

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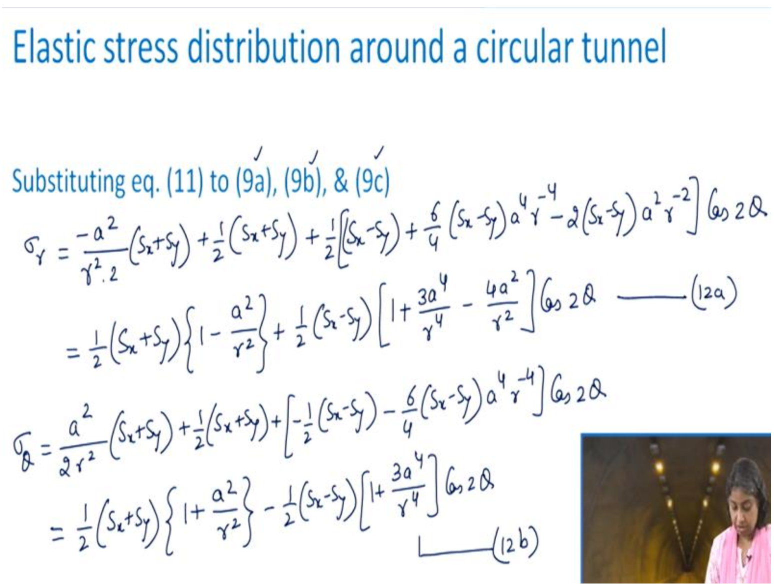

So, if I substitute this equation number 11 to earlier equations 9a, 9b, and 9c, this is what that we are going to get as σ_r to be equal to:

$$\sigma_r = -\frac{a^2}{2r^2}(s_x + s_y) + \frac{1}{2}(s_x + s_y) + \frac{1}{2}\left[(s_x - s_y) + \frac{6}{4}(s_x - s_y)a^4r^{-4} - 2(s_x - s_y)a^2r^{-2}\right]cos2\theta$$

$$\sigma_r = \frac{1}{2}(s_x + s_y)\left\{1 - \frac{a^2}{r^2}\right\} + \frac{1}{2}(s_x - s_y)\left[1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right]cos2\theta \qquad 12(a)$$

I will mark this equation as equation 12a. Now similarly, we can substitute this equation to the respective equation for σ_θ. So, I am going to get in this particular manner that is:

$$\sigma_\theta = \frac{a^2}{2r^2}(s_x + s_y) + \frac{1}{2}(s_x + s_y) + \left[-\frac{1}{2}(s_x - s_y) - \frac{6}{4}(s_x - s_y)a^4r^{-4}\right]cos2\theta$$

$$\sigma_\theta = \frac{1}{2}(s_x + s_y)\left\{1 + \frac{a^2}{r^2}\right\} - \frac{1}{2}(s_x - s_y)\left[1 + \frac{3a^4}{r^4}\right]cos2\theta \qquad 12(b)$$

This will be equation number 12b. Now, we have another equation which is for $\tau_{r\theta}$.

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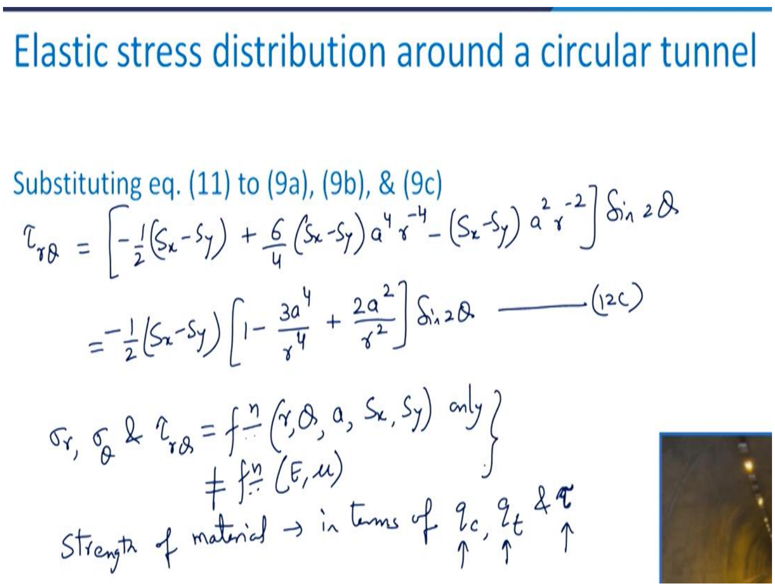

So, let us do that as well so we have this $\tau_{r\theta}$ will be equal to:

$$\tau_{r\theta} = \left[-\frac{1}{2}(s_x - s_y) + \frac{6}{4}(s_x - s_y)a^4r^{-4} - (s_x - s_y)a^2r^{-2}\right] sin2\theta$$

$$\tau_{r\theta} = -\frac{1}{2}(s_x - s_y)\left[1 - \frac{3a^4}{r^4} - \frac{2a^2}{r^2}\right] sin2\theta \qquad 12(c)$$

Mark this equation as equation number 12c.

Now, kindly note here that σ_r, σ_θ, and this $\tau_{r\theta}$; they all are the function of r, θ, a, s_x, and s_y only. What I mean here? That they are not the function of E and μ, that is the elastic modulus and Poisson's ratio of the rock. So, what does this mean? That whether these stresses are independent of the elastic properties of the medium that is what it shows mathematically, but what happens is?

It is not that these stresses are independent of the material properties. They come in picture, in an indirect fashion, so the strength of the material is defined in terms of q_c, q_t, and τ. So, this q_c is UCS, q_t is the tensile strength, and τ is the shear strength. So, how these really affect if it is not showing mathematically in the expression of σ_r, σ_θ, and $\tau_{r\theta}$.

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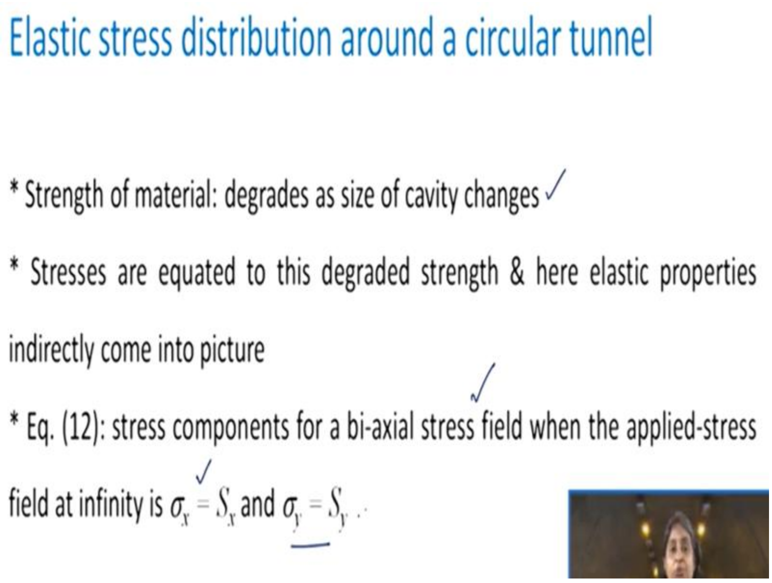

Let us see that this strength of the material it degrades as the size of the cavity changes. These stresses are equated to the degraded strength and hence the elastic properties. They come into picture in an indirect manner, so it is not that the stresses are not related to the elastic properties of the rock medium. Now, these equations 12a, 12b, and 12c. They give us the stress components for the biaxial stress field where the applied stress field at the infinity was taken as σ_x to be equal to s_x and σ_y to be equal to s_y.

So, we have now the general form of the biaxial stress field. Let us try to understand in more detail that what happens in case if you have the uniaxial state of stress or the hydrostatic state of stress. So, these 2 states of stresses, they are going to be the particular cases of the general biaxial stress field.

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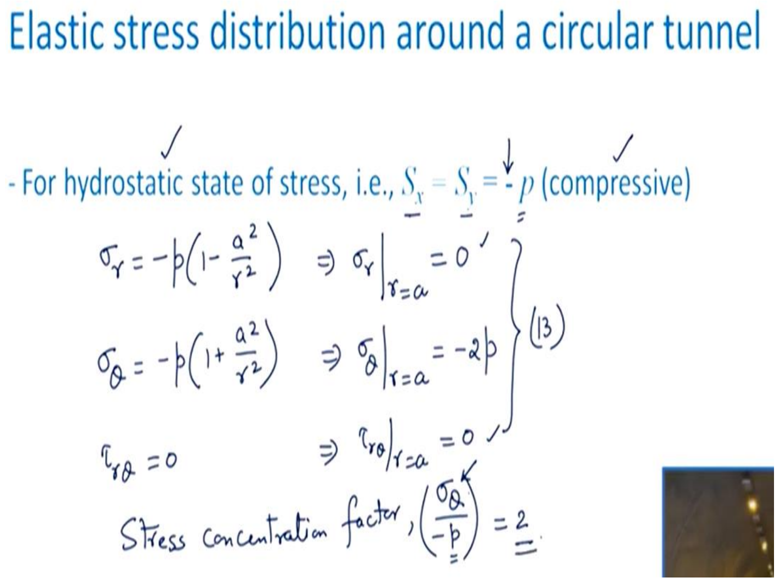

Take a look here first, for the hydrostatic state of stress, that is, in this case, your s_x and s_y they both will be equal, and when they are compressive. I will just put a negative sign with their magnitude. Because if you recall, in the previous class, I mentioned to you that all these are generated with tension as positive. So, when it is compressive there is going to be a negative sign associated with that.

So just substitute the values of s_x and s_y, which is equal to -p in equation number 12, and you will get the state of stress for hydrostatic state of stress. So, you will have σ_r as - p (1 – a^2 upon r^2), and if you just substitute σ_r, which is at r = a, that means just substitute r to be equal to a. So, this is what that you are going to get as 0, then σ_θ will be –p (1 + a^2 upon r^2).

And if we want to find out σ_θ at r = a, what we have is - 2 times the value of in-situ stress, that is, p. Now, $\tau_{r\theta}$ is going to be equal to 0, and $\tau_{r\theta}$ of course at r = a will be equal to 0. So, all these three equations, I mark it as equation number 13.

$$\left.\begin{aligned} \sigma_r &= -p\left(1-\frac{a^2}{r^2}\right) \Rightarrow \sigma_r|_{r=a} = 0 \\ \sigma_\theta &= -p\left(1+\frac{a^2}{r^2}\right) \Rightarrow \sigma_\theta|_{r=a} = -2p \\ \tau_{r\theta} &= 0 \Rightarrow \tau_{r\theta}|_{r=a} = 0 \end{aligned}\right\} \quad 13$$

Now, I define the stress concentration factor, which is σ_θ upon -p. Now basically, this stress concentration factor, in general its definition, is going to be any particular stress and its ratio with the in-situ stress that is p.

Let us say if I want to find out the stress concentration factor with respect to σ_r, so that is going to be σ_r upon p. So, in this case, this stress concentration factor with respect to σ_θ this works out to be equal to 2. So, for the hydrostatic state of stress, see what happens at the tunnel periphery; σ_r = 0, $\tau_{r\theta}$ = 0, and the stress concentration factor σ_θ upon p with a negative sign, of course, is equal to 2.

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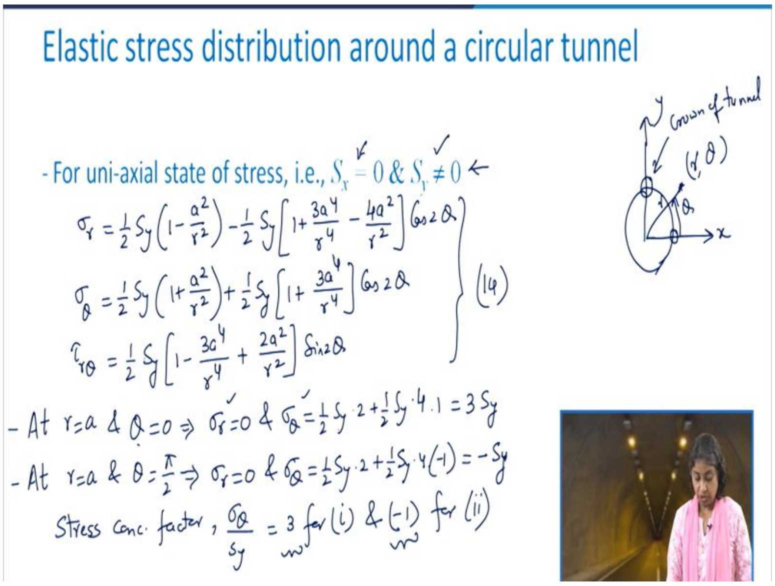

Now, coming to the uniaxial state of stress, so in this case, I have the applied stress only in y-direction, and in x-direction, this is equal to 0. That means the stresses are applied only in the vertical direction. So, what is going to be the expression for σ_r, σ_θ, and $\tau_{r\theta}$. Just substitute these values in the equation 12. So, this is what that we are going to get:

$$\left.\begin{aligned} \sigma_r &= \frac{1}{2}\, s_y\left(1-\frac{a^2}{r^2}\right)-\frac{1}{2}\, s_y\left[1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right]cos2\theta \\ \sigma_\theta &= \frac{1}{2}\, s_y\left(1+\frac{a^2}{r^2}\right)+\frac{1}{2}\, s_y\left[1+\frac{3a^4}{r^4}\right]cos2\theta \\ \tau_{r\theta} &= \frac{1}{2}\, s_y\left[1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right]sin2\theta \end{aligned}\right\} \quad 14$$

So, this equation I call as equation number 14. Now, what will happen at r = a, which is at the tunnel periphery, and θ to be equal to 0, that means along the x-axis. So here, just substitute it, so this first term will become equal to 0, and this cos0 will be there; that will give me σ_r to be equal to 0, and your σ_θ will be equal to just substitute. So that is going to be half s_y into 2 + half s_y. This is going to be you substitute r to be equal to a. This is going to be 4, and then cos of 2θ = 0. So, we have here as 1, this is going to be equal to this is S_y and $2S_y$.

This will become $3S_y$, now that is the first position that where we want to find out σ_r, σ_θ. Then the second location is at r = a, and θ is equal to π by 2. So, remember, these were the axis x and y-axis, and we had the circular tunnel. So, this was any point here, and so this was represented by r, θ; r was this distance, and θ was this angle. So, when I say θ to be equal to 0, this means it is this location and for θ = 90 degrees is this location.

So, this is typically the crown of the tunnel, so is just substitute the values what you will get is again σ_r to be equal to 0 and σ_θ, you will get as half s_y into 2 + half s_y into 4 into -1 because here θ = 90 degrees. So, you will have that as to be -1 so this is going to be - S_y. So, in this case, the stress concentration factor which is σ_θ upon s_y that is equal to 3 for the first condition and -1 for the second condition.

$$At\, r = a \;\; \& \; \theta = 0 \;\; \Rightarrow \; \sigma_r = 0 \;\; \& \, \sigma_\theta = \frac{1}{2} s_y \times 2 + \frac{1}{2} s_y \times 4 \times 1 = 3\, s_y$$

$$At\, r = a \;\; \& \; \theta = \frac{\pi}{2} \;\; \Rightarrow \; \sigma_r = 0 \;\; \& \, \sigma_\theta = \frac{1}{2} s_y \times 2 + \frac{1}{2} s_y \times 4 \times (-1) = -\, s_y$$

Now note that here it is minus, and here this is positive only. So, what does this signify?

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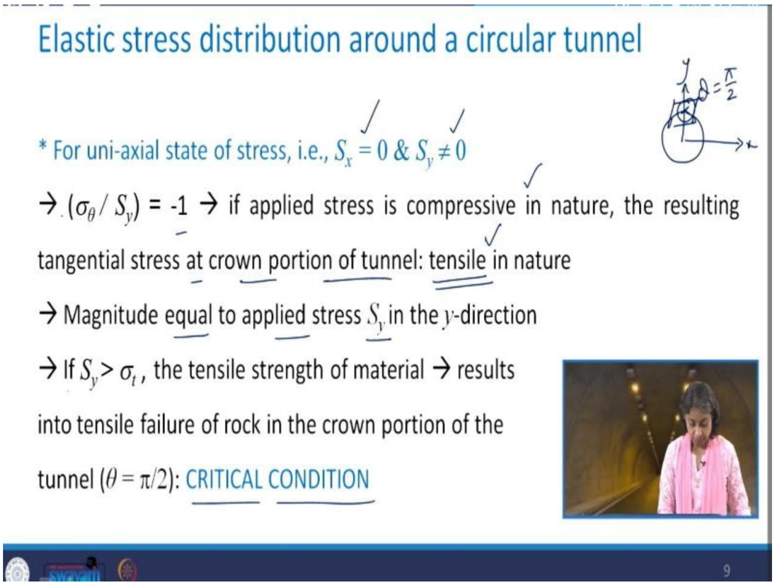

So, for the uniaxial state of stress, which is represented by s_x = 0 and s_y non-zero, we saw that sigma theta upon s_y works out to be -1. So, if the applied stress is compressive, the resulting tangential stress at the crown portion of the tunnel is going to be tensile in nature because there is the change of the sign. So, if it is here positive, this is here negative. We need to be careful here that we have the tensile stresses in the crown portion of the tunnel.

Now, the magnitude of this is equal to the applied stress in the y-direction but then equal and opposite magnitude. If s_y is greater than σ_t which is the tensile strength of the material, what will happen? This will result into the tensile failure of the rock in the crown portion of the tunnel, which is this position this is y-axis, x-axis, and this is the crown position where θ equal to π by 2.

So, this becomes a critical condition. So, you should keep in mind that the uniaxial state of stress. In case you have the tunnel subjected to only the vertical stresses, in that case, the crown portion of the tunnel, there is going to be the presence of the occurrence of tensile stresses, and that makes that portion very critical. If this stress becomes higher than the tensile strength of the rock and you know that the tensile strength of the rock is less than the uniaxial compressive strength.

So, we need to be extremely careful that the applied stresses should not become more than the tensile strength of the material. And if it is becoming, that means the tensile failure is going to occur, and we need to be careful, especially here in the crown portion where such a condition can occur.

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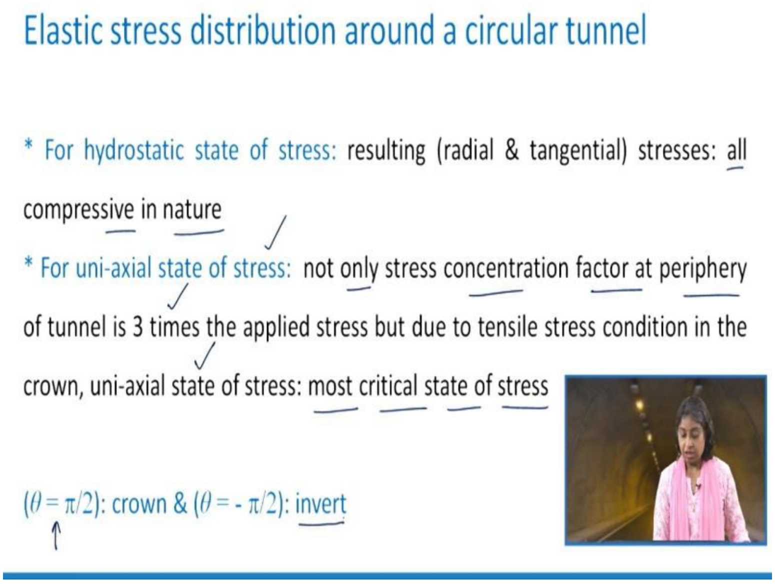

Now, in case if you have the hydrostatic state of stress resulting stresses, both radial as well as the tangential stresses, we have seen that they were all compressive in nature. So, for the uniaxial state of stress, it is not only the stress concentration factor at the periphery of the tunnel is 3 times the applied stress, but due to the tensile stress condition in the crown portion, this uniaxial state of stress becomes a most critical state of stress. Please keep this in mind.

Now, when I say crown portion means essentially, I am referring to theta equal to π by 2, and later on, I will also use the term invert. So, invert of the tunnel corresponds to theta equal to -π by 2.

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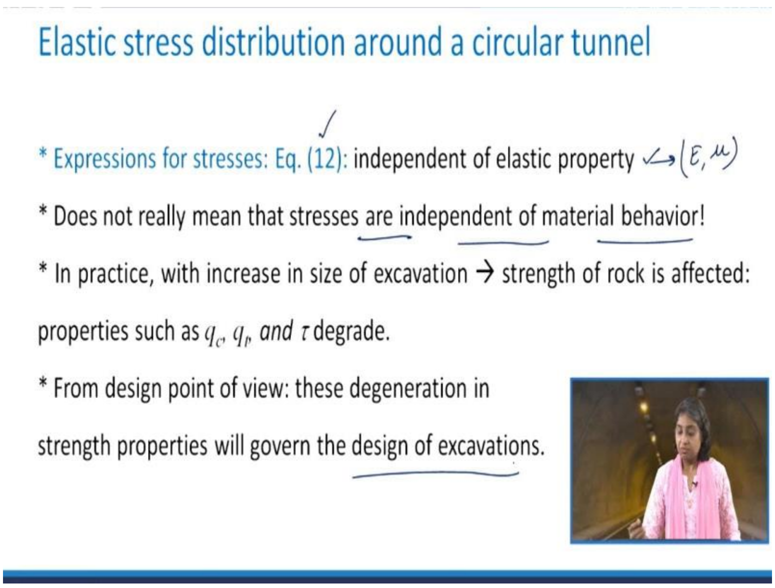

Now, these expressions for these stresses which are given by equation number 12. They are independent of the elastic property, which was E and μ which we saw earlier. So, this does not mean that the stresses are going to be independent of the material behavior that happens in practice that when you increase the size of the excavation, the strength of the rock is influenced.

And the mechanical properties, such as its uniaxial compressive strength, tensile strength, and the shear strength they degrade. So, from the design point of view, these degraded values in these strength characteristics, these govern the design of excavation. So indirectly, that is how the properties elastic properties come into the picture.

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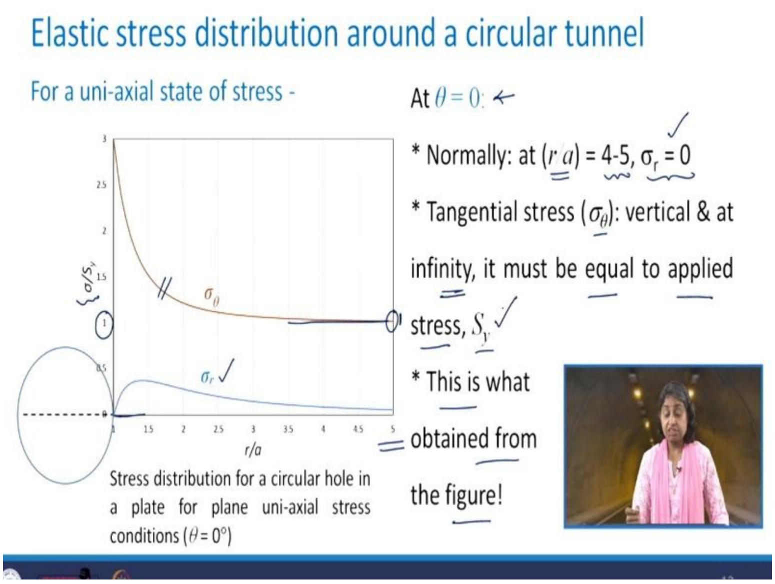

Now, let us take a look at the stress distribution. As of now, we have seen the various expression for a uniaxial state of stress as well as a hydrostatic state of stress. So, how does this look like? How does its variation look with the increase in the distance from the periphery of the tunnel?

So, we are going to discuss that so normally, at theta equal to 0, this means it is this axis that we are talking about.

Normally, at r by a when it is very large, that means to the extent of 4 to 5, this sigma r becomes equal to 0. That is what that you see when I plot the variation of σ_r. So, you see, normally at this, it is very small negligible for all practical purposes. We can consider this to be equal to 0. The tangential stress, which is the σ_θ, and the variation has been shown by this curve.

We can consider this to distribution as of now. We have seen the various expression for uniaxial state equal to 0, the tangential stress, which is the σ_θ, and the variation has been shown by this curve at the vertical and infinity. It must be equal to the applied stress because that is what is the boundary condition is, and you see that this is what that we obtain from this figure that when it goes beyond.

So, you see that here this stress concentration factor becomes equal to 1, so this corresponds to stress concentration factor of 1. So, what does that mean that the $\sigma_\theta = s_y$, which is what should happen at the large distance. So exactly the similar situation, the same variation that we are getting from this particular figure.

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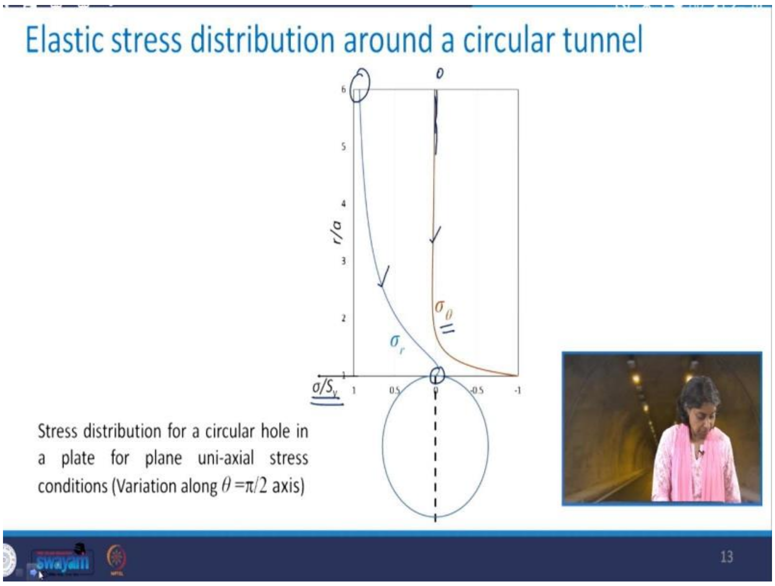

Now, here we see the stress distribution concerning the distance in the vertical direction that is away from the crown portion. So, you see that σ_r variation is this, and σ_θ variation is this. So here you can see that the σ_r at a large distance this sigma by s_y becomes equal to 1, and at a large distance, your σ_θ becomes equal to 0. Here this is the zero axis, so when we go to the large distance, this value becomes equal to zero.

So, this is how we can have the elastic stress distribution around the circular tunnel. So right now, we took an axis theta to be equal to 0, and this is the θ for π by 2. What about its circumference? How this is going to behave?

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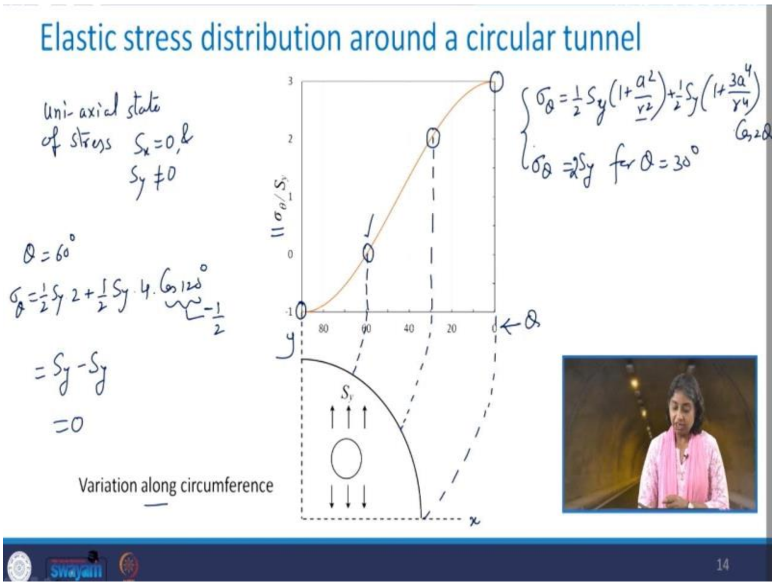

So let us take a look, so here I have tried to plot the variation of these stresses along the circumference. Kindly note that we are only considering right now the uniaxial state of stress with s_x to be equal to 0 and s_y non-zero. So that is what has been shown here now. How this has been plotted, you see that theta equal to 0. So, this is kind of your x-direction and y-direction. So, θ equal to 0 will correspond to this. So, this has been mapped in this particular manner to this value, and this is θ by s_y.

So, what did we get in case of the uniaxial state of stress at theta equal to zero. It was three, so that is what that you are getting here, so these are the values of basic θ. So similarly, when θ is equal to 90, you are getting -1. So likewise, let us take this θ, maybe say 30˚, so see if you just substitute in the expression. So, you have this:

$$\sigma_\theta = \frac{1}{2}\, s_y \left(1 + \frac{a^2}{r^2}\right) + \frac{1}{2}\, s_y \left[1 + \frac{3a^4}{r^4}\right] cos2\theta$$

So just substitute θ to be equal to 30˚, and you will see that σ_θ will work out to be equal to 2 times sy for θ equal to 30˚. Similarly, maybe you can consider θ to be equal to 60˚. Some typical values I am just taking so you see here just substitute θ equal to 60˚. Here so. you will have here σ_θ as, of course, we are considering at the circumference, so r = a.

So, this first term is going to be half s_y into 2, and then the next term is going to be half s_y into 4 into cos of 120˚, which is half with a negative sign. So, what does this give us? So, this is going to be s_y, and this is also s_y, so with the negative sign so this becomes equal to 0. So, this is what that we are getting in the variation. So, once we have the various expressions so for the typical situation, we can determine its variation along maybe θ equal to zero axis, θ equal to π by 2 axis, and all along the circumference of the circular tunnel. So, this is how we can get the elastic stress distribution all around the circular tunnel. What about the displacements? Now that we will see in the next class. Thank you very much.