Underground Space Technology
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Module No # 06
Lecture No # 28
Elasto-Plastic Analysis of Tunnels: Tresca Yield Criterion-01

Hello everyone, in the last class we had the discussion on openings in laminated rocks and we discussed that, depending upon how these lamina form in the laminated rocks they are oriented. Based upon that, we defined various cases and we saw that, what all are the appropriate support system in each of those that can be provided in the roof, as well as in the side walls. So, today we will start a new topic which is the elastoplastic analysis of tunnels.

So, in this one we will be doing the analysis following 2 criteria, one is the Tresca yield criterion and another one is the Mohr-Coulomb failure criterion. So, as of now whatever, that we discussed we considered the rock to be an elastic medium, which may not be the situation. So, in case if the rock is behaving like an elastoplastic material, how to do the analysis for the tunnels that we are going to learn. So, as a first step we are going to consider the circular tunnel and, today we will take up the part of the analysis using Tresca yield criterion.

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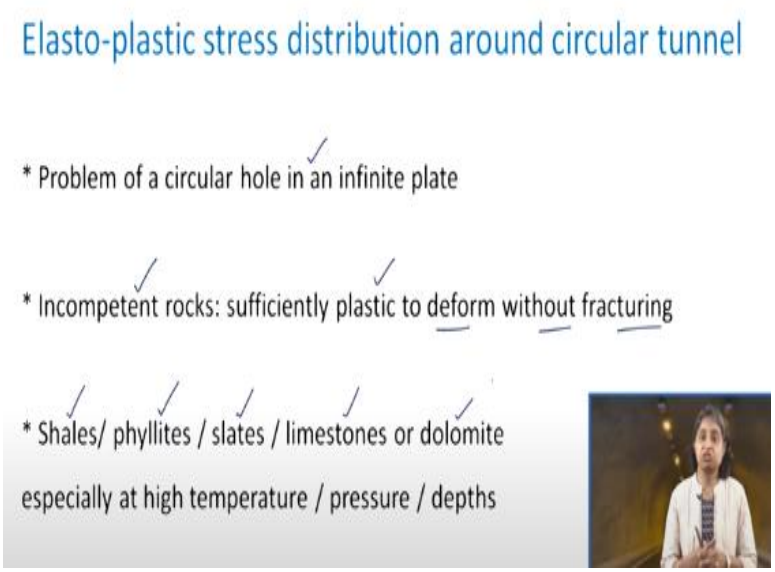

So, first we learn about the problem then we will discuss about the geometry loading condition, material property, and the boundary conditions, as far as this analysis is concerned. So, when we say that we are going for the elastoplastic analysis so, basically our focus is on the determination of the stress distribution around a circular tunnel following the elastoplastic analysis. That means that, we are going to consider the rock as the elastoplastic material.

So, this problem corresponds to a problem of a circular hole in an infinite plate. Here, we are going to consider that the rock is incompetent rock and, hence we can safely assume that this type of rock is sufficiently plastic to deform without fracturing. So, what kinds of rock can exhibit such type of behavior? Some of these include shales, phyllites, slates, limestones, or dolomite, especially at high temperature pressure and depths. So, therefore it is extremely important for us to have the idea about this elastoplastic analysis.

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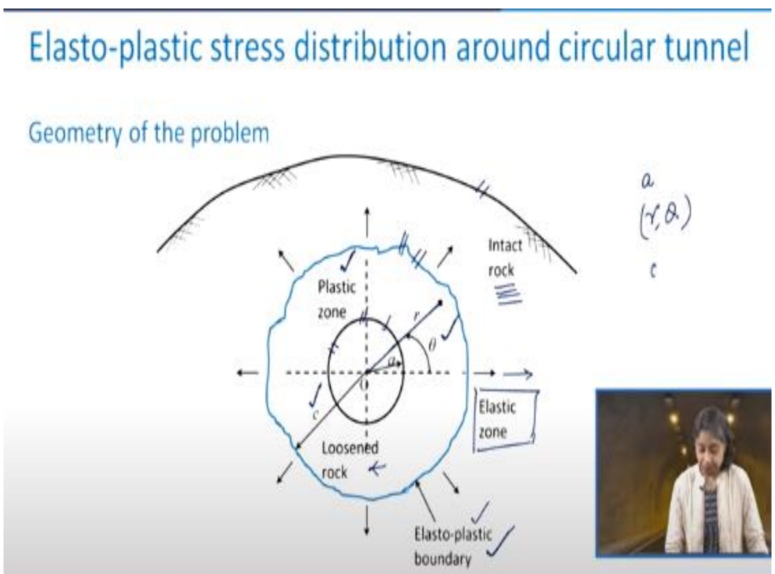

So, first coming to the geometry of the problem, so, here you see that we have the ground surface and this is assumed to be an intact rock. Which is not the situation in the field, in the field what you get is, the rock mass but then to start with we are making this assumption that the excavation is made in intact rock.

The excavation is circular in shape having the radius a, as has been shown in the figure any point in the rock medium is represented by the coordinate system r, theta, where r is the radial distance as has been shown by this line here. And theta be the angle that is measured from this horizontal axis in the anticlockwise direction. What happens when you excavate the opening may be using the blasting, or some other way some other method.

There is going to be a zone, where the rock will be loosened because of the excavation of this opening. So, that area I am calling as the loosened rock, and we assume that the zone between this excavation circular opening and up to the end of this loosened rock, it is in plastic zone. And beyond this, there is again the existence of the elastic zone, and therefore, this blue color line, it shows you the elastoplastic boundary as has been written here.

Now, we have some assumptions which are made I will come to that little later, but then, first assumption that we are going to have here is that because, of the symmetry of loading boundary condition geometry. What we are expecting, is that this elastoplastic boundary will also have a circular shape and therefore, its radius is defined by this alphabet c.

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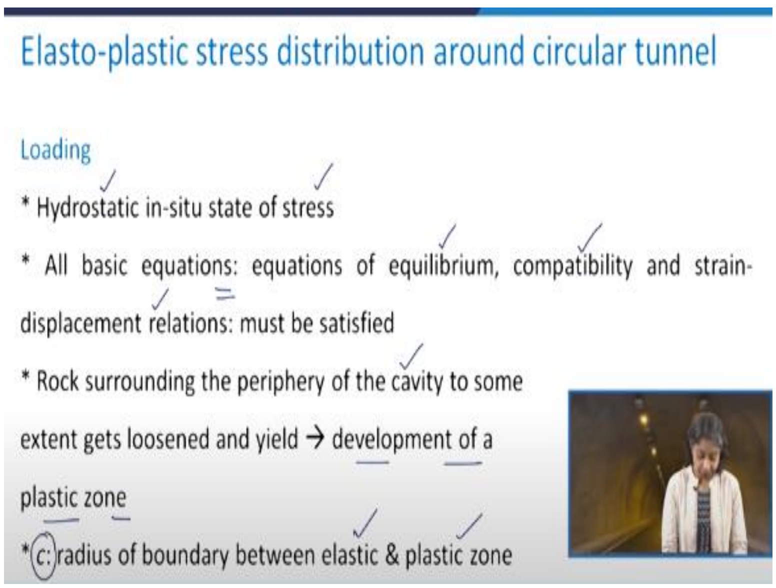

Coming to the loading aspect, it is considered that there exists the hydrostatic in situ state of stress. So, all the basic equations are going to be the equations of equilibrium, compatibility, and strain displacement relations, and these all must be satisfied. If you recall when we discussed about the elastic analysis, or when we were finding out the stress distribution for the elastic case for a circular tunnel, there also we considered the similar type of basic equations like equations of equilibrium, compatibility, and strain displacement relation.

So, we are going to make use of the similar equations, the only difference is going to be that now, we are going to use some kind of elastoplastic yield criterion to define the behavior of the rock. In the case of this situation, elastoplastic analysis what we are considering is that the rock, which is surrounding the periphery of the cavity or the opening, it gets loosened to some extent and it yields and therefore there is a development of a plastic zone. And, as I explained that we define the radius of boundary between elastic and plastic zone by the alphabet c.

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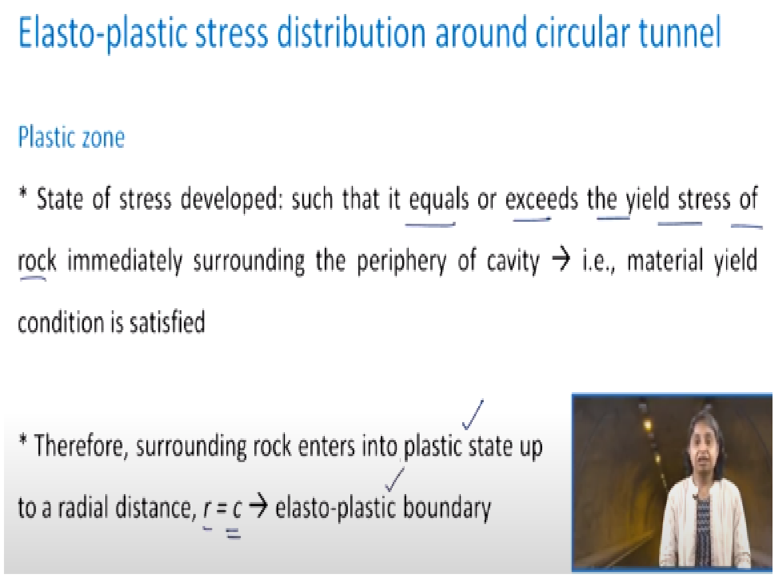

As far as plastic zones zone is concerned, a state of stress which is developed is such that, it equals or exceeds the yield stress of the rock immediately surrounding the periphery of the cavity. That means, that the material yield condition is satisfied and therefore, the rock enters into the plastic state up to a radial distance r = c which is defined by the elastoplastic boundary.

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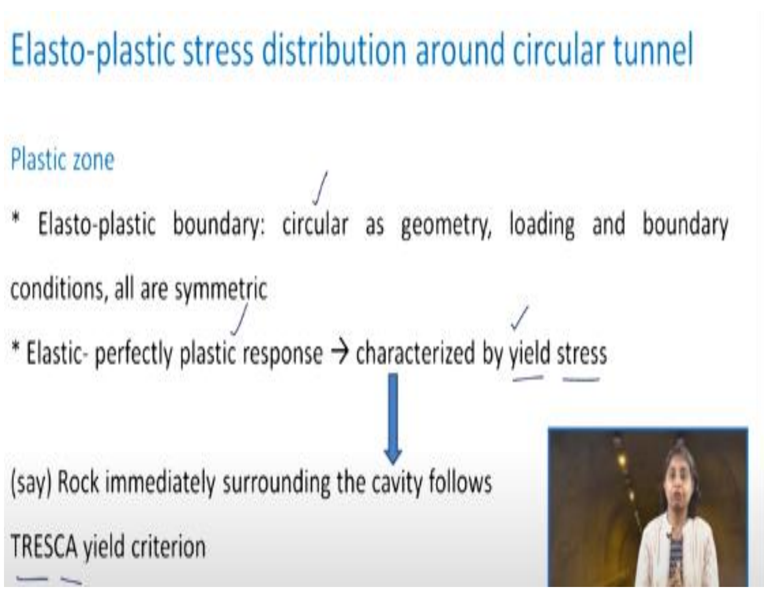

Now, as I mentioned that being the geometry loading and the boundary condition all symmetric, we can safely assume this elastoplastic boundary to be circular in nature, and the elastic perfectly plastic response is assumed which is characterized by a yield stress. So, how are we going to define this yield stress? So, say rock immediately surrounding the cavity, it follows the Tresca yield criterion, it is not necessary that always it will follow this yield criterion.

There, there are n number of failure criterion, that we have discussed somewhere in some of the earlier lectures in this course, like Tresca then Mohr-Coulomb, Hoek and Brown and there are many. So, here in this particular case what we are going to take is that, that the rock is following the Tresca yield criterion.

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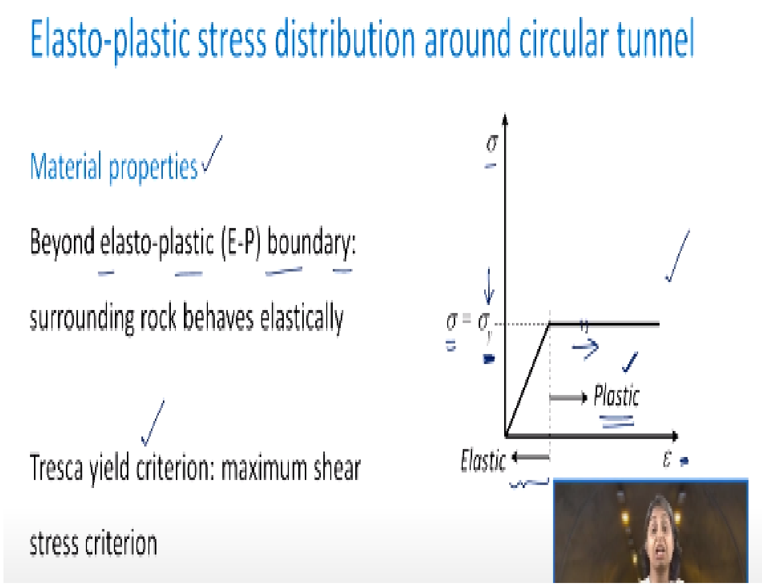

Now, how should we define this yield criterion that will say about the property of the material, so here comes the material properties that is beyond the elastoplastic boundary the surrounding rock will behave in an elastic manner. But, in the plastic zone, there is going to be the Tresca yield criterion which is the maximum shear stress criterion. And, you can see that in this figure I have shown the stress versus strain relationship that, how this is going to look like.

So, here up to the yield stress this zone is elastic and beyond this it is in the plastic zone and why I am calling this as the maximum shear stress criterion, Means, that this plastic zone is going to be defined, when this sigma that is the stress reaches to the maximum value of this as the yield stress. As, has been shown you see if you go beyond this strain is increasing but the stress remains the same, that is equal to the maximum stress or the yield stress.

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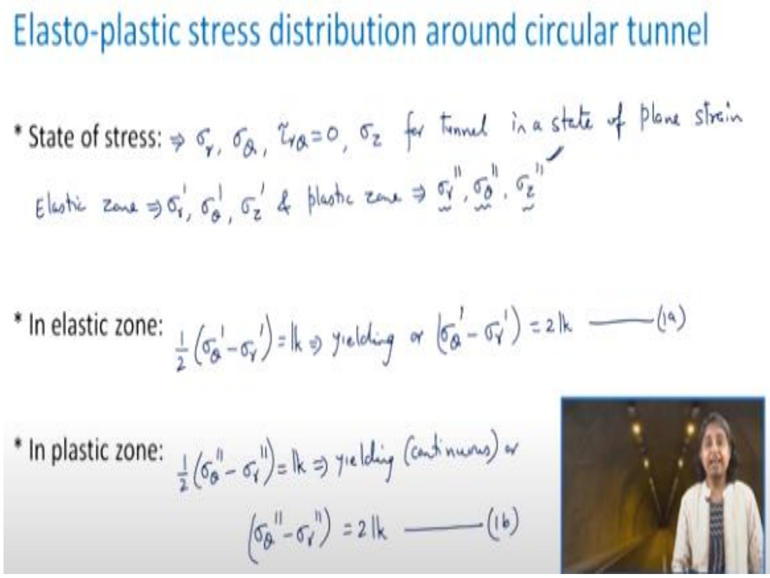

So, what are going to be the state of stress in elastic zone, and plastic zone, let us try to understand these. So, the state of stress is going to be

$\sigma_r, \sigma_\theta, \tau_{r\theta} = 0$ And, σ_z for the tunnel in a state of plane strain. And, in the elastic zone, I am going to define these, state of stresses as $\sigma_r', \sigma_\theta', \sigma_z'$ And, in the plastic zone I am going to define these as $\sigma_r'', \sigma_\theta'', \sigma_z''$.

So, this is what is going to be the difference in the stresses, so, stresses in the elastic zone will have the single prime, and stresses in the plastic zone will be represented by the typical notation, but having the double prime. So, in the elastic zone what we are going to have as far as the yielding is concerned that is going to be

$\frac{1}{2}(\sigma_\theta' - \sigma_r') = lk$

which will define the yielding.

Or, we can write

$(\sigma_\theta' - \sigma_r') = 2lk$

that is 1a. But, in case of the plastic zone what we are going to have is

$\frac{1}{2}\left(\sigma_\theta'' - \sigma_r''\right) = lk$

in this case this is going to be again yielding, which is going to be continuous. Or we can write

$$\left(\sigma_\theta'' - \sigma_r''\right) = 2lk$$

this is equation number 1b.

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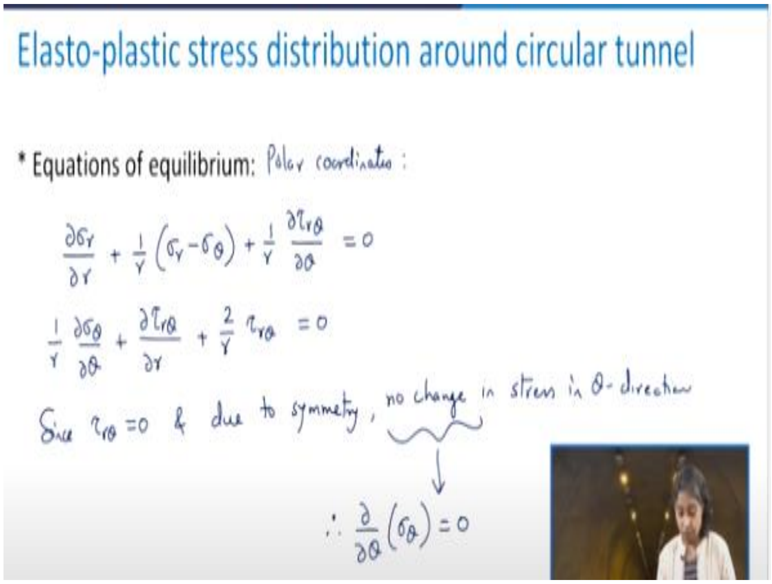

So, let us write the equations of equilibrium in case of the polar coordinates, so, we use the polar coordinate system here. So, what we are going to have is

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} = 0$$

And we have,

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} = 0$$

Now, we have seen that

$$\tau_{r\theta} = 0$$

and also due to symmetry, there will be no change in the stress in theta direction. So, what does this mean, this will give me

$$\frac{\partial}{\partial \theta}(\sigma_\theta) = 0$$

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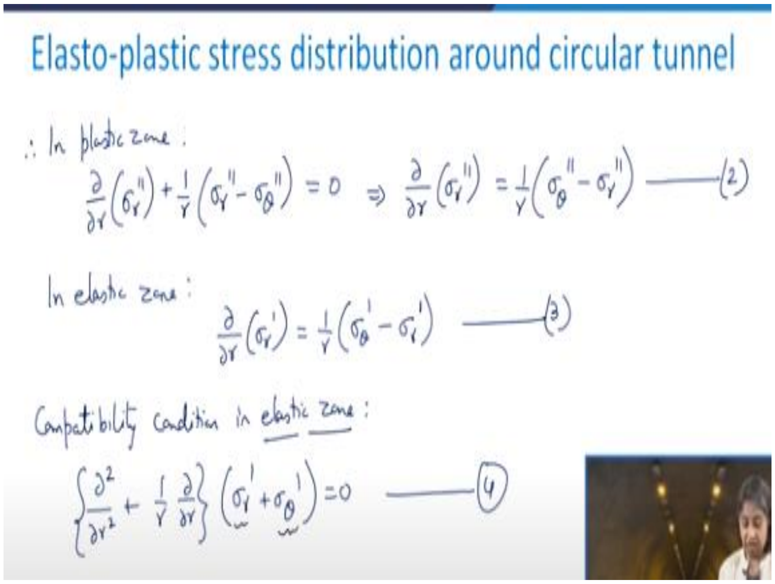

So, what will happen in the plastic zone; what we will have as

$$\frac{\partial}{\partial r}\left(\sigma_r''\right)+\frac{1}{r}\left(\sigma_r''-\sigma_\theta''\right)=0$$

remember we assume that in plastic zone how we are going to present the state of stress, so there we decided that we will use double prime. So, this is going to be equal to 0, because $\tau_{r\theta}=0$.

Or we can have

$\frac{\partial}{\partial r}\left(\sigma_r''\right)=\frac{1}{r}\left(\sigma_\theta''-\sigma_r''\right)$so, I will mark this as equation number 2.

Now, what will happen in elastic zone, so, in the similar fashion can we not write this

$$\frac{\partial}{\partial r}\left(\sigma_r'\right)=\frac{1}{r}\left(\sigma_\theta'-\sigma_r'\right)$$

Now, what is going to be the compatibility condition in elastic zone. Let us see, compatibility condition in elastic zone, that is going to be

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right\}\left(\sigma_r'+\sigma_\theta'\right)=0$$

because we are talking about the elastic zone, so I am using these stresses having the single prime.

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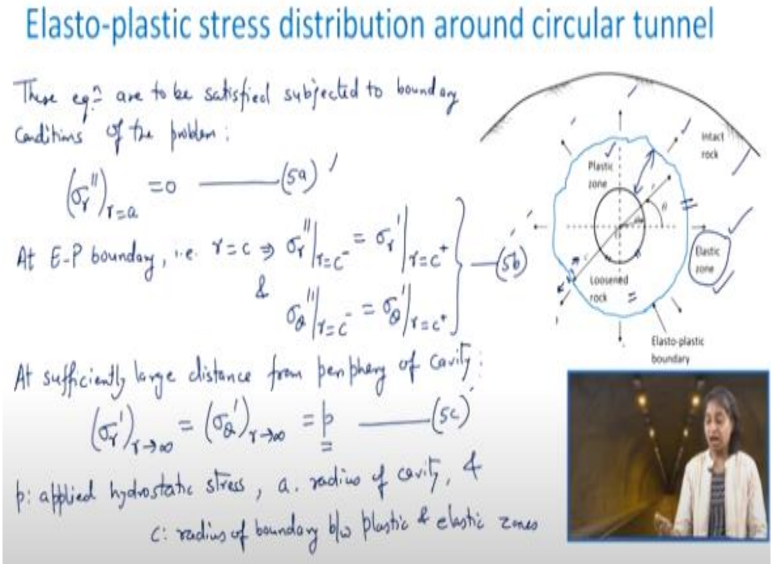

Now, these equations are to be satisfied subjected to appropriate boundary conditions, so with respect to the geometry that I have shown here let us try to define those boundary conditions. So, these equations that we had in the previous slides, these are to be satisfied subjected to boundary conditions of the problem. So, let us define these, so first of all if we just take see here you have the plastic zone, and beyond r = c you have the elastic zone.

So, what will happen because it is the stress-free boundary, because it is excavated, so, we are going to have

$(\sigma_r'')_{r=a} = 0$

make this as equation number 5a. Then, at elastoplastic boundary, so in short I am writing as e dash p boundary, what does this mean what is the value of r here? So, at elastoplastic boundary means this boundary and you see the value of r is c.

So, at r = c, what we have, there has to be the continuity of stresses so, therefore we will have whatever are these stresses just inside this boundary the same are going to be just outside the boundary. So, what we are going to have is see inside the boundary you have the plastic zone and outside the boundary you have the elastic zone. So, we have here

$\sigma_r''|_{r=c^-}\ \sigma_r'|_{r=c^+}$

that means, it is although it is c but little bit inside this elastoplastic boundary so it is just c-.

So, this is going to be sigma r prime at r = c + so, that the c + and c - they say that a point which is very very close to the elastoplastic boundary, so that for all practical purposes we can take it at the elastoplastic boundary. But, in one case it is in the plastic zone and in another case, it is in the elastic zone so therefore, we are writing it like this. Similarly, we will have the same situation for

$$\sigma''_\theta\,|_{r=c^-}\ \sigma'_\theta|_{r=c^+}$$

this is going to be, sigma theta prime at r equal to c + i will mark these 2 equations together as 5b.

Now what will happen if you go at sufficient large distance from the periphery of the cavity? So, at sufficiently large distance from periphery of the cavity what we are going to have is the elastic domains only see because the plastic zone will exist only up to certain extent beyond that it is again the elastic zone. So, when we have the sufficient large distance from the periphery of the boundary, we will have the elastic situation.

So, in this case what we are going to have is that

$$(\sigma'_r)_{r\to\infty} = (\sigma'_\theta)_{r\to\infty} = p$$

because, if you recall when we discussed about the loading condition, I mentioned to you that we are going to have the hydrostatic state of stress. So, in this case, as has also been shown in the figure that all around this in situ stresses they are there.

So, in r direction as well as in the theta direction we have this as p so this is going to become the third boundary condition. Where this p is the applied hydrostatic stress of course we know, a, is the radius of cavity, and c is the radius of boundary between plastic and elastic zones. So, this is how we can define all the boundary conditions, and now using these boundary conditions we have to solve all those equations of equilibrium and compatibility equation to obtain the stresses all around the periphery and in the raw.

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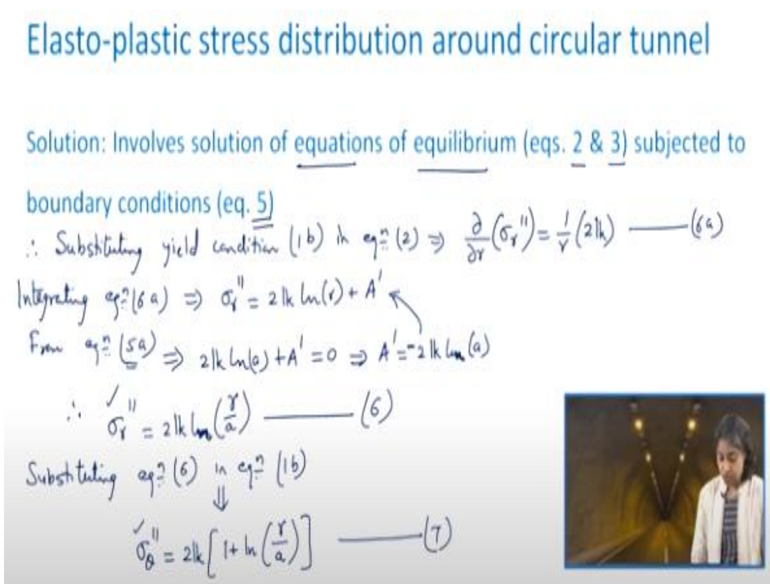

So, this solution will involve the solution of equations of equilibrium, which were given by equations 2 and 3 and subjected to the boundary conditions, which we just saw that given by equation number 5. So, what we are going to do is we substitute the yield condition in the equation of equilibrium which is equation number 2, and see how it looks like. So, what we do is we substitute the yield condition which is 1b in equation number 2, so, what we get is

$$\frac{\partial}{\partial r}\left(\sigma_r''\right) = \frac{1}{r}(2lk)$$

make this as 6a.

Then if I integrate this equation what we will get? So, integrating this equation 6a, what we have is

$$\sigma_r'' = 2lk\ln(r) + A'$$

A prime is the constant of integration which is to be obtained from appropriate boundary condition. So, let us apply the boundary conditions, so I use the equation 5a, for this purpose, so, what we have is

$2lk\ln(a) + A' = 0$ because 5a stated that sigma r prime at r = a, is going to be equal to 0.

So, that is what that we have here so from here I can say that

$$A' = -2lk\ln(a)$$

with the negative sign here. So just substitute it a prime here in this equation, so, what we are going to get is,

$$\sigma_r'' = 2lk \ln\left(\frac{r}{a}\right)$$

make this equation as may be say equation number 6. Now I substitute this equation in equation number 1b, so that I can obtain sigma theta double prime also.

So, substituting this equation 6 in equation 1 b, so, what I am going to get here as

$$\sigma_\theta'' = 2lk\,[1 + \ln\left(\frac{r}{a}\right)]$$

mark this equation as equation number 7. Now, having once known this σ_r'' and σ_θ'' of course these are state of stress in the plastic zone.

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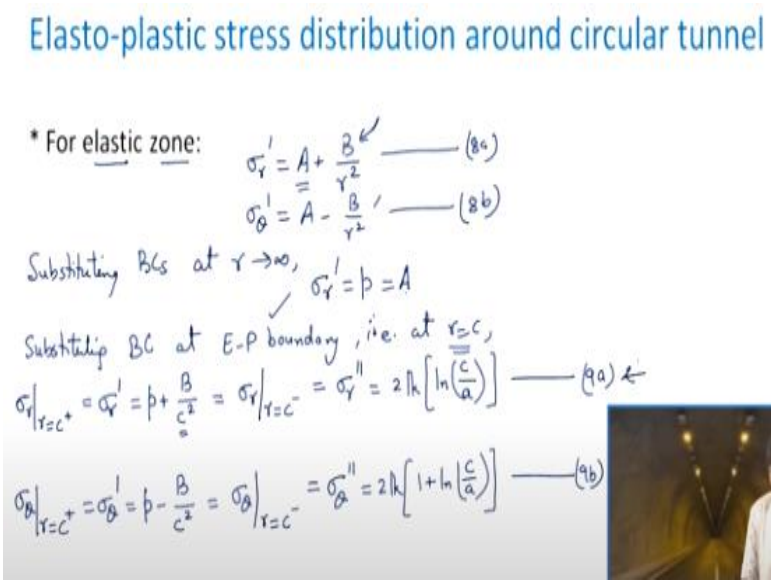

So, let us talk about what happens in the elastic zone so, here I will take you back to the elastic analysis of the circular tunnel, and there we saw that we were getting these stresses of this form that is

$$\sigma_r' = A + \frac{B}{r^2}$$

And, we had this

$$\sigma_\theta' = A - \frac{B}{r^2}$$

so you can make it as 8a, and 8b. So, from where these are coming, maybe, you can refer to our discussion on the elastic stress distribution around circular tunnel, and you will get the idea.

Now, you we substitute the boundary conditions at r tending to infinity because there it is in the elastic zone. So, substituting boundary conditions at r tending to infinity, what we have here is that

$\sigma_r^{'} = p$

So, you just substitute it here, so, you see the first term will become a, and the second term in this equation will become equal to 0 because r being tending to infinity. So, therefore this

$\sigma_r^{'} = p = A$

Then, I substitute the boundary condition at elastoplastic boundary that is at r = c, so what we have that is

$\sigma_{\frac{r}{r=c+}} = \sigma_r^{'}$

means, we are in the elastic zone. So, I am going to have this as

$\sigma_{\frac{r}{r=c+}} = \sigma_r^{'} = p + \frac{B}{c^2}$

because I will substitute it here

a = p so that comes here; + b upon r = c so, that is b upon c square. And, this is going to be equal to $\sigma_{\frac{r}{r=c-}} = \sigma_r^{''}$

what is this? This is sigma r double prime, because now it we are in plastic zone.

And, how did we define this we just derived

$2lk\left[\ln\left(\frac{c}{a}\right)\right]$

this is what that we are going to have now 9a. And then we can write the similar expression with reference to

$\sigma_{\frac{\theta}{r=c+}} = \sigma_\theta^{'}$

being in the elastic zone. And, this is going to be

$\sigma_{\frac{\theta}{r=c+}} = \sigma_\theta^{'} = p - \frac{B}{c^2} = \sigma_{\frac{\theta}{r=c-}} = \sigma_\theta^{''} = 2lk\left[1 + \ln\left(\frac{c}{a}\right)\right]$

So, this is how we apply the boundary condition at elastoplastic boundary where r = c so focus on this equation 9a, and we try to simplify this.

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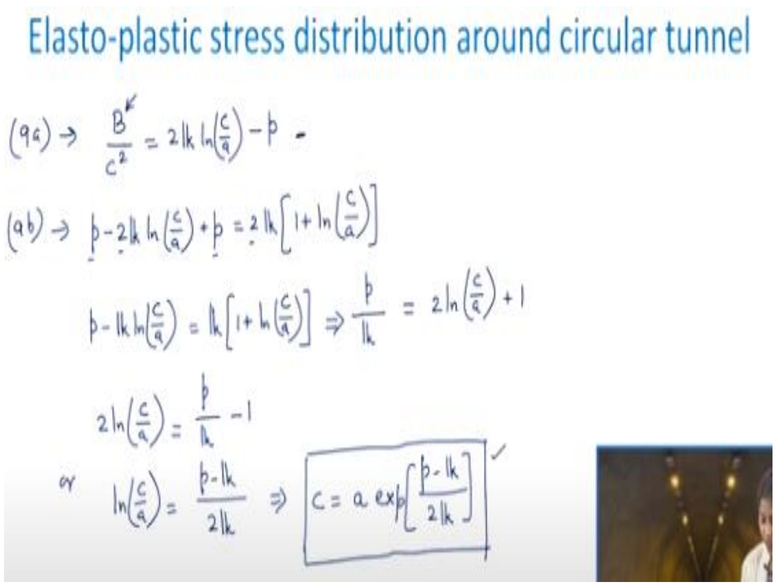

We can determine the constant b, so I take this equation 9a, and what I have from this is

$$\frac{B}{C^2} = 2\text{lk}\ln\left(\frac{c}{a}\right) - p$$

And from 9b what we have is, just substitute this B/c^2 in equation 9b, so we will have

$$p - 2\text{lk}\ln\left(\frac{c}{a}\right) + p = 2\text{lk}[1 + \ln\left(\frac{c}{a}\right)]$$

Now if you just try to simplify this, so what we will have is, so this will go p, so this will become 2p and 2 is here, 2 is here as well.

So, throughout it will get cancel so what we have is

$$p - \text{lk}\ln\left(\frac{c}{a}\right) = \text{lk}\left[1 + \ln\left(\frac{c}{a}\right)\right]$$

or we can write

$$\frac{p}{\text{lk}} = 2\ln\left(\frac{c}{a}\right) + 1$$

Or we have

$$2\ln\left(\frac{c}{a}\right) = \frac{p}{\text{lk}} - 1$$

or

$\ln\left(\frac{c}{a}\right) = \frac{p-lk}{2lk}$

or we can have

$c = a \exp\left[\frac{p-lk}{2lk}\right]$

So, this is how we can determine the radius of the elastoplastic boundary, that is c now I just substitute these, the the expression of the c here in this equation.

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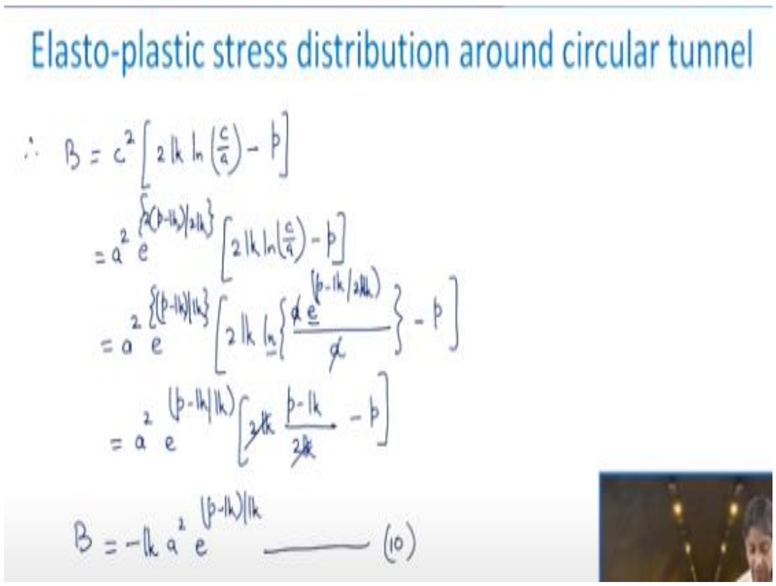

What I am going to get is;

$$B = c^2[2lk\, ln\left(\frac{c}{a}\right) - p]$$

so, this is going to be

$$= a^2 e^{\{\frac{2(p-lk)}{2lk}\}}[2lk\, ln\left(\frac{c}{a}\right) - p]$$

So, this will become

$$= a^2 e^{\{\frac{(p-lk)}{lk}\}}[2lk\, ln\{\frac{ae^{(p-\frac{1k}{2lk}}}{a}\} - p]$$

So, this will be

$$= a^2 e^{\frac{(p-lk)}{lk}}[2lk\, \frac{p-lk}{2lk} - p]$$

So, here we will get a very simplified expression that is

$$B = -lk\ a^2 e^{(p-lk)/lk}$$

so make this equation as equation number 10.

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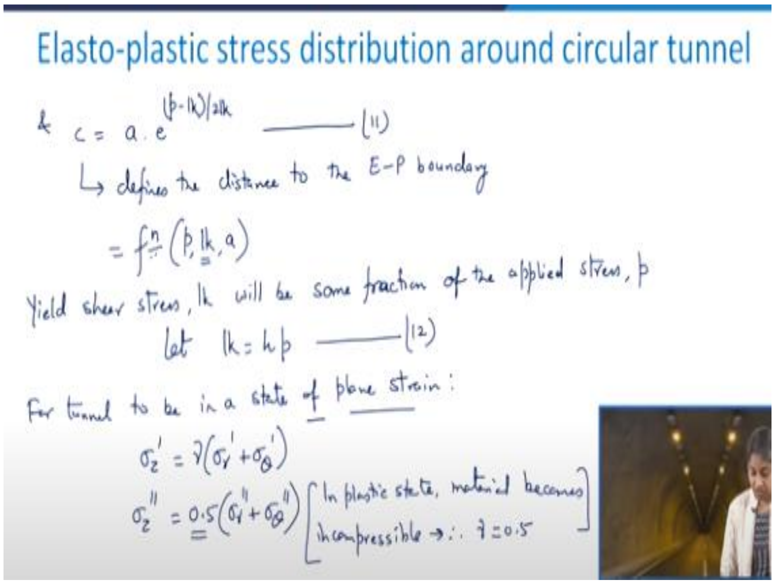

And, we already have this

$$c = a.e^{\frac{(p-lk)}{2lk}}$$

this is equation number 11. So, basically, this c it defines the distance to the elastoplastic boundary, and we can see that it is the function of p kappa and a. Now let us focus on this yield, kappa how to get this, so the yield shear stress which is kappa, this will be some fraction of the applied stress which is the p. So, what we do is we assume that let this

$lk = hp$, make this equation as equation number 12.

So, for the tunnel to be in the state of plane strain, so for the tunnel to be in a state of plane strain this

$$\sigma_z^{'} = \gamma(\sigma_r^{'} + \sigma_\theta^{'})$$

And

$$\sigma_z^{''} = 0.5(\sigma_r^{''} + \sigma_\theta^{''})$$

Why this poisons ratio becomes equal to 0.5, because in the plastic state the material becomes incompressible and corresponding to an incompressible material this Poisson's ratio becomes equal to 0.5.

So, maybe you can write here that in the plastic state the material becomes incompressible and therefore; what you have is the Poisson's ratio as 0.5. So, this is how the state of stress in the plane strain situation for a tunnel can be written, so this is how the state of stress can be written for a tunnel in the plane strain condition. So, what we have seen that how using the appropriate boundary condition, we can solve the equations of equilibrium and the compatibility condition.

And, now we will further see this analysis and then try to plot these stress distributions, all around the periphery of the tunnel. And then we will try to see some aspects that what happens at the elastoplastic boundary, or what happens when we move away from the elastoplastic boundary to the stress distribution, radial as well as the tangential stress distribution. So, we will continue with our discussion in the next class, thank you very much.