Underground Space Technology
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Lecture No # 20
Stress Distribution around Non–Circular Openings in Elastic Ground Conditions: Greenspan Method

Hello everyone, in the previous class, we discussed the elastic analysis of the concrete lining of a circular tunnel, and this problem was taken care of by a problem in the theory of elasticity of thick cylinders, and then I mentioned to you that till now, we discussed the circular tunnels only. But then, in practice, you have various shapes of the tunnels, which may be non-circular. So, today we will see how we can determine the stress distribution around the non-circular openings in elastic ground conditions.

So, the method that is adopted to obtain the stress distribution is called as Greenspan method. Let us try first to understand that what are the various shapes of the tunnels that are used in the field. And then, we will learn about the Greenspan methods and using which how we can obtain the stress distribution around these non-circular openings, especially in elastic ground conditions, so you take a look here.

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In this slide, I have pasted a few pictures of non-circular tunnels. So, the first picture gives you the idea about the horseshoe-shaped tunnel. You can see why it is called this. Because of the shape of the tunnel, the cross-section of the tunnel is the shape of a horseshoe. Then take a look at this figure this looks like the shape D. So, this is called a D-shaped tunnel. This is an eggshaped tunnel while this one is giving you the idea that what the elliptical-shaped opening will look like. Then this is the rectangular shape tunnel.

So, till now, what we saw was related to the circular tunnel. We saw how we can determine the stress distribution along with the displacement along the tunnel periphery. So, today we will see how we can handle such regular shapes.

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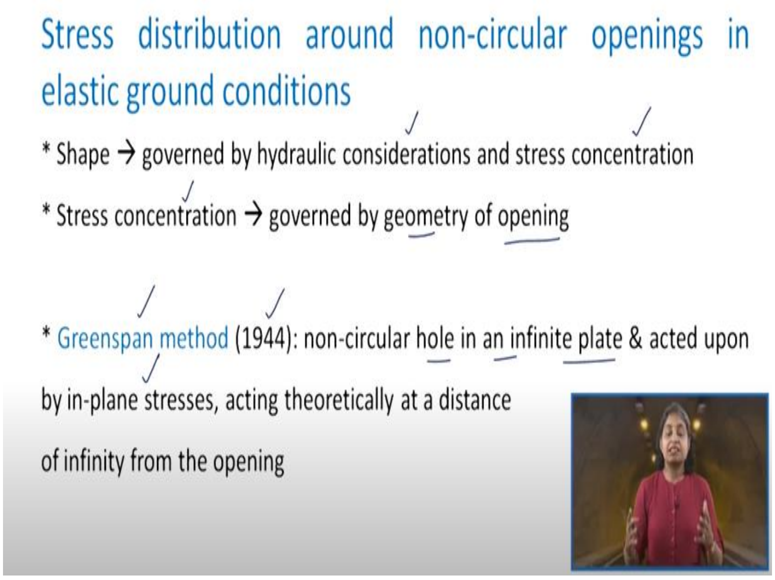

So, basically, the shape of the cross-section of the tunnel is governed by hydraulic consideration and also the stress concentration. On the other hand, the stress concentration they are governed by the geometry of the opening. So, non-circular openings analysis in elastic ground conditions were given by Greenspan in 1944, where they considered the non-circular hole in an infinite plate, which is acted upon by the in-plane stresses. And these were acting theoretically at a distance of infinity from the opening.

It is exactly the same as we did in the case of the circular tunnels, where we considered that the applied stresses are acting at a large distance or maybe at an infinite distance for all practical purposes.

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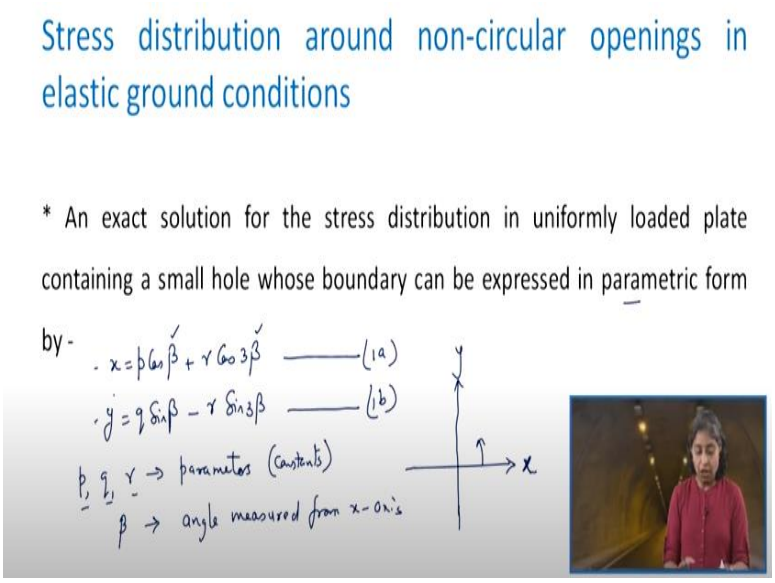

So, the exact solution for the stress distribution in such a uniformly loaded plate, which contains a small hole whose boundary, can be expressed in the parametric form by these two expressions, which are given as:

$$x = pcos\beta + rcos3\beta \tag{1a}$$

$$y = qsin\beta - rsin3\beta \tag{1b}$$

Where this p, q, and r are the parameters, and for a particular shape, these are constants, and β is the angle which is measured from the x-axis.

So basically, this is how the axis is taken, so this is what is your x-axis, and this is the y-axis, and this β angle will be measured from the x-axis. So, if we use various values of β and the particular parameter for a shape p, q and r, then we can get corresponding to each value of β. We can get a set of coordinate systems, which is x, y, and therefore we will be able to generate that shape of the opening.

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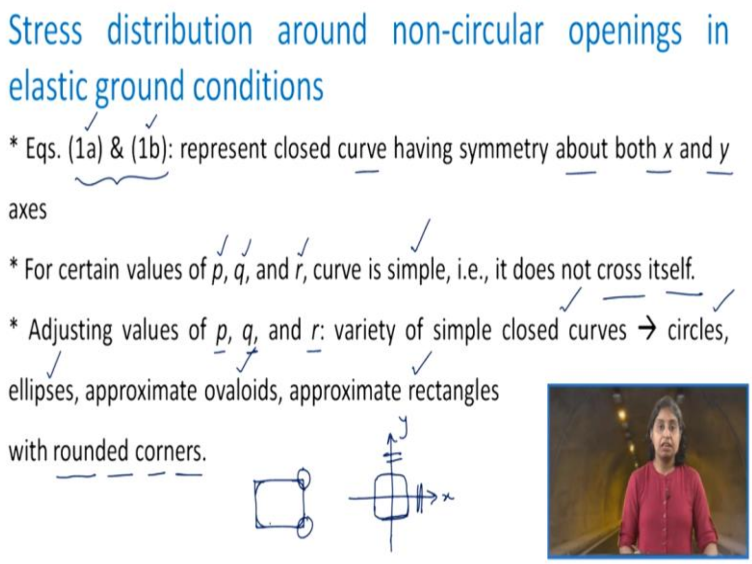

So, the shape which is given by equations 1a and 1b, we basically represent the closed curve which has symmetry about both the x and y-axis. So, for example, let us say if you have this as the x-axis and y-axis and say if we have the square type of the opening. So basically, the coordinate system is considered in such a manner that the shape is symmetrical about the x-axis as well as about the y-axis. So, for certain values of these parameters p, q, and r, the curve that will be generated by these two equations is simple and does not cross itself.

So, it is a close curve. Now, if we adjust the values of these parameters p, q, and r, we can generate a variety of simple closed curves, and these include saying circles, ellipses, approximate ovaloid, and approximate rectangles with rounded corners. Now, here you see, one has to take a note of this term that is rounded the corner. We do not provide this type of the corners, which are sharp, but we go for this kind of rounded corners, which you will learn in a short while.

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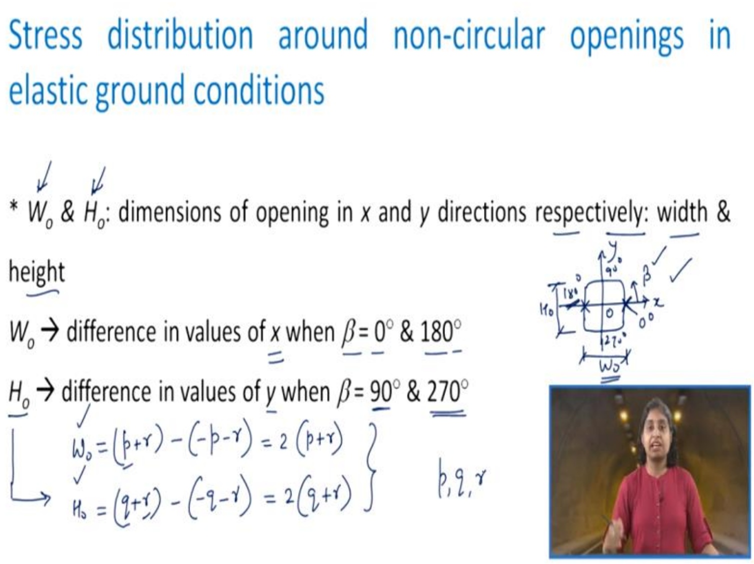

So here, I take the notation where W_o and H_o are the dimensions of the opening in x and y direction, respectively, or we are calling W_o as the width of the opening and H_o as the height of the opening. So, in other words, we can say that W_o is the difference in the value of x when the β is equal to 0 degrees and 180 degrees, and H_o is the difference in the values of y when the β is equal to 90 degrees and 270 degrees. See how so; this is x, and here it is y. This is considered to be the origin, and as I mentioned that it is going to be the symmetry.

So, let us say this is the shape, so β is measured from the x direction. So, here it is 0 degrees, here it is 90, this is 180 degrees, and here it is 270 degrees. So basically, W_o is this dimension, and H_o is this dimension. So, you see that if you just subtract or take the difference of the value here at β = 0 and the value here at 180 degrees, β = 180 degrees. Then you will get this W_o that geometrically also you can see from this figure. So basically, W_o will be equal to:

$$W_0 = (p + r) - (-p - r) = 2(p + r)$$

So, this we are getting by substituting the respective value of β in our equations 1a and 1b. So, W_o you will get here as twice (p + r). Similarly, this H_o is the difference in the value of y when the β is 90 degrees and the β is 270 degrees. So, I will get H_o as:

$$H_0 = (q + r) - (-q - r) = 2(q + r)$$

So, having known the values of these parameters corresponding to a particular shape, this is how we can find out the width and height of that shape of the opening.

Now, the question comes here what will be the value of these parameters p, q, and r? So, as I mentioned that for a particular shape, it will have one set of values.

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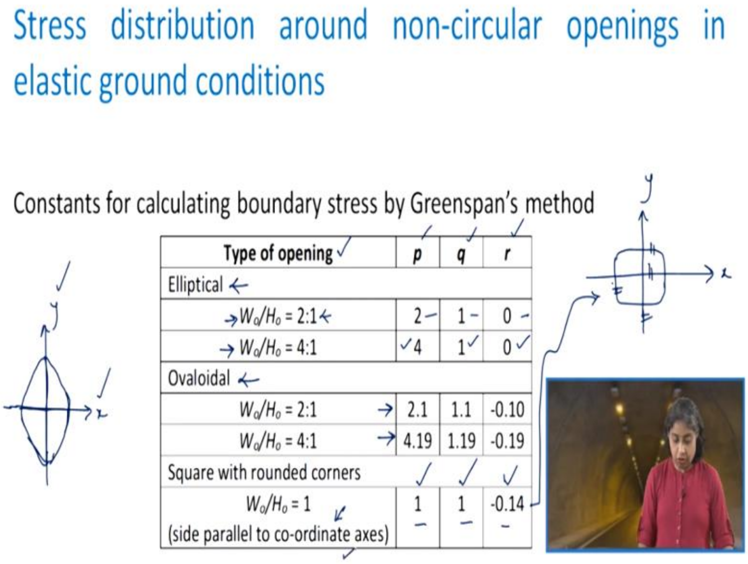

Type of opening	p	q	r
Elliptical			
W_o/H_o = 2:1	2	1	0
W_o/H_o = 4:1	4	1	0
Ovaloidal			
W_o/H_o = 2:1	2.1	1.1	-0.10
W_o/H_o = 4:1	4.19	1.19	-0.19
Square with rounded corners			
W_o/H_o = 1 (side parallel to co-ordinate axes)	1	1	-0.14

So, the table here gives you the idea that for different types of opening, what are going to be the parameters p, q, and r? So, the first one deal with the elliptical shape of the opening. Now in the case of the elliptical shape, you know that you have the major axis and the minor axis of the ellipse. So, if the width-to-height ratio of this ellipse is 2:1, in that case, p takes the value as 2, q is 1, and r becomes equal to 0.

However, in case if width to height ratio is 4 to 1, the corresponding value of p, q, and r become 4, 1, and 0, respectively. In case you have the ovaloidal shape, then again, for 2 different ratios of width to height, you have the values which are given by these two rows. In case you have the square shape opening with rounded corners. So, in that case, the height and width ratio is going to be equal to 1, and the value of p will be 1, q will be 1, and r will be -0.14.

Now, here the condition is that side should be parallel to the coordinate axis. This means that if this is the coordinate axis, the sides should be parallel. This should be parallel to the x-axis, and this side should be parallel to the y-axis. You can have another combination, which can be something like this, so this is x, and this is y, where you have the diagonals parallel to this axis. So here, you have these two diagonals which are parallel to or maybe coinciding with the x and y-axis.

So, these are the constants for this condition. That is why we are writing a side parallel to the coordinate axis.

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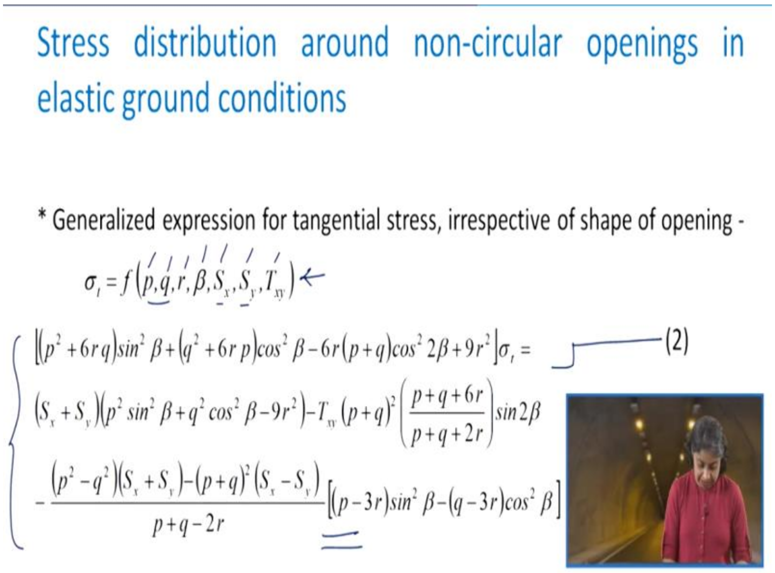

Then, the generalized expression for the tangential stress, irrespective of the shape of the opening, was derived by these research workers, and it was seen that this is the function of these parameters p, q, r, angle β, and the stresses S_x, S_y, and T_{xy}. So, S_x and S_y are the normal stresses in x and y directions, respectively. So, this is again a complicated and long-expression in terms of these quantities, and we are making this equation as equation 2.

Please note that you do not need to remember this expression. Because usually, you know that the various parameters like p, q, r are given to us, and then to be specific, we do not talk in terms of the most general state of stress. So, we consider some particular state of stress, and for that situation, you get a very simple expression for the tangential stress. So, you do not need to bother about remembering this long-expression. This is only for your information that such things exist.

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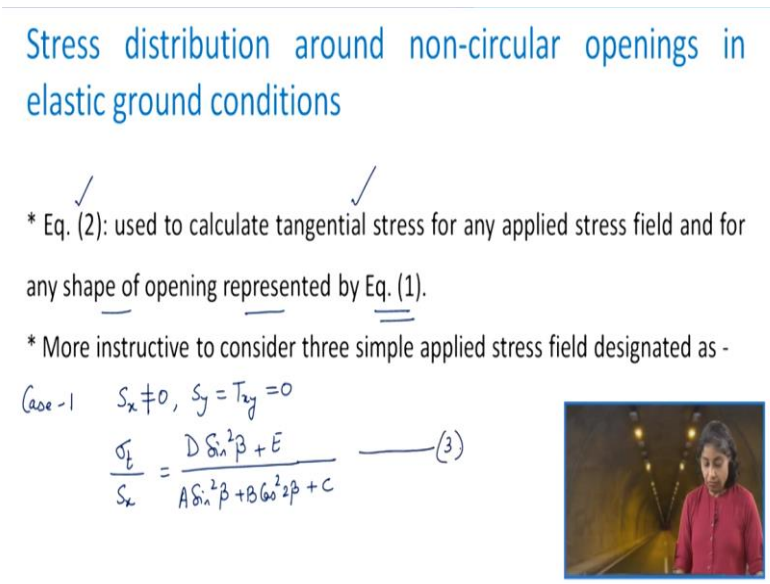

Now, this equation 2, as I mentioned that for any shape of the opening, which is represented by equation 1. We can use equation 2 to calculate the tangential stress for any applied stress field. However, it is more instructive to consider basically three simple applied stress fields; what are those? Let us take a look so. First, we define case 1 here, you have S_x not equal to 0, but S_y and T_{xy} are equal to 0.

So, in this case, if you substitute these values in the previous expression which was given by equation 2. What you are going to get is σ_t upon S_x will be equal:

$$\frac{\sigma_t}{S_x} = \frac{D sin^2\beta + E}{A sin^2\beta + B cos^2 2\beta + C} \tag{3}$$

So, this is equation number 3.

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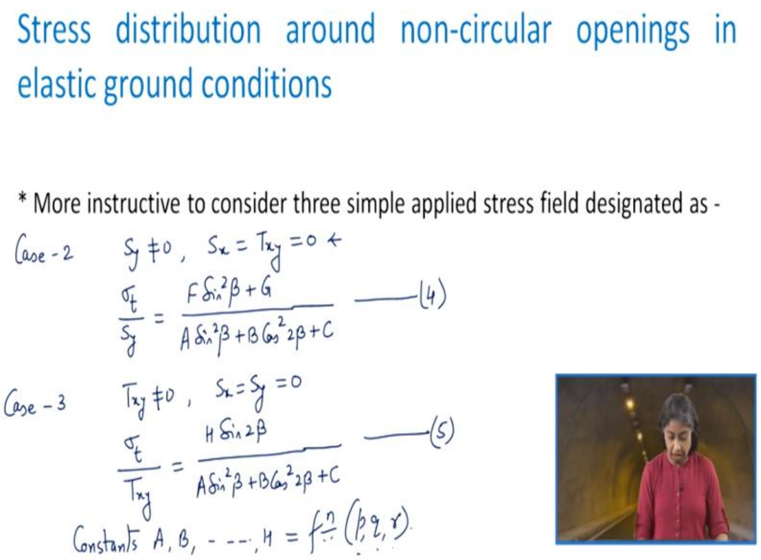

Similarly, we have the other 2 cases, such as case 2, where you have S_y not equal to 0 and S_x and T_{xy} they are equal to 0. So, what we get here is σ_t upon S_y as:

$$\frac{\sigma_t}{S_y} = \frac{F sin^2\beta + G}{A sin^2\beta + B cos^2 2\beta + C} \tag{4}$$

So, this is what is going to be the resulting expression, in case you have this type of state of stress, and then finally the third case, which I represent case 3, in this case, T_{xy} is not equal to 0. But, S_x and S_y both are equal to 0. So accordingly, what we have is σ_t upon T_{xy}, this is equal to:

$$\frac{\sigma_t}{T_{xy}} = \frac{H sin 2\beta}{A sin^2\beta + B cos^2 2\beta + C} \tag{5}$$

This is equation number 5. So, these constants A, B, and so on up to H. These are the functions of p, q, and r. So, let us take a look that how these are expressed in terms of p, q, and r.

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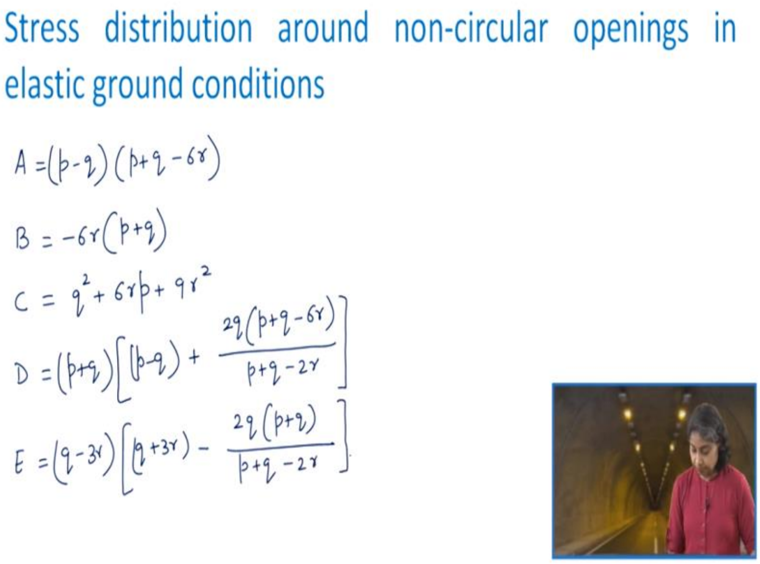

So, here we have:

$$A = (p - q)(p + q - 6r)$$

$$B = -6r(p + q)$$

$$C = q^2 + 6rp + 9r^2$$

$$D = (p + q)\left[(p - q) + \frac{2q(p + q - 6r)}{p + q - 2r}\right]$$

$$E = (q - 3r)\left[(q + 3r) - \frac{2q(p + q)}{p + q - 2r}\right]$$

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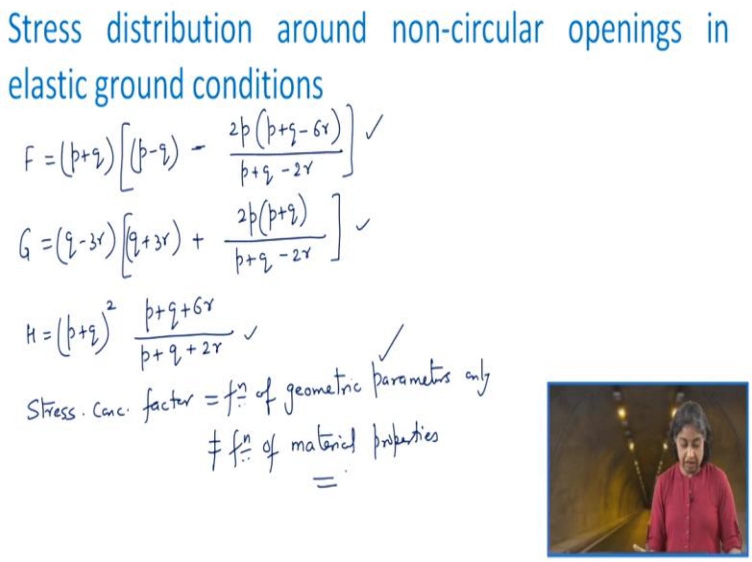

$$F = (p+q)\left[(p-q) - \frac{2p(p+q-6r)}{p+q-2r}\right]$$

$$G = (q-3r)\left[(q+3r) - \frac{2p(p+q)}{p+q-2r}\right]$$

$$H = (p+q)^2\left[\frac{p+q+6r}{p+q+2r}\right]$$

So, this is how all these constants or parameters A to H are obtained. So basically, here the stress concentration factor is the function of geometric parameters only, and it is not the function of material properties.

See, it was all the functions of A, B, C, D to H, and you can see that these are all the functions of p, q, and r only, which depends only on the geometric parameters and not on the material properties.

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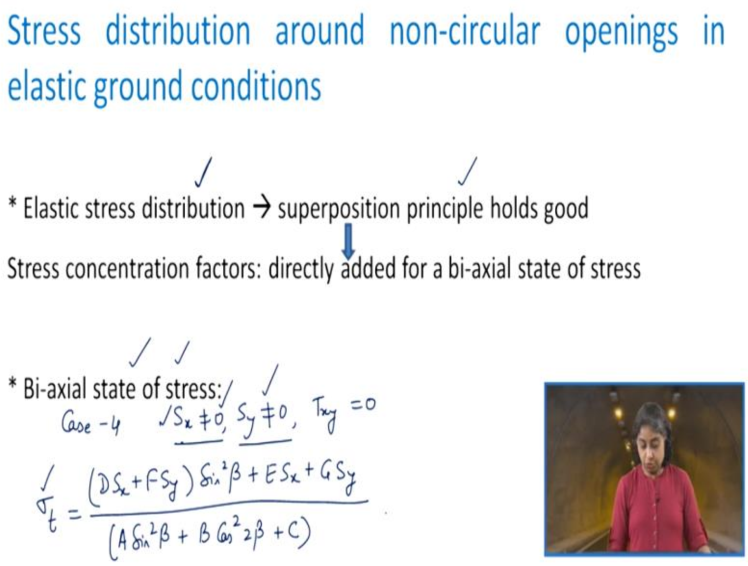

So, what happens, in case you have the elastic stress distribution? In case of such a situation, the superposition principle holds good. So, in case if you have the biaxial state of stress, then in that case stress concentration factors can be obtained directly by adding to the uniaxial state of stress situation. Like previously, we saw that in case 1, case 2, and case 3, all the 3 were in a uniaxial state of stress. Now, in the case of the situation, S_x and S_y both are present, and T_{xy} = 0.

Then you can apply the principle of superposition and can obtain these stress concentration factor. So, let us take a look that if we consider the biaxial state of stress. So accordingly, we define the fourth case here, where you have S_x not equal to 0, S_y not equal to 0, but T_{xy} = 0. That means that the stresses are applied in the x as well as in the y-direction. So, in that case, you sum the expression for σ_t in the case of when it was the uniaxial state of stress with $S_x = 0$ and when it was for $S_y = 0$.

So ultimately, what you get here as:

$$\sigma_t = \frac{(DS_x + FS_y)sin^2\beta + ES_x + GS_y}{Asin^2\beta + Bcos^2 2\beta + C}$$

So, this is nothing different than what we did in case of the uniaxial state of stress with reference to case 1, 2, or 3. So, when you have the presence of S_x as well as S_y, simply you have to do is find out σ_t as if it is only S_x which is presently found out σ_t as if only S_y is present.

Then sum these two, and you will be able to get the tangential stress in case of the biaxial state of stress. But please note that this superposition principle holds good when we have this elastic stress distribution. If it is not so, then we will not be able to do such a simple superposition.

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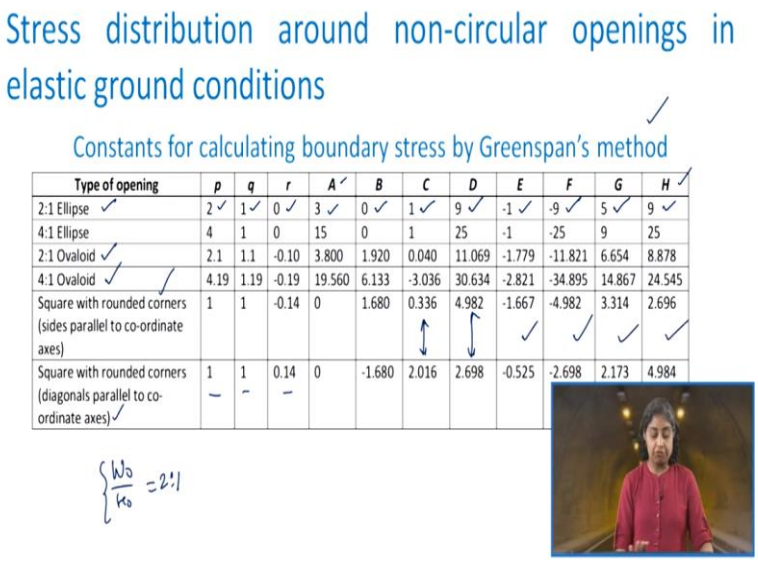

Type of opening	p	q	r	A	B	C	D	E	F	G	H
2:1 Ellipse	2	1	0	3	0	1	9	-1	-9	5	9
4:1 Ellipse	4	1	0	15	0	1	25	-1	-25	9	25
2:1 Ovaloid	2.1	1.1	-0.10	3.800	1.920	0.040	11.069	-1.779	-11.821	6.654	8.878
4:1 Ovaloid	4.19	1.19	-0.19	19.560	6.133	-3.036	30.634	-2.821	-34.895	14.867	24.545
Square with rounded corners (sides parallel to co-ordinate axes)	1	1	-0.14	0	1.680	0.336	4.982	-1.667	-4.982	3.314	2.696
Square with rounded corners (diagonals parallel to co-ordinate axes)	1	1	0.14	0	-1.680	2.016	2.698	-0.525	-2.698	2.173	4.984

Now, the constants which are there for a calculation of the boundary stress by this Greenspan method for various types of opening. So, here you see that for the 2 is to 1 ellipse. So, when I say 2 are to 1 ellipse means W_o upon H_o equal to 2 is to 1 and the shape is elliptical. So, in that case, what we have as p as 2, q 1, r = 0, and the constants A to H, are given in respective columns. For example, for the first case that is 2 is to 1 ellipse; A is 3, B is 0, C is 1, D is 9, E is – 1, F is– 9, G is 5, and H is 9.

So, likewise, if you have different shapes such as ovaloid again with 2 ratios of W_o upon H_o, then the square shape, of course, you need to have here the rounded corners, and these 2 situations can be there as I already explained you in the first situation. The sides are parallel to the coordinate axis, and in another situation, the diagonals are parallel to the coordinate axis. So, you see that in both the cases, although it is the square opening in both the cases, but then the constants are different.

Although p, q, and r parameters are the same, you see here the value of C is different D, E, F, G, and H. They are all different for these two situations.

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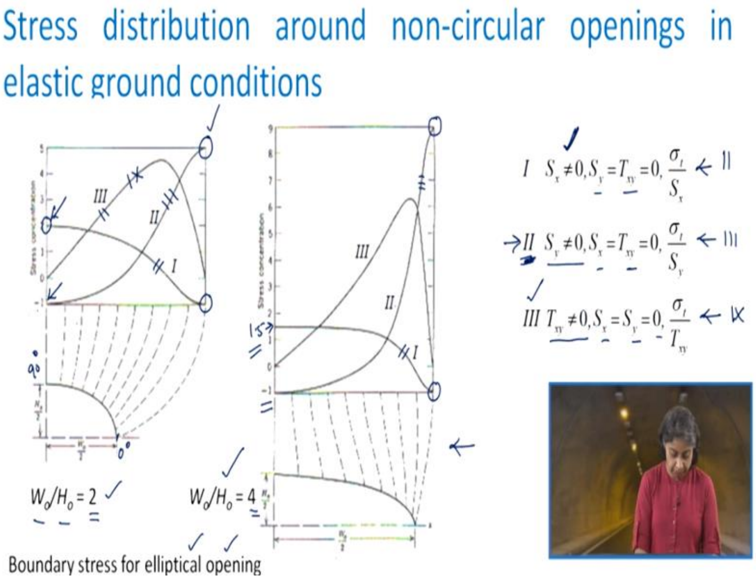

Now, based upon this for various cases as we considered and I explained to you, that the case 1 corresponds to when you have the uniaxial state of stress with the stress in the x-direction only, and there is no stress in the y-direction and $T_{xy} = 0$. So, in that case, the stress concentration factor is given by σ_t upon S_x. If you have the second case, then in that case, S_y is non-zero, and the remaining two stresses are equal to 0, and the stress concentration factor is defined by σ_t upon S_y.

And in case, you have the third situation there, T_{xy} is non-zero, and S_x and S_y both are 0, and therefore, the stress concentration factor is defined by σ_t upon T_{xy}. So, these three curves are there in each of the figures for different shapes of the opening, where these cases are defined. For example, here we are dealing with the elliptical opening, so the stress distribution around the periphery of this ellipse.

So, that is from here, it starts with 0 degrees and then it goes up to here as 90 degrees. So, this curve corresponds to this first condition. This is for the second condition, and this is for the third condition. So, likewise a similar way, we are going to discuss the boundary stress for different types of openings. So here, the first figure deals with the ratio W_o by H_o to be equal to 2, and then you can see that at the boundary here at $\beta = 0$.

How this stress concentration is – 1, and at β = 90, it is 2. For example, in the case of the second case, is this one when you have the stress applied in the direction of the y-axis? Then, in that case, at β equal to 0, you will have the stress concentration factor as 5, and for β equal to 90, you have the stress concentration factor as - 1. Similarly, for the third case also, the variation can be seen in this manner for various values of β all along the boundary of the opening.

So, similar is the situation here, but then you can see that there is a difference in the magnitude of the stress concentration factor. For example, just take the same stress state, that is let us say, state 1 case 1 I consider. So, β is equal to 0. It is -1 here. But, see β equal to 90, it somewhat approximately says 1.5 here, and here it is 2. So, when W_o upon H_o is increased from 2 to 4.

You see that the stress concentration factor at β is equal to 90. It reduces from 2 to 1.5, and similar type of comparison you can make for all the respective cases. For example, let us take case 2 here, see β equal to 0 and stress concentration factor you get as high as 9. While it was 5 in this case, although at β equal to 90, you have the same stress concentration factor that is -1. So, this was about the elliptical opening.

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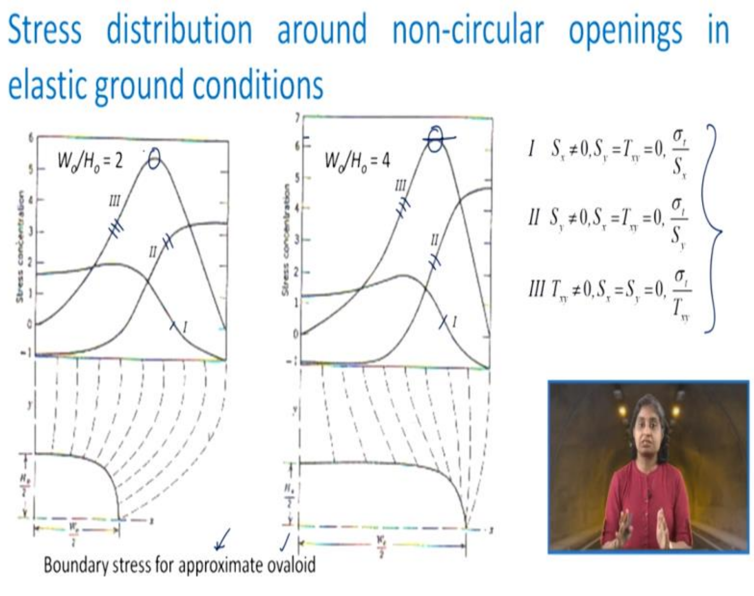

Now, in case if you have the ovaloidal shape of the opening so, we write here as approximate ovaloid. Because of the fact that those equations 1a and 1b, if you just substitute the particular values of p, q, and r, you can only generate the approximate ovaloid. So, that is why we are calling here an approximate ovaloid. But it is as good as ovaloid for all practical purposes. So, a similar type of comparison and the distribution can be seen in these two figures, which typically represent the stress distribution all along the boundary of the ovaloidal shape of the opening for two situations, for width-to-height ratio as 2 and 4.

So again, in this case here, all the 3 cases have been shown again, the 3 cases correspond to these states of stress. So, if you just compare these 2, probably let us take case 3. So, you see, the maximum stress concentration factor is somewhat here, which is maybe larger than 6. However, in this case, you see it is a little larger than 5. So, the pattern may be the same, but the magnitude is not. It depends upon the size of the opening.

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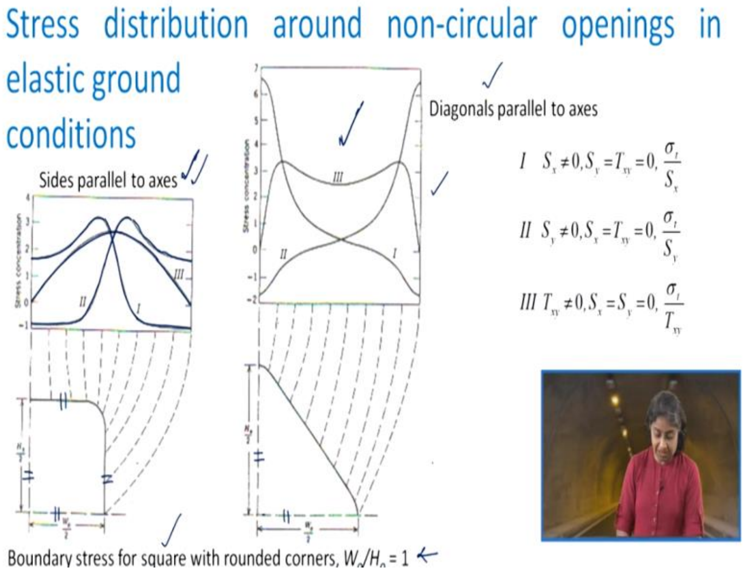

Similar observations are made here, with reference to the square opening with rounded corners and since it is the square opening. So, you have W_o upon H_o to be equal to 1 so, be careful here that this is what is for situation 1. This is for the state of stress designated as case 2, and this one is for case 3. Similar is the situation here so the difference between the 2 situations is in this case, the sides are parallel to the axis you see here, but in this case, diagonals are parallel to the axis.

So, this is the diagonal, which is maybe merging here with the y-axis, and this is another half diagonal, which is merging with the x-axis. So, based on how the shape is placed accordingly, you will have different stress distributions. Please note that here we are only talking about the boundary stress distribution. Now, if you just take a look, the magnitude of these stresses or the stress concentration factor is larger here in this case as compared to when you have these sides which are parallel to the axis.

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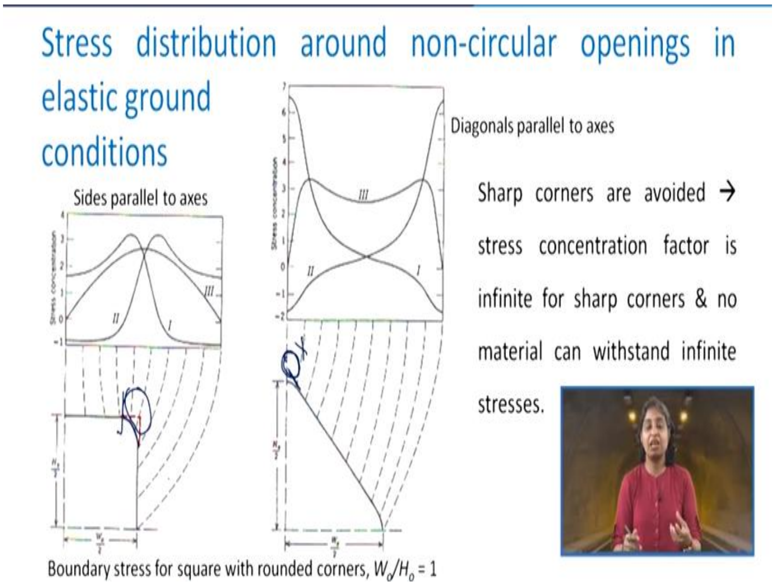

So here, as I have been mentioning that we do not have sharp corners, for example, let us say I have here this type of situation very sharp corner. So, this situation is avoided the reason is that the stress concentration factor at such sharp corners becomes infinite, and it is impossible that any material can withstand that kind of infinite stress or huge stresses. That is why to do away with such type of situation, the corners of the opening are made a little rounded like this.

This is what is needed, not this is not correct. This type of sharp corner they are not used not adopted. So, if we take a look at these figures close look corresponds to any particular shape but different aspect ratios or if you compare 2 different shapes with the same aspect ratio. For example, if you compare 2 to 1 ellipse with 2 is to 1 one ovaloid.

Then by studying this distribution for the boundary stresses for different shapes of the openings, then we can come out with the same general guideline for the design of these openings. So, this we will discuss in detail in the next class. So, what we discussed today was that what is the Greenspan method. How it helps us in obtaining the stress distribution around the non-circular openings such as elliptical, square, or rectangular openings?

And how for different shapes you can get the boundary stress distribution, and how they compare with each other? So, we will deduce the general guideline for the design of such underground excavations in the next class. Thank you very much.