Underground Space Technology
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Module No # 04
Lecture No # 19
Thick Wall Cylinder in the Biaxial Stress Field

Hello everyone, in the previous class, we discussed the elastic analysis of the circular tunnels. And we saw how various components of these stresses, like radial, tangential, and the shear stresses can be determined along with the displacements that are the radial displacement and the tangential displacements. We also saw their variation all along the tunnel periphery. Now, note that we did not have any lining in the case of the circular tunnel in that earlier class.

But then, let us say that if we have the thick concrete lining for the circular tunnel. So, in that case, the question is whether it is possible to carry out the elastic analysis for that. So, the answer is yes. We have the theory related to thick wall cylinder in the theory of elasticity. And by making the use of this, we can address the problem of thick concrete lining in the case of the circular tunnel and how this is done?

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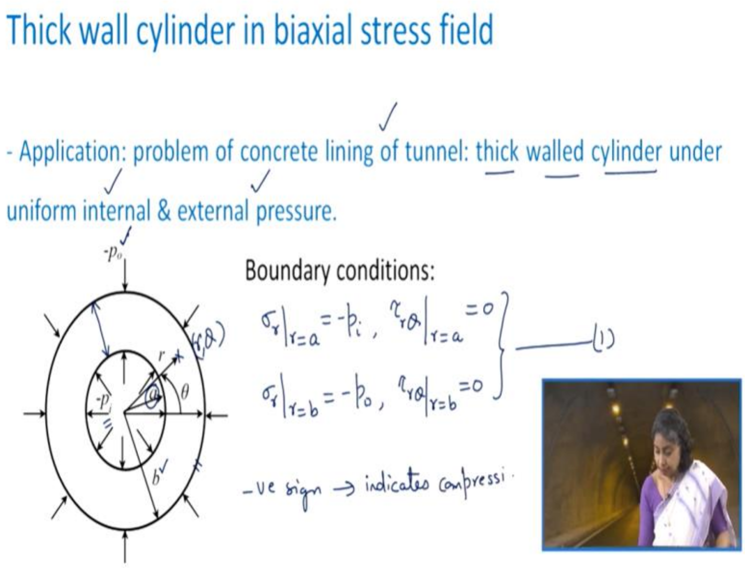

See here that the problem of the concrete lining of the tunnel, so how this is done is that take a look here that the problem of the concrete lining of the tunnel can be addressed by considering it as a thick-walled cylinder under uniform internal and external pressure. So, this has been shown here with the help of this figure, that it is the thick wall cylinder with this as the thickness of this cylinder.

This is subjected to an internal pressure p_i, and an external pressure p_o, and you can see that I have put a negative sign representing the compressive nature of these pressures. The internal radius is a, and the external radius is b, and the angle θ; I am measuring in this direction from the horizontal axis. Any point here can be represented by the coordinate r comma θ. So first, let us take a look that what are going to be the boundary conditions for these.

So, we have here σ_r at r = a; this is going to be - p_i, and $\tau_{r\theta}$ at r = a will be equal to 0. Now, what about the outer periphery? So, we have σ_r, r = b = - p_o, and $\tau_{r\theta}$ at r = b; this will be equal to 0.

$$\left.\begin{aligned} \sigma_r|_{r=a} &= -p_i \,, \tau_{r\theta}|_{r=a} = 0 \\ \sigma_r|_{r=b} &= -p_0 \,, \tau_{r\theta}|_{r=b} = 0 \end{aligned}\right\} \quad (1)$$

So, in a combined manner, mark these equations as equation number 1, so here negative sign indicates the compression.

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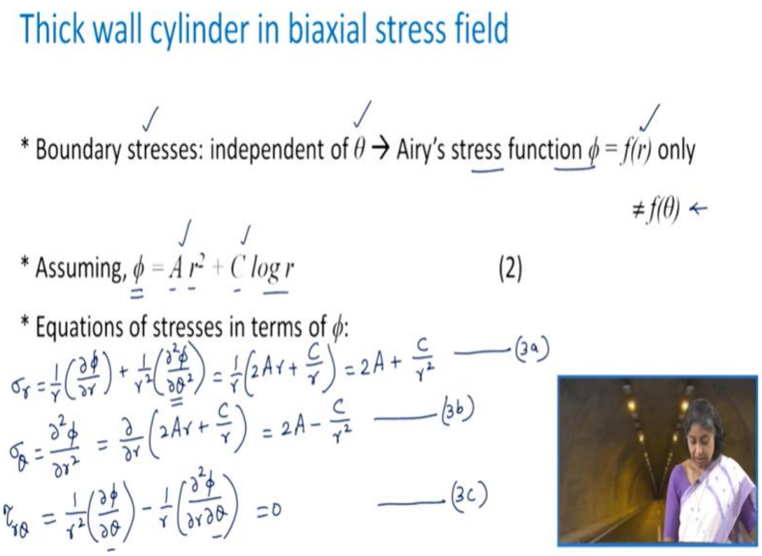

So, how should we approach here. So, this boundary stresses, we saw that they are independent of θ. So, we can say that the Airy's stress function φ, which will represent this type of problem, can be the function of r only and not the function of θ. So accordingly, this φ can be assumed to be:

$$\varphi = Ar^2 + C\log r \quad (2)$$

where A and C are the constants that will be determined from the boundary conditions.

So, let us try to write the equations of stresses in terms of φ then, we will have this σ_r as:

$$\sigma_r = \frac{1}{r}\left(\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right) = \frac{1}{r}\left(2Ar + \frac{C}{r}\right) = 2A + \frac{C}{r^2} \quad (3a)$$

Make this equation 3a. Then we have sigma θ that is equal to:

$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2} = \frac{\partial}{\partial r}\left(2Ar + \frac{C}{r}\right) = 2A - \frac{C}{r^2} \qquad \text{(3b)}$$

So, this is equation 3b, and then finally, we have the expression for $\tau_{r\theta}$, and this will be equal to:

$$\tau_{r\theta} = \frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right) - \frac{1}{r}\left(\frac{\partial^2\varphi}{\partial r\partial\theta}\right) = 0 \qquad \text{(3c)}$$

Since here, in both the terms you have the derivative with respect to θ. So, this will be equal to 0, so this equation I will write it as 3c.

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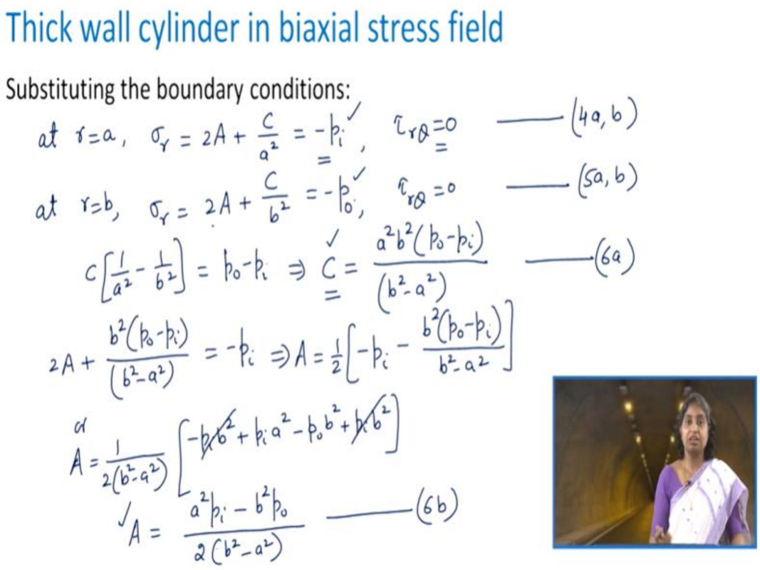

Now, let us substitute these boundary conditions and see how we can go ahead with the derivation. So, what we have is that:

$$at\ r = a, \qquad \sigma_r = 2A + \frac{C}{a^2} = -p_i\ , \quad \tau_{r\theta} = 0 \qquad \text{(4a, b)}$$

So, I write these equations as 4a, and b. So, this equation is 4a, and this equation has 4b. Then, we have:

$$at\ r = b, \qquad \sigma_r = 2A + \frac{C}{b^2} = -p_0\ , \quad \tau_{r\theta} = 0 \qquad \text{(5a, b)}$$

Mark this equation as 5a, and 5b. Now, from these 2 equations, if you just subtract C what you will get? This 2A, and 2A will get cancelled out, so what you will have here is:

$$C\left[\frac{1}{a^2}-\frac{1}{b^2}\right] = p_0 - p_i \quad \Rightarrow \quad C = \frac{a^2b^2(p_0 - p_i)}{(b^2 - a^2)} \tag{6a}$$

Mark it as equation 6a. Now, once I obtain the expression for C, from this 6a just substitute it in either 4a, or 5a. So, you will be able to get the expression for A. So, to let us say that, I put it in for A so that is going to be:

$$2A + \frac{b^2(p_0 - p_i)}{(b^2 - a^2)} = \ -p_i \qquad \Rightarrow \ A = \ \frac{1}{2}\left[-p_i - \frac{b^2(p_0 - p_i)}{(b^2 - a^2)}\right]$$

So, you see this, and this will get cancelled out, and finally, what you will get A as:

$$A = \ \left[\frac{a^2p_i - b^2p_0}{2(b^2 - a^2)}\right] \tag{6b}$$

So, this will be equation number 6b. So, this is how we can derive these two constants A, and C. Now, what we do is we substitute this back to the expression for the stresses.

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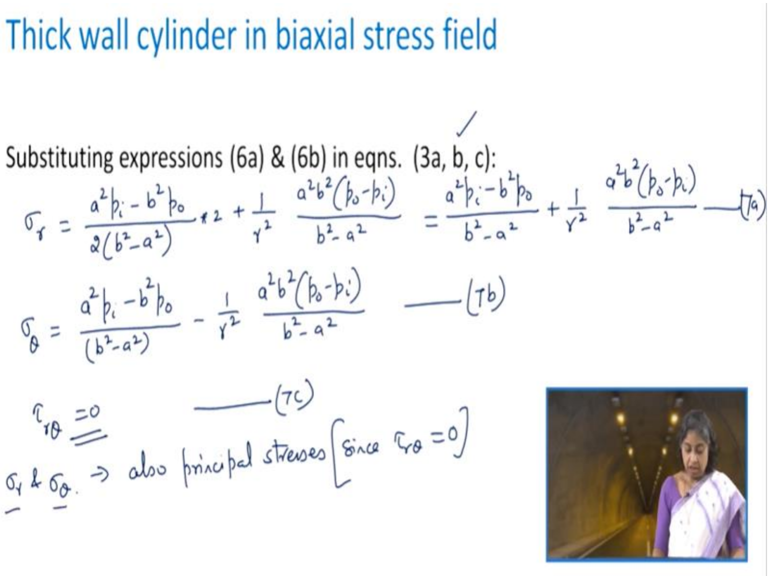

So, substituting this in equations 3a, 3b, and 3c, what we are going to get is σ_r as:

$$\sigma_r = \frac{a^2p_i - b^2p_0}{2(b^2 - a^2)} * 2 + \frac{1}{r^2}\left(\frac{a^2b^2(p_0 - p_i)}{b^2 - a^2}\right) = \frac{a^2p_i - b^2p_0}{b^2 - a^2} + \frac{1}{r^2}\frac{a^2b^2(p_0 - p_i)}{b^2 - a^2} \tag{7a}$$

This equation will be 7a. Similarly, we can determine sigma θ. So, that is going to be:

$$\sigma_\theta = \frac{a^2p_i - b^2p_0}{(b^2 - a^2)} - \frac{1}{r^2}\frac{a^2b^2(p_0 - p_i)}{b^2 - a^2} \tag{7b}$$

So, this is going to be 7b, and of course, my $\tau_{r\theta}$ is going to be:

$$\tau_{r\theta} = 0 \tag{7c}$$

So, this is equation 7c. Now, this σ_r and σ_θ are also the principal stresses, since $\tau_{r\theta}$ is equal to 0. See here, this is equal to 0 so these will also be the principal stresses.

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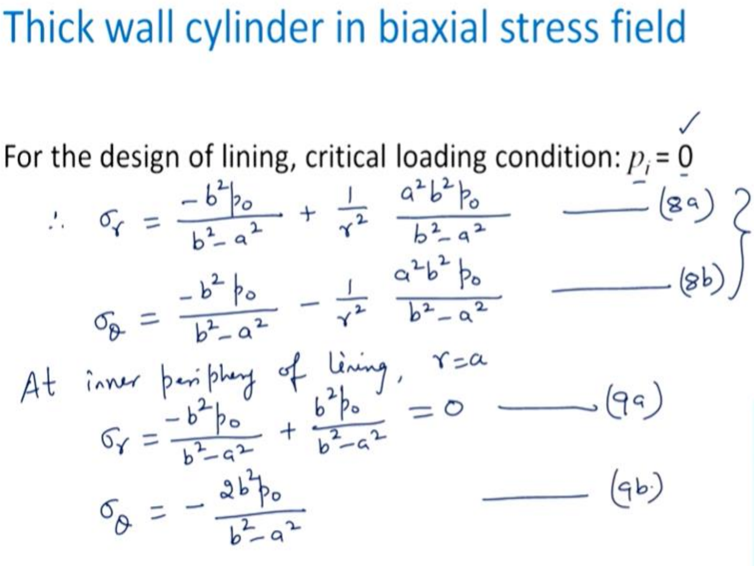

Now, for the design of the lining, the critical loading condition will be when this internal pressure is equal to 0. So, let us substitute this p_i to be equal to 0, in our earlier expressions. So, what we will get is σ_r will be equal to:

$$\sigma_r = \frac{-b^2 p_0}{b^2 - a^2} + \frac{1}{r^2}\left(\frac{a^2 b^2 p_0}{b^2 - a^2}\right) \quad (8a)$$

And σ_θ will be:

$$\sigma_\theta = \frac{-b^2 p_0}{b^2 - a^2} - \frac{1}{r^2}\left(\frac{a^2 b^2 p_0}{b^2 - a^2}\right) \quad (8b)$$

This is equation 8b.

Now, what happens at the inner periphery of the lining? So, what will be the value of r, this will be a. So, σ_r will be equal to, so just substitute r to be equal to a in these two expressions and see what we get is:

$$\sigma_r = \frac{-b^2 p_0}{b^2 - a^2} + \frac{b^2 p_0}{b^2 - a^2} = 0 \quad (9a)$$

$$\sigma_\theta = \frac{-2b^2 p_0}{b^2 - a^2} \quad (9b)$$

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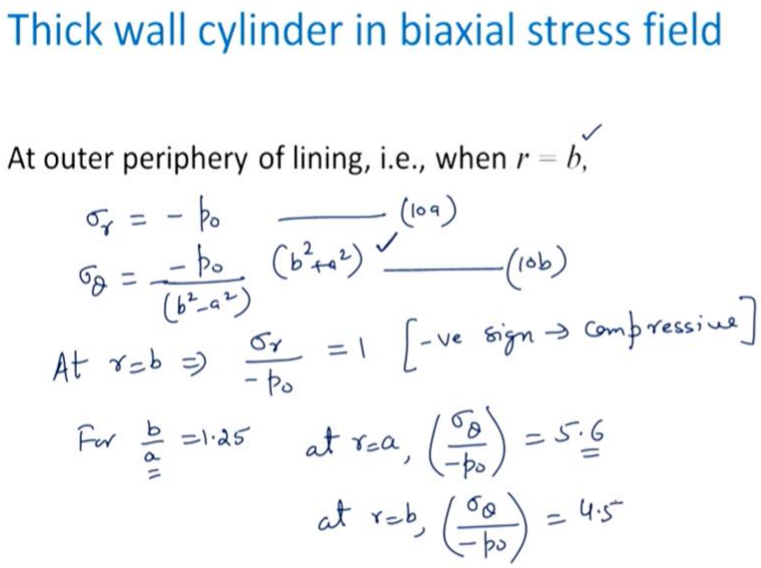

Now, what will happen at the outer periphery when r = b. So, I have this σ_r that will become:

$$\sigma_r = -p_0 \tag{10a}$$

And σ_θ will be:

$$\sigma_\theta = \frac{-p_0}{(b^2 - a^2)}\,(b^2 + a^2) \tag{10b}$$

Now here, again negative sign is representing the compressive nature of the stress, and σ_θ will be given by this expression. So, let us say I take, for example, b by a, to be equal to say 1.25.

So, what will happen at r = a, my σ_θ upon - p_o will work out to be 5.6. You can just substitute the value of b by an in these expressions and you will be able to get this term 5.6, and at r = b, what you will have as σ_θ upon - p_o as 4.5.

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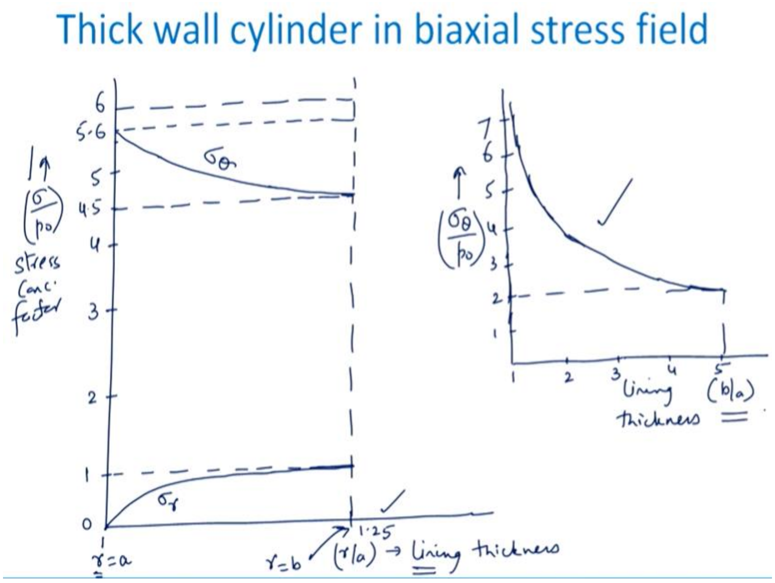

So, let us try to plot this and try to see the variation of it. So, let me plot it here for you so on the x-axis, I have r upon a, that is going to be representing lining thickness. On the y-axis, let us have sigma upon p_o which is the stress concentration factor. You recall, I mentioned to you how we were defining the stress concentration factor. Exactly, in a similar manner that is the stress divided by the external pressure in this case, and in the earlier case we had that as in-situ stresses.

So, b by a = 1.25. So, let us say, I write it this point as may be 1.25 and here this point is 1. So basically, this will represent r = a, and this point will represent r = b. So, I just draw a vertical line, what did we see that the σ_θ was 5.6 at r = a. so, let us first try to draw the variation for σ_θ. So, say this is starting from 0 and then may be this is 3. So roughly, I do it 1, 2 then 3, 4, 5, and then 6. So, we had here as 5.6.

So, somewhere here it is going to be, so this value is 5.6 and r = b, when b by a was 1.25 that was 4.5. So, 4.5 is somewhere here. So, I draw another line, and then the variation is going to be something like this. So, this is going to be the variation of σ_θ along with the lining thickness. Similarly, if we try to draw the variation for σ_r, so at r = a, that was equal to 0 and for r = b that was equal to 1.

So, this is how is going to be the variation for σ_r. So, it will approach a kind of 1 at r = b. So, this is the variation for σ_r. Now, the question is here, we took the b by a value to be 1.25. What if we vary the values of b by a, then how this sigma θ upon p_o will vary, take a look. So, here we have the lining thickness in terms of saying b by a. So, I have here 1 then 2, 3, 4 may be this is 5.

And then on this axis. I have σ_θ upon p_o maybe 1, 2, 3, 4, 5, 6, and 7 so on. So basically, this will be something like this. So, the variation, in this case, will look like this with reference to the lining thickness, which is b by a. Now, how to determine the displacement? So, this was all about the stresses and their variation. What about the displacements take a look?

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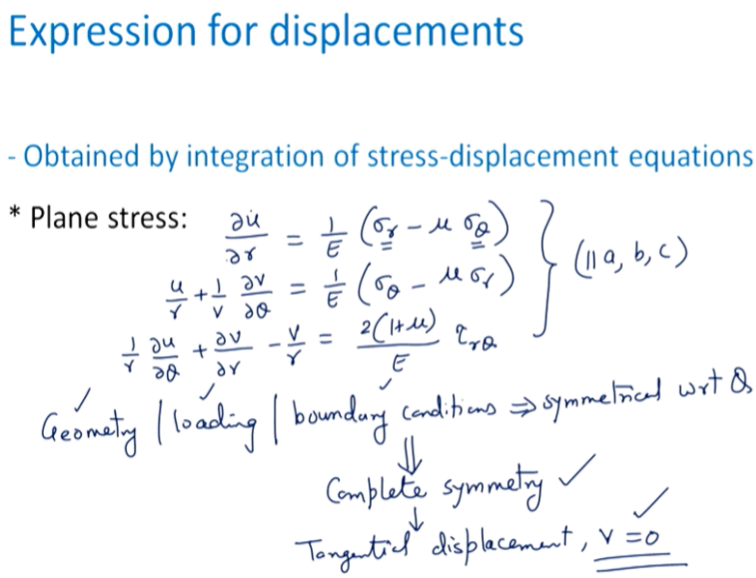

As, we know that it is we are going to get these by integrating these stress displacement equations. So, in case we have the plane stress situation so, what we have is:

$$
\left.
\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{1}{E}(\sigma_r - \mu\sigma_\theta) \\
\frac{u}{r} + \frac{1}{v}\frac{\partial v}{\partial \theta} &= \frac{1}{E}(\sigma_\theta - \mu\sigma_r) \\
\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} &= \frac{2(1+\mu)}{E}\tau_{r\theta}
\end{aligned}
\right\} \quad (11a,b,c)
$$

I mark these equations as equation 11a, b, and c.

Now, in our case, geometry loading as well as the boundary conditions is all symmetrical with respect to θ. And we can therefore say that we have the complete symmetry, and therefore we can have the tangential displacement which we represent by v to be equal to 0. Now, if there is not the symmetry in either of these conditions that is either geometry or loading or boundary then we cannot say this.

So, until unless there is the complete symmetry tangential displacement, we cannot say that it will be equal to 0. But in our case, this is what is the situation? So, we can just say that the tangential displacement is going to be equal to 0. Now, what we do is we know the expression for σ_r, and σ_θ. We just substitute it here and then try to integrate this equation.

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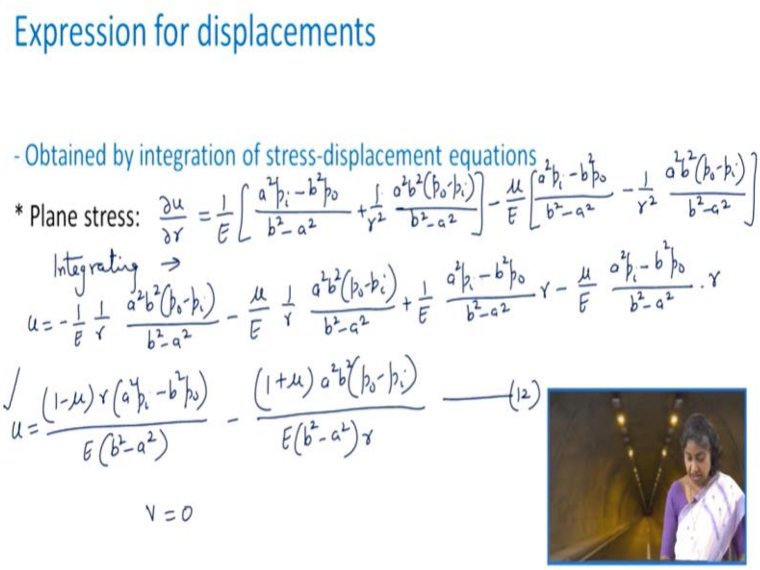

So, here what we are going to have is:

$$\frac{\partial u}{\partial r}=\frac{1}{E}\left[\frac{a^2p_i-b^2p_0}{(b^2-a^2)}+\frac{1}{r^2}\frac{a^2b^2(p_0-p_i)}{b^2-a^2}\right]-\frac{\mu}{E}\left[\frac{a^2p_i-b^2p_0}{b^2-a^2}+\frac{1}{r^2}\frac{a^2b^2(p_0-p_i)}{b^2-a^2}\right]$$

So, this is what we are getting. Now, if I integrating this what we will get is that:

$$u=-\frac{1}{E}\frac{1}{r}\frac{a^2b^2(p_0-p_i)}{b^2-a^2}-\frac{\mu}{E}\frac{1}{r}\frac{a^2b^2(p_0-p_i)}{b^2-a^2}+\frac{1}{E}\frac{a^2p_i-b^2p_0}{b^2-a^2}r-\frac{\mu}{E}\frac{a^2p_i-b^2p_0}{b^2-a^2}r$$

Or if you just simplify this, you will get ultimately as:

$$u=\frac{(1-\mu)r(a^2p_i-b^2p_0)}{E(b^2-a^2)}-\frac{(1+\mu)a^2b^2(p_0-p_i)}{E(b^2-a^2)r} \qquad (12)$$

Make this equation as equation number 12. So, this is how, we can find out the displacement u and due to complete symmetry displacement v. We have seen that it is already equal to 0.

$$v=0$$

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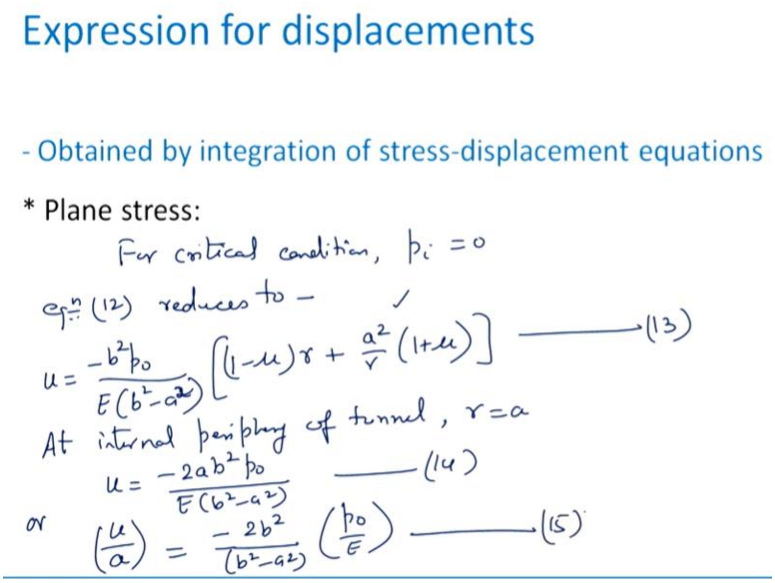

Now, for the critical condition, we have p_i to be equal to 0. So, our equation 12 reduces to u as:

$$u = \frac{-b^2 p_0}{E(b^2 - a^2)}\left[(1-\mu)r + \frac{a^2}{r}(1+\mu)\right] \tag{13}$$

So, this is how we can determine for the critical condition. What is going to be the expression for displacement, now what will happen? At the internal periphery of the tunnel, we have r equal to a, so just substitute r = a here. What you will get is, u as:

$$u = \frac{-2ab^2 p_0}{E(b^2 - a^2)} \tag{14}$$

And we can write this in the non-dimensional form, that is as if it is u by a will be:

$$\frac{u}{a} = \frac{-2b^2}{(b^2 - a^2)}\left(\frac{p_0}{E}\right) \tag{15}$$

So, this is how, you can determine the expression for displacement and also for stresses, in case of a problem with the concrete lining of the circular tunnels. So today, we learned about this elastic analysis of the concrete lining for the circular tunnel.

And we took the help of the theory of elasticity and the problem of thick wall cylinder to solve the problem. Now, there can be tunnels that are not circular in shape. So, how can we carry out the analysis of the non-circular shape of the tunnels? That we will learn in the next class. Thank you very much.