Underground Space Technology
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Module No # 05
Lecture No # 23
Stress Distribution for Multiple Openings

Hello everyone, in the previous class we discussed the governing principle for the stress distribution for non-circular openings such as elliptical, ovaloid, squares, and rectangular with rounded corners. And there we took 3 types of stress conditions, which were uniaxial, biaxial and hydrostatic state of stress. So, today we will discuss how the stress distribution is going to be for multiple openings.

As of now, we have discussed stress distribution or the displacement variation all around the boundary of any opening, when you had a single opening. But then, there can be a situation where you have more than one opening then in that case, how does the stress distribution changes let us try to take a look today.

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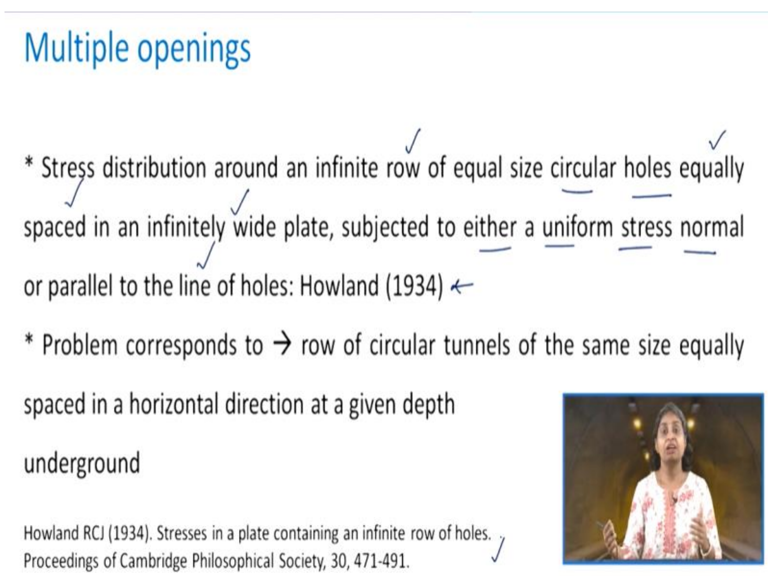

The stress distribution around an infinite row of equal size circular holes, that is what, we are going to consider. So right now, we will focus only on these circular openings of equal size. And another condition which is going to be there, that these are equally spaced in an infinitely wide plate and these are also subjected to either uniform stress, normal, or parallel to the line of holes.

Now, the solution for such a problem was proposed by, Howland in 1934, and here is the complete reference.

So, this problem corresponds to the row of circular tunnels, having the same size and, equally spaced in the horizontal direction at any given depth. That means the level at which the excavation is made for both openings, or if you have more than 2 openings, it is going to be the same from the ground surface.

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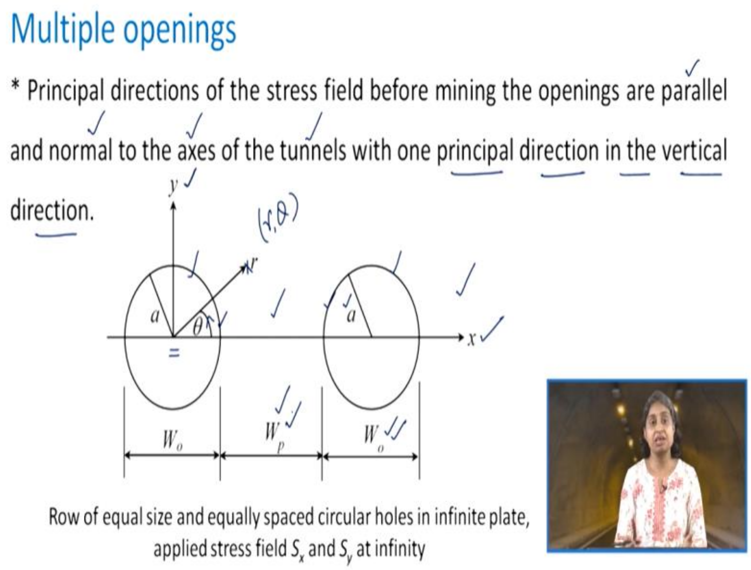
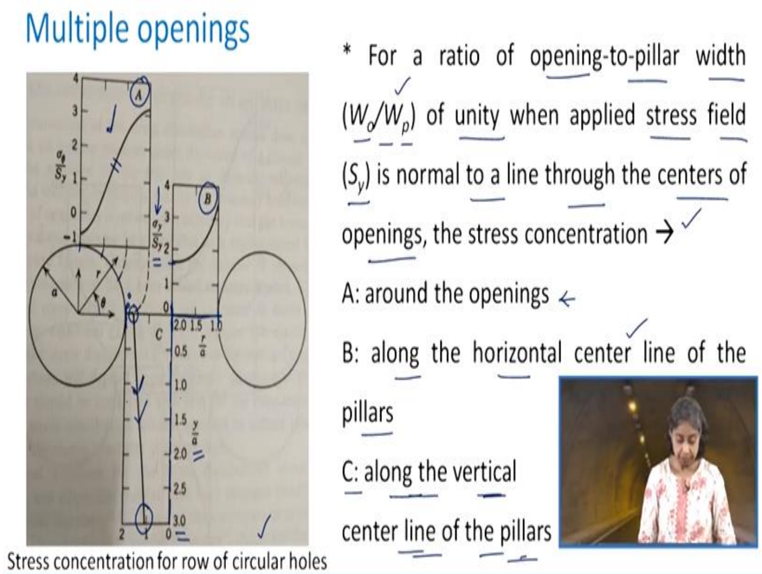


So, we are going to consider such a situation which is shown here, in this figure. So, you can see that we have 2 openings, both are circular and are of the same size have the radius, as small a and its width is represented by W_o is the width of the opening. Now you see that there is going to be an unexcavated space between these 2 circular openings. The width of which is represented by W_p, this p stands for pillar. So, there is the unexcavated space, and it will kind of form a pillar, between the 2 circular openings. So, the width of the opening is W_o, and the width of the pillar is W_p.

Now, another thing that you need to keep in mind is that the principal directions of the stress field, before mining the opening, or before excavating the opening are parallel and normal to the axes of the tunnels with the 1 principal direction being in the vertical direction. So, you see that we have the axis as, the x-axis here as a horizontal one and y-axis as the vertical axis. So, any point in the space will be represented by the coordinate, r, θ; r being the radial distance from this opening, and θ being the inclination of this radial distance from the x-axis.

Now, the authors obtained the solution we will not be going into the mathematical details of it. But, we will directly try to study, that, how the stress distribution is going to be, and how it compares with the situation when you had only a single opening that was circular in nature. **(Refer Slide Time: 05:13)**



So, this is what is the stress concentration for the row of circular holes? So, in this case, the opening to pillar width ratio is considered to be a unity that is, (W_o/W_p) that is (W_o/W_p) is equal to unity. When the applied stress field S_y is normal to a line, through the centers of the opening. So, in this case, the stress concentrations were plotted as a part of 3 variations. The first one is this variation A, which is written here, this is around the opening, which means that if you take the angle theta, here it is 0^0, this is 30^0, this is 60, and here it is 90.

So, accordingly, the stress distribution was plotted and it looks like this. Then, the second one is along the horizontal center line of the pillars. So, this is what is the pillar, So, the center line so, is the horizontal line, so the variation along this is given by B. So, this is what is (σ_y/S_y). So, everything about all the stress distribution, we are plotting with respect to the stress concentration, and how did we define the stress this concentration factor?

That was the stress divided by the applied stress. So, in this case, this is going to be (σ_y/S_y) and, you can see how the variation is somewhere here at the center line it is approximately maybe 1.7

or so, and then, it increases as you go beyond the center line. Then, the third one is along the vertical center line of the pillar. So, this is the width of the pillar and you draw the vertical center line.

So, which is this, and see, how the variation looks like as you go in the vertical direction. So, it starts from here at the center line, and then when you go move in the vertical direction. See how it changes and when it is (y/a) becomes approximately equal to 3, this stress concentration is close to 1. So, here this is the stress concentration with respect to the vertical center line of the pillars. That is (σ_y/S_y) here this variation.

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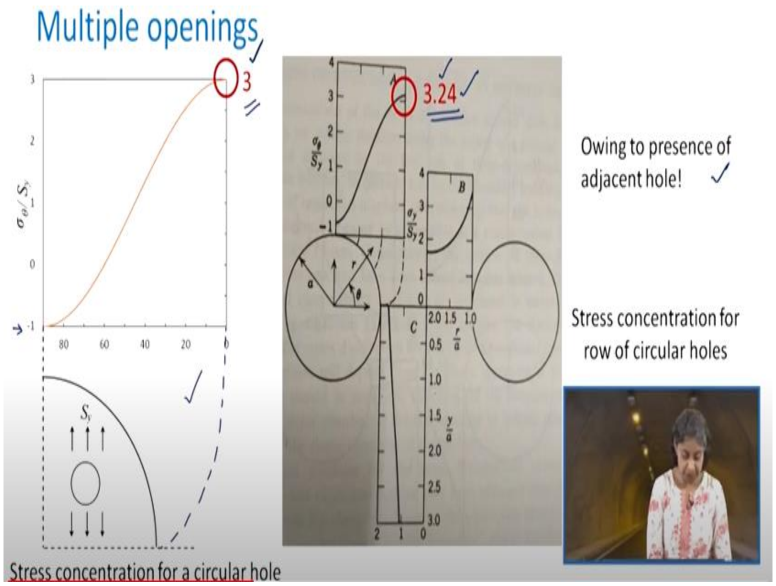

Now, if I just compare the case of a single circular opening with that of the 2 circular openings. This figure, you are aware of, we discussed in some of our earlier classes. So, you see that if I just take this at the boundary or $\theta = 0^0$, it was coming out to be 3. In the case of the uniaxial state of stress, and you remember I mentioned to you that, at the crown portion it is going to be critical, because of the presence of this tensile stress situation.

So, the similar thing that we get here is, at $\theta = 0$, what we get here is 3.24. Now, this is owing to the presence of the adjacent hole. So, the stress concentration factor increases from 3 to 3.24, in case, you have 2 circular openings.

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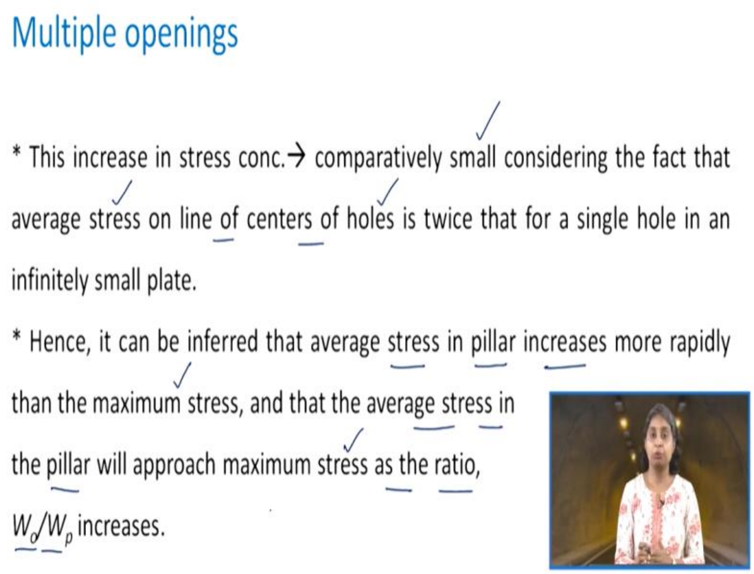

Now, this increase in the stress concentration is relatively quite small, considering the fact that the average stress on the line of centers of the hole is, approximately twice that of a single hole in an infinite small plate. So, you see that although the average stress has increased twice on the line of centers of the hole. But then, the increase in decent stress concentration is relatively quite small.

And, therefore, we can infer that, the average stress in the pillar increases, more rapidly than the maximum stress and that the average stress in the pillar will approach the maximum stress as the ratio W_o by W_p increases. So, please keep this in mind.

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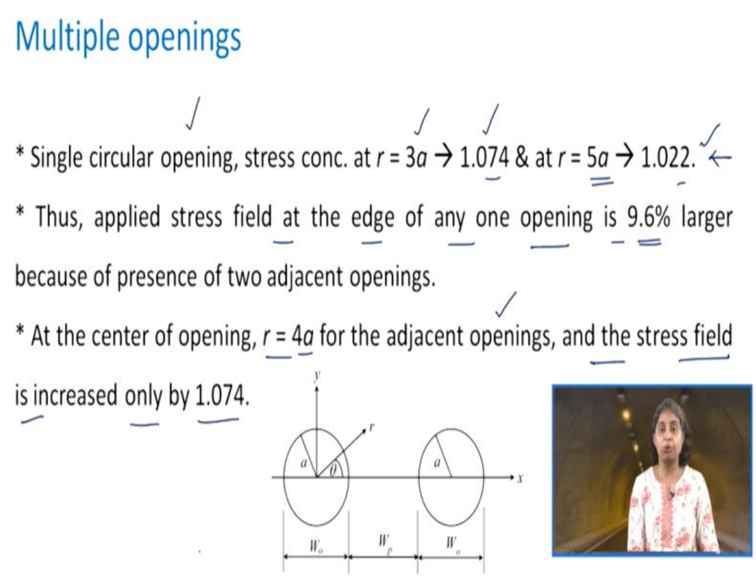

Now, we have seen that in the case of the single circular opening the stress concentration at r = 3a was found to be 1.074, and at 5a it was 1.022. This, you can refer to one of our earlier lectures where we discussed the single circular opening. So, these values have been picked from there only. So, therefore the applied stress field, at the edge of any, one opening. Any, one of the openings is going to be about 9.6% larger, because of the presence of 2 adjacent openings. But, at the center of the opening, what will happen is when you have r = 4a. So, in that case for the adjacent opening and the stress field, it will be increased only by 1.074.

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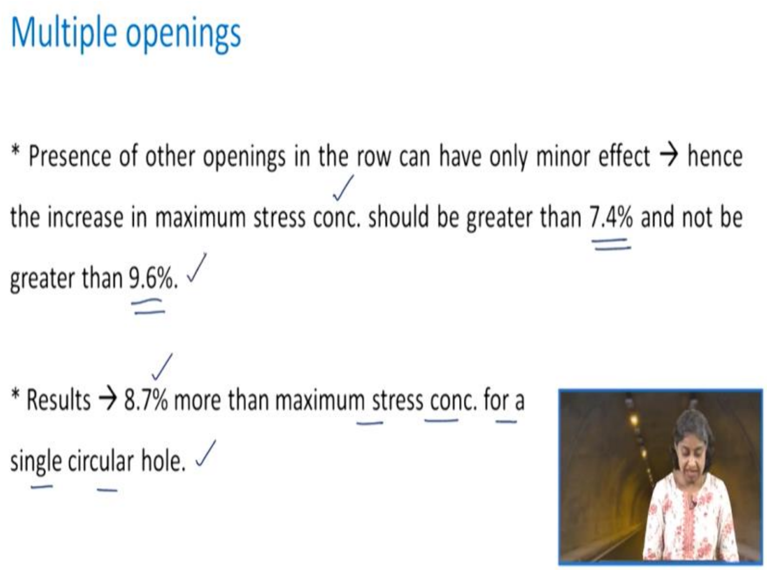

Now, the presence of the other openings in the row can have only a minor effect. And, therefore the increase in the maximum stress concentration, should be greater than 7.4% and not be greater than 9.6%. That means, it should be somewhere between 7.4 and 9.6% and, the result we saw is 8.7% more than the maximum stress concentration for a single circular hole. So, remember that it has to be greater than 7.4% and less than 9.6%.

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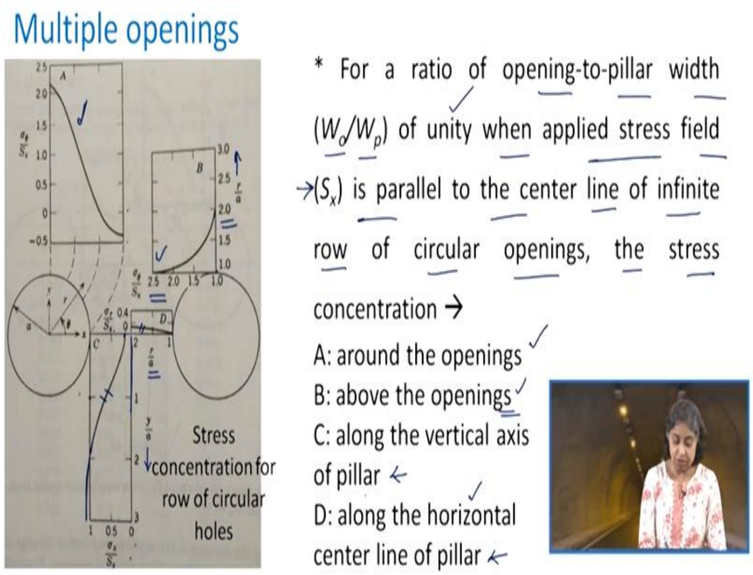

Now in case, if we have the opening to the pillar with the ratio, that is (W_o/W_p) equal to unity, and when the applied stress field is parallel to the centerline of the infinite row of the circular opening. So, in this case, here it is the S_x, that is present in the earlier case, it was S_y that was present. So, the stress concentration was drawn around the opening, above the opening, then along the vertical axis of the pillar, and along the horizontal center line of the pillar.

So, let us take a look, when it is around the opening it is this figure and sees how it is changing like. At 0, it is somewhere between 0 and -0.5, and at $\theta = 90^0$, it is somewhat greater than 2. When you have this stress concentration, curve b, which is above the opening, so, there you see that it is just above the opening here, and what you have is (σ_θ/S_x) here. So, in that case, see just at the crown you have this as 2, and then here it is 2.5.

As you have this variation of (r/a). Then, along the vertical axis of the pillar, that is here, this one vertical axis of the pillar is this. So, this is the variation is (y/a). So here, it is varying as, 1, 2 and, then 3, this is 0. So, this variation starts from here and it becomes or it follows this particular path. D is the stress concentration along the horizontal center line of the pillar and, then you can see that, it is starting from (r/a) equal to 1, 2 it is going up to 2.

And the variation (σ_r/S_y), which is on this axis has been shown by this particular plot. So, this is between 0 and 0.4. So, in this case, if again I compare it with the case of a single circular hole, so in the single case it was 3, at the boundary, at $\theta = 0^0$, however in this case it is 2.16. why are we considering? Because, you see that in this case it was S_y, but when you have the S_x or in the absence of $S_y = 0$ and, if only S_x will be there because it is symmetric.

So, you are going to get the similar stress concentration factor, so in this case, here we are getting 2.16, that is at θ will be equal to 90 degrees. So, here whatever was there at θ = 0, the same will correspond to θ = 90, in such type of loading condition because in this case, it is S_x, which is applied. So, this reduction is the result of the shielding effect of the adjacent holes in the row.

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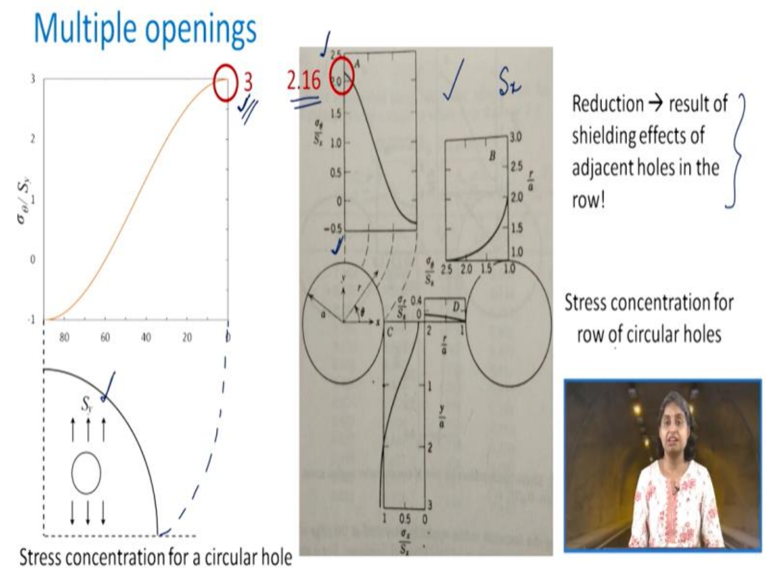

So, in this case, if again I compare it with the case of a single circular hole, in the single case it was 3, at the boundary at $\theta = 0^0$. However, in this case, it is 2.16. Why are we considering it, because you see that in this case, it was Sy, but when you have the S_x or in the absence of S_y, $S_y = 0$, and if only S_x will be there because it is symmetric? So, you are going to get a similar stress concentration factor. So, in this case, here we are getting 2.16, which is at theta will be equal to 90^0.

So here, whatever was there at theta was equal to 0. The same will correspond to θ = 90, in such type of loading condition because in this case, it is S_x that is applied. So, this reduction is the result of the shielding effect of the adjacent holes in the row.

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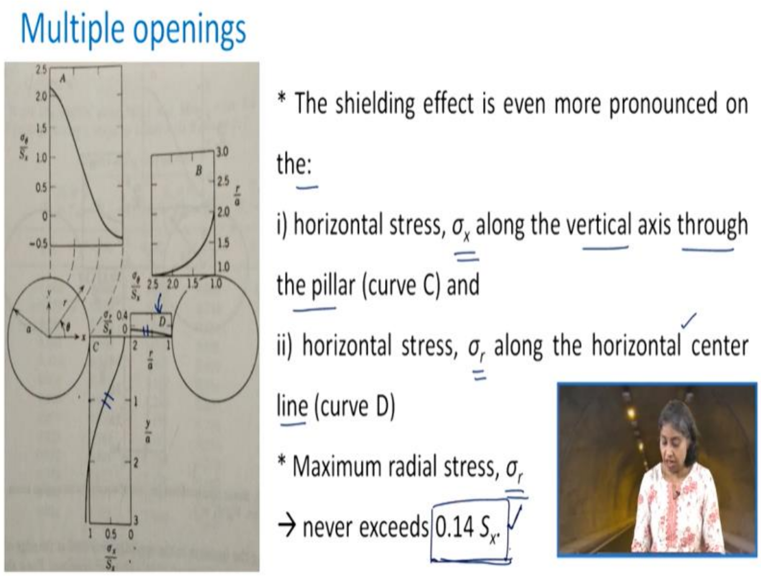

Now, this shielding effect is even, more pronounced on the horizontal stress σ_x along the vertical axis through the pillar, which is this curve and the horizontal stress σ_r along the horizontal center line which is this D. So, focus on these 2, you see that the maximum radial stress you focus on this curve D, the maximum radial stress σ_r it never exceeds 0.14 times S_x. You see that here it is sigma r upon S_x is given, so the maximum radial stress is 0.14 S_x.

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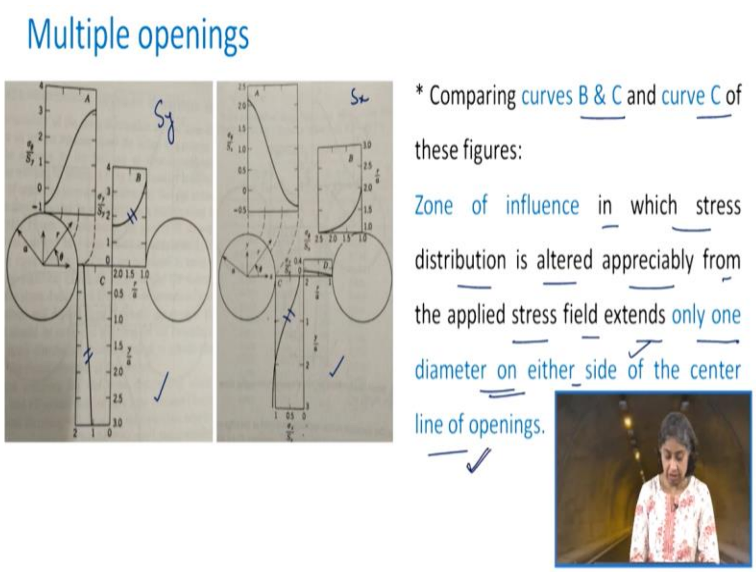

Now, if we compare the curves B and C and curve C of these 2 figures where, in one case you have S_y, and in another case, you have S_x. That is the loading condition, loading direction is different in both situations. So, the zone of influence in which the stress distribution is altered appreciably from the applied stress field it extends only one diameter on either side of the center line of the opening. So, you just take a look here, B, C, and the curve C here.

So, you can just take a look that, in either of the condition, wherever there is a change in the stress distribution. It is only up to 1 diameter on either side of the center line of the opening. So, kindly keep this in mind.

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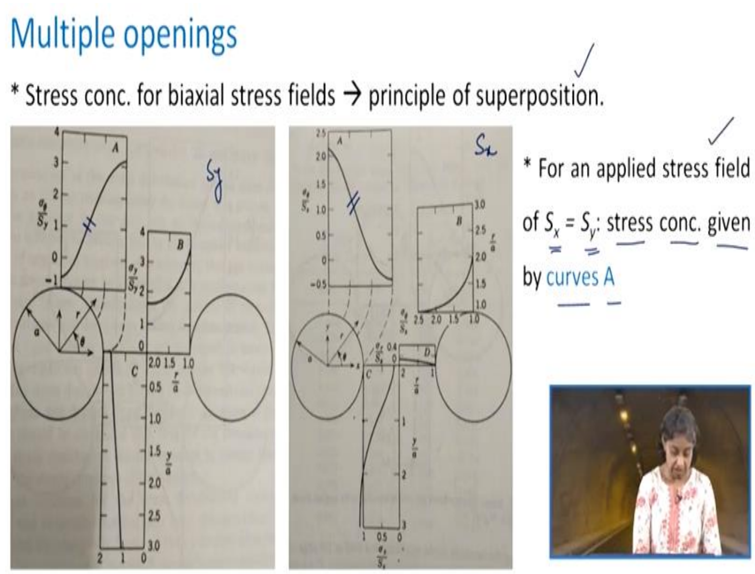

Now, the stress concentration for the biaxial stress field as of now, we discussed when S_y is present and in this case, when S_x is present. In case, if you have the biaxial stress field that means, both S_x and S_y, they are present and say for example that if these 2 are equal. Then what, we can do is these stress concentration factors can be given by the curves A. That is curve A, in this case and curve A in this case.

So, what we will do is because we are talking about the elastic ground condition and therefore the principle of superposition will be holding good. In this case, if you just try to combine these 2 then, in that case, you will be able to get the stress concentration for an applied stress field which is biaxial.

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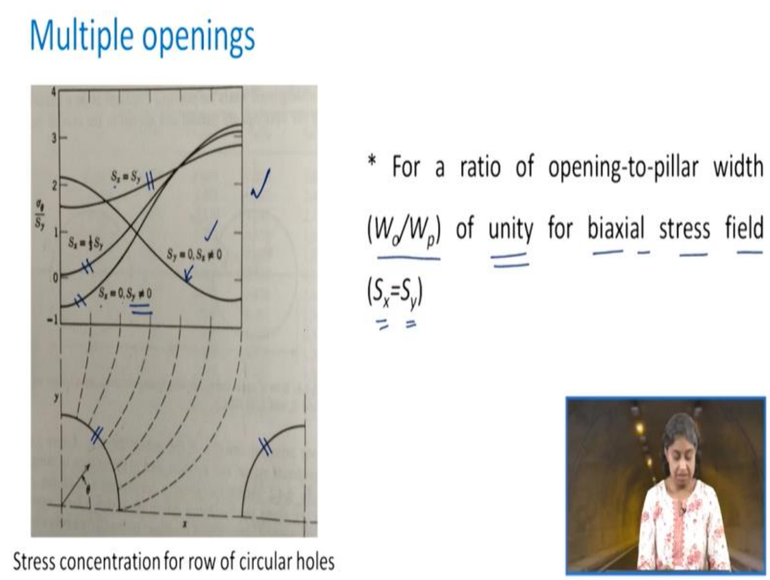

Take a look, at what it looks like. So, you see that we have a row of circular holes, which means 2 we are considering here. This is the first one, and this is the second one, we are considering only one-quarter of these holes. Now, these results they are for W_o/Wp to be equal to unity, and we are considering the biaxial stress field, with the condition that $S_x = S_y$. So, as I mentioned just superimposed these curves, so, you see this is what it looks like.

So, this was the curve which is for $S_y = 0$ and S_x not equal to 0. This is the curve where $S_x = 0$, and S_y is not equal to 0. This is the curve when you have $S_x = S_y$ and, this is the curve when you have S_x=(S_y/3). So, you just simple superposition and, you will be able to get the stress distribution corresponding to the biaxial state of stress.

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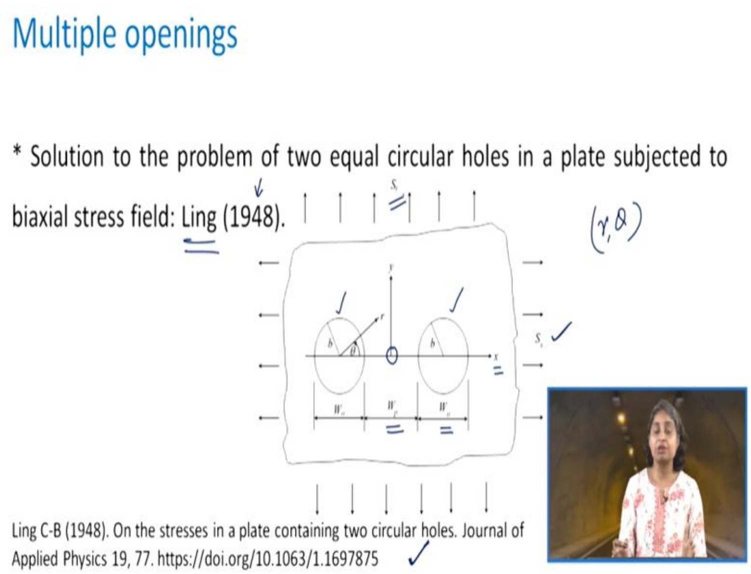

Now, the solution to the problem of 2 equal circular holes in a plate subjected to a biaxial stress field, was directly solved by this author without superposition in 1948, and the details of this have been picked from this journal paper. So here, you see, how the geometry looks like. So, you have the 2 circular openings with the radius B, and any point in the space is again represented by r, θ. Where, theta is measured in the anticlockwise direction, from the x-axis and note here, that the origin is at the center of the pillar width of the opening is W_o, and the width of the pillar is W_p.

And, here it is subjected to a biaxial state of stress, where it is S_x in the x direction and S_y in the y direction. So, this problem was solved directly for the biaxial state of stress.

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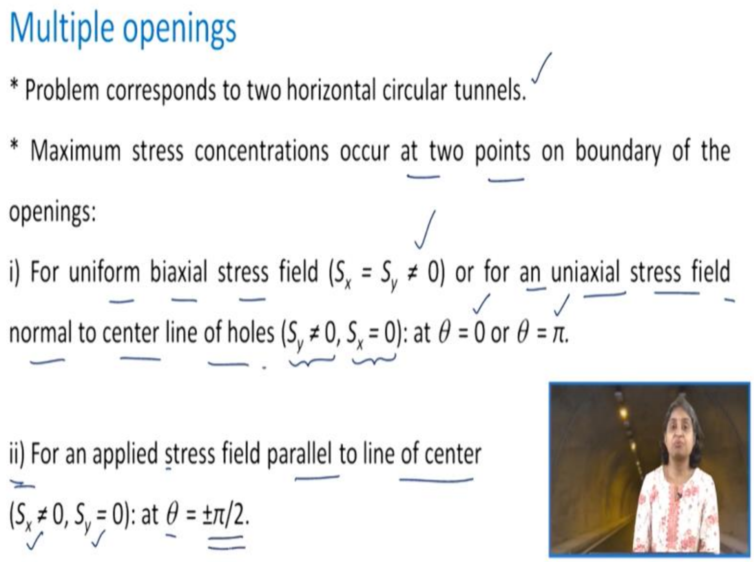

So, this problem corresponded to the 2 horizontal circular tunnels, and it was observed that the maximum stress concentrations occur at 2 points on the boundary of the openings. What are those 2 points? So, in case, if you have the uniform biaxial stress field, what do we mean by this mathematically? Is that $S_x = S_y$, and both are non-zero or for a uniaxial stress field, which is normal to the center line of the hole?

That means, normal to the center line of the whole means, that there is going to be the y component of the stress, but not the S component of the stress. So, that mathematically shows that $S_y \neq 0$, and $S_x = 0$, and then the location that you will have is $\theta = 0$, and $\theta = \pi$ is 180^0. Now, in the case of the second situation, for an applied stress field parallel to the line of the center, that means here, S_x is non-zero and $S_y = 0$. These maximum stress concentrations will be occurring on the boundary at $\theta = \pm \pi/2$.

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Multiple openings

Maximum stress concs. for two circular holes in a plate (after Ling 1948).

W_p/W_o	$S_x = S_y \neq 0$		$S_y \neq 0, S_x = 0$		$S_x \neq 0, S_y = 0$
	σ_t/S_y		σ_t/S_y		σ_t/S_y
	$\theta = 0$	$\theta = \pi$	$\theta = 0$	$\theta = \pi$	$\theta = \pm\pi/2$
0	2.894	0.000	3.869	0.000	2.569
0.5	2.255	2.887	3.151	3.264	2.623
1.0	2.158	2.411	3.066	3.020	2.703
2.0	2.080	2.155	3.020	2.992	2.825
4.0	2.033	2.049	3.004	2.997	2.927
7.0	2.014	2.018	3.001	2.999	2.970
10.0	2.000	2.000	3.000	3.000	3.000

Now, the maximum stress concentration for 2 circular holes in a plate, again these values has been pegged from this reference only. So, you see that, when the pillar-to-opening width ratio changes from 0 to 10, how the various stress concentration, they change for different loading conditions. So, the first column tells us about the biaxial state of stress in which situation S_x and S_y both are equal and non-zero.

While, the second column gives you the idea about the stress field, when there is the stress applied in only the y direction and not in the x direction. And, the third one represents the values of the maximum stress concentration factor, when you have the stress field only in x direction and not in the y direction. So, we have the variation at theta equal to 0 and π for these 2 situations, as I mentioned with reference to the previous slide, and here in this case, it is $\theta = \pm \pi/2$.

And, you see how the maximum stress concentration for 2 circular holes in a plate, they vary as the pillar width to opening width ratio increases from 0 to 10. See, at θ equal to 0 for the first situation biaxial state of stress. This is varying from 2.894 to 2 and at 180^0, it is varying from 0 to 2. Similarly in case, if you have S_y non-zero and $S_x = 0$, that means, it is a uniaxial state of stress.

So, again here these are the 2 points, where you are going to have the maximum stress concentration, that is at $\theta = 0$ and $\theta = 180^0$, this is how the variation will look like, and in case you have the stress applied in the x direction and not in the y direction. In that case, the maximum stress concentration is going to be at $\theta = \pm 90^0$, and it varies from 2.569 and increases to 3.

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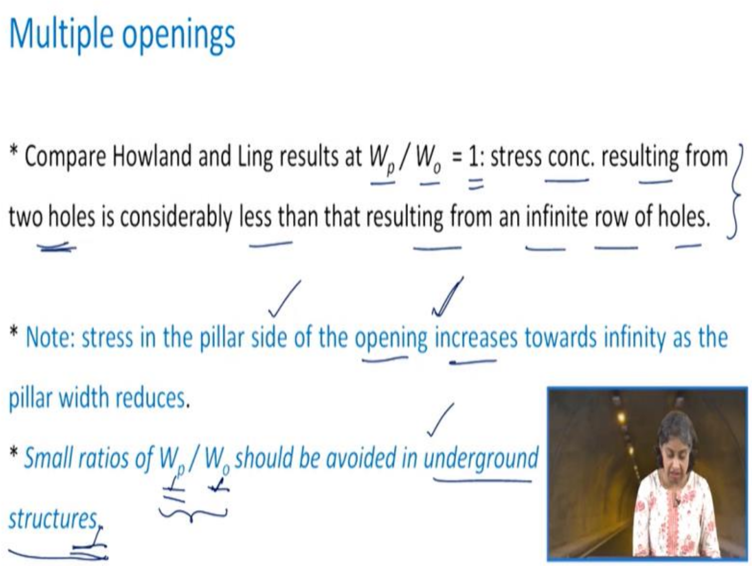

So, this is how we can find out the stress distribution directly, by the method that was given by Ling in 1948. When he solved them directly, these problems were subjected to a biaxial state of stress. Now, if you compare the results of both of these authors at W_p/W_o to be equal to 1. The stress concentration resulting from the 2 holes is considerably less than the resulting from the infinite row of holes.

So, kindly keep this in mind that in case, if you have 2 holes and, if you have an infinite row of holes, there is going to be a difference in the result. Now, there are a few things that you need to keep in mind, the stress in the pillar side of the opening increases towards infinity, as the pillar width reduces. That means when this W_p reduces then this condition is satisfied, and therefore the small ratios of the width of the pillar to the width of the opening should be avoided in underground structures.

So, this has reference with respect to the stress concentration at various locations. So, when you go for the underground structures where you have multiple openings. So, you should always keep that in mind that you should avoid a small ratio of W_p/W_o. So, this is how you can deal with the multiple opening. So, we discussed today the multiple openings having a circular shape. As far as this course is concerned, other shapes of the opening are beyond the scope.

However, there are many numerical studies which are available, which handle the non-circular shapes as well. So, this is what, was all about the multiple openings and their stress distribution. So, in the next class, we will start our discussion on a new topic, that is opening in laminated rocks, thank you very much.