Underground Space Technology
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Module No # 07
Lecture No # 30
Basics of Rock Engineering: Coring, Sampling, UCS of intact Rock

Hello everyone, in the previous class we discussed about the elastoplastic analysis of a circular tunnel. And the tunnel was excavated in the rock which was following Tresca yield criterion. So, today we will learn about another failure criterion which is Mohr-Coulomb criterion. So, what we are going to learn is the elastoplastic analysis of the circular tunnel in the rock following Mohr-Coulomb failure criterion.

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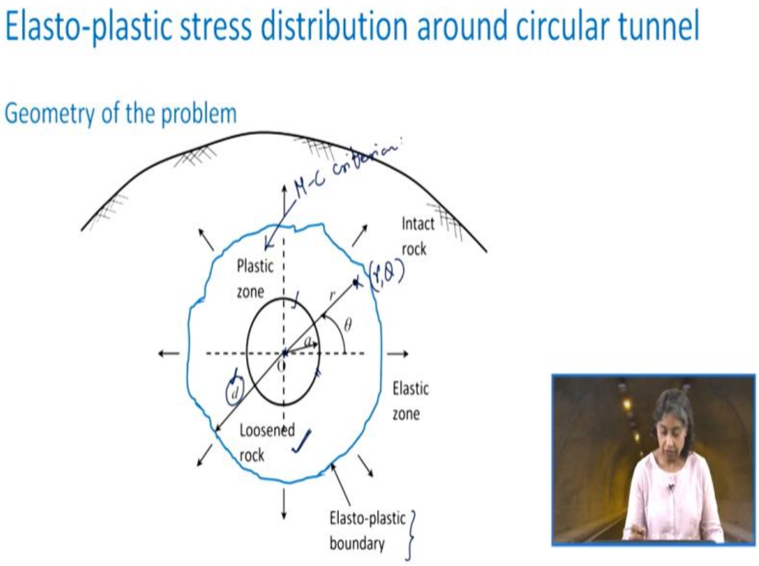

So, we are going to consider the similar geometry, except for the fact that here we are going to consider the elastoplastic boundary radius as d. Rest everything remains the same, loading, geometry, boundary conditions, everything being symmetric, and therefore we are assuming that the shape of the last 2 plastic boundary is going to be circular. Any point in the domain we can represent by r, theta, where r is the radial distance of this point from the center of the circle, and theta being measured in the anti-clockwise direction, from the horizontal axis.

So, upon the excavation of this tunnel, there is going to be the formation of the plastic zone in the surrounding of the periphery of this cavity. And, that is going to be the loosened rock and we are representing the rock here by Mohr-Coulomb criterion

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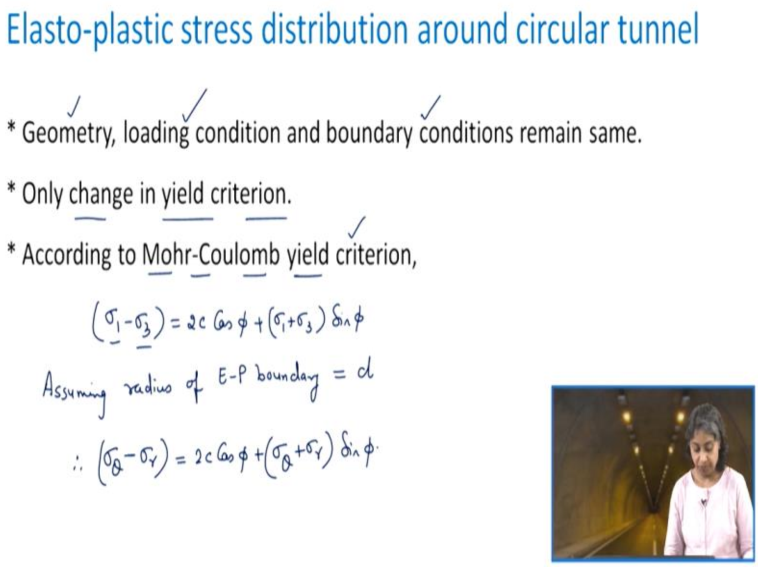

So, as I mentioned to you everything remains the same such as geometry, loading condition, and boundary condition, except for that there is change in the yield criterion. So, we are going to have the rock following Mohr-Coulomb yield criterion. So, according this we have

$$(\sigma_1 - \sigma_3) = 2c\ cos\emptyset + (\sigma_1 + \sigma_3)\ \mathrm{sin}\emptyset$$

Now we assume the radius of elastoplastic boundary to be equal to d.

So, what we have here is sigma theta minus sigma r since the material is following Mohr-Coulomb yield criterion so,

sigma 1 and sigma 3 they will be replaced by sigma theta and sigma r respectively. So, this is

$$(\sigma_\theta - \sigma_r) = 2c\ cos\emptyset + (\sigma_\theta + \sigma_r)\ \mathrm{sin}\emptyset$$

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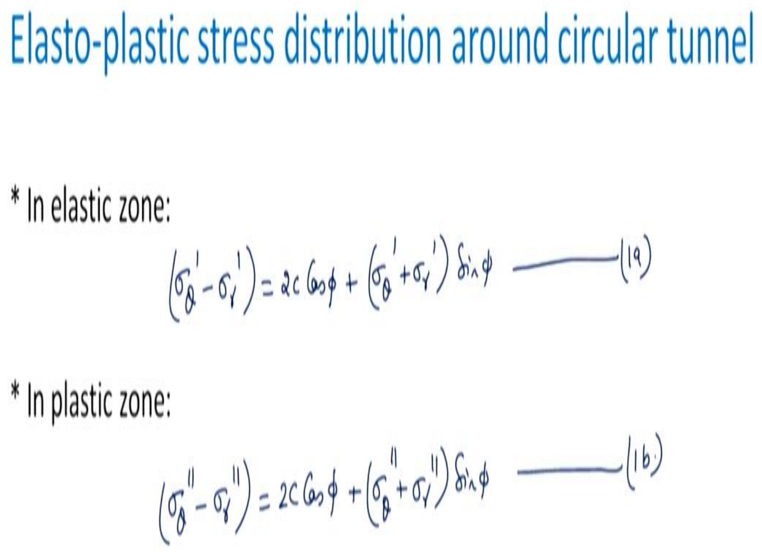

Now, in the elastic zone we can write the yield condition by single prime in this manner that is

$$(\sigma_\theta^{'} - \sigma_r^{'}) = 2c\ cos\emptyset + (\sigma_\theta^{'} + \sigma_r^{'})\ \sin\emptyset$$

This equation, I will write as 1a, and in the plastic zone we will represent these stresses using this double prime. So, accordingly, we will write in this manner that is

$$(\sigma_\theta^{''} - \sigma_r^{''}) = 2c\ cos\emptyset + (\sigma_\theta^{''} + \sigma_r^{''})\ \sin\emptyset$$

that is 1b.

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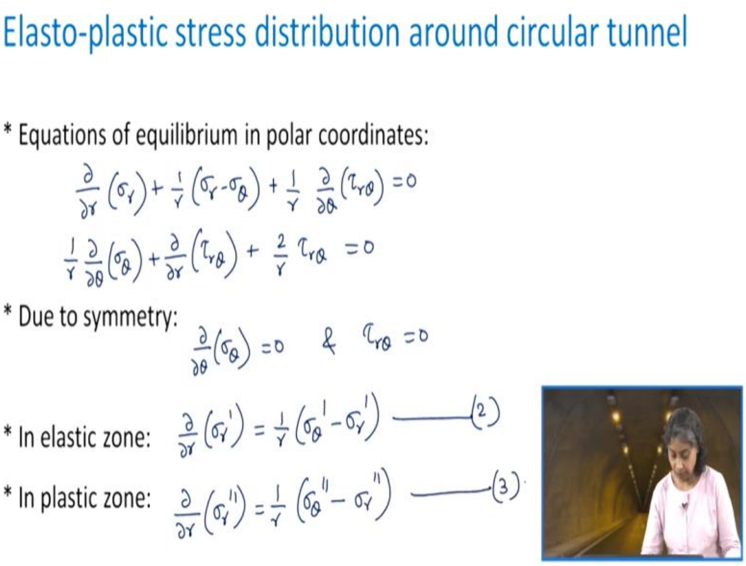

We have the equations of equilibrium in polar coordinates, we have discussed this when we were studying this Tresca yield criterion-related, elastoplastic stress distribution. So, the same equations are going to be here that is

$$\frac{\partial}{\partial r}(\sigma_r) + \frac{1}{r}(\sigma_r - \sigma_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta}) = 0$$

And, we had

$$\frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_\theta) + \frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{2}{r}\tau_{r\theta} = 0$$

Now, what happens due to symmetry

$$\frac{\partial}{\partial \theta}(\sigma_\theta) = 0\ \ \&\ \ \tau_{r\theta} = 0$$

So, therefore in the elastic zone the equations are going to be,

$$\frac{\partial}{\partial r}(\sigma_r^{'}) = \frac{1}{r}(\sigma_\theta^{'} - \sigma_r^{'})$$

and in the plastic zone we will have

$$\frac{\partial}{\partial r}(\sigma_r^{''}) = \frac{1}{r}(\sigma_\theta^{''} - \sigma_r^{''})$$

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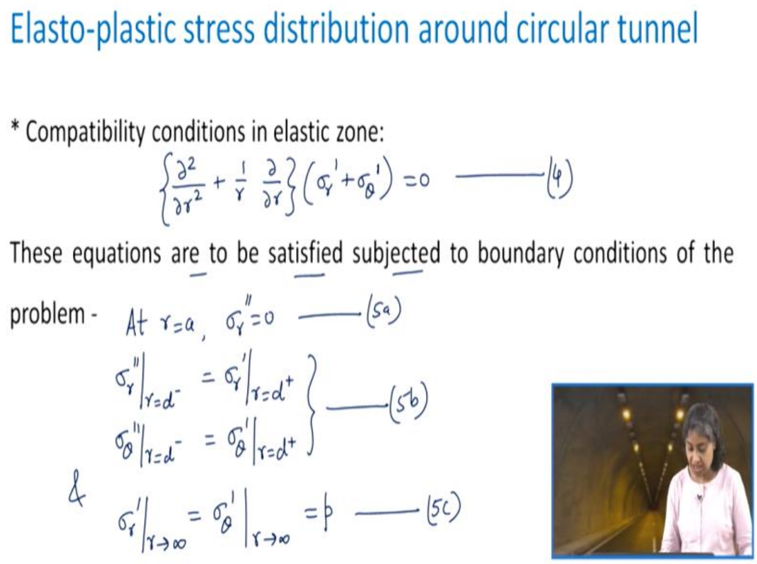

What about the compatibility conditions in the elastic zone, this is going to be given by this equation

$$\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\}(\sigma_\theta^{'} + \sigma_r^{'}) = 0$$

I will make this as equation number 4, now these all these equations are to be satisfied subjected to the boundary condition of the problem. So, let us define the boundary conditions so we have

$$At\ r = a, \sigma_r^{'} = 0$$

to be equal to 5a.Then,

$$\sigma_r^{''}|_{r=d^-} = \sigma_r^{'}|_{r=d^+}$$

So, this is exactly on the similar lines as we did in the previous case. Because at the elastoplastic boundary just outside the boundary, you have the elastic case, and just inside the boundary you have the plastic domain. So, accordingly, these equations are written, then we have

$$\sigma_\theta^{''}|_{r=d^-} = \sigma_\theta^{'}|_{r=d^+}$$

And finally, we will have

$$\sigma_r^{'}|_{r\to\infty} = \sigma_\theta^{'}|_{r\to\infty} = p$$

and we make this as 5c. Now we have the yield conditions which were represented by 1a, and 1b, so, we take the help of these and then try to go ahead with the analysis.

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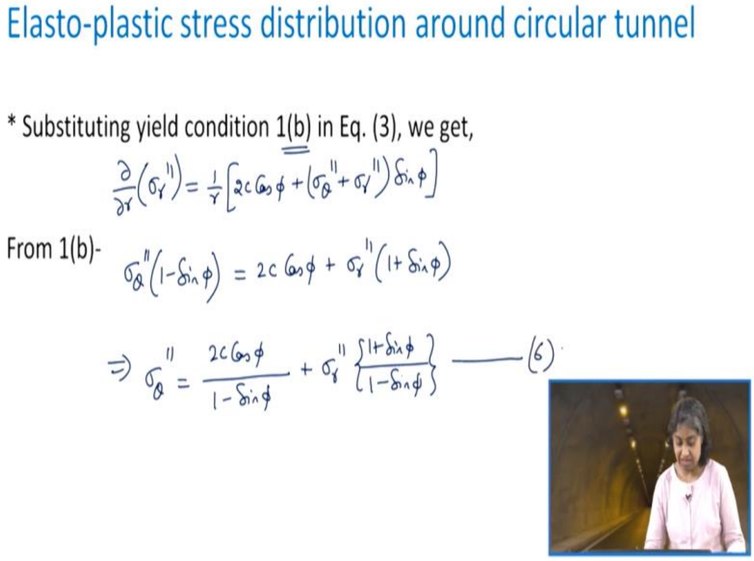

So, if we substitute this yield condition 1b in equation number 3, what we get is,

$$\frac{\partial}{\partial r}(\sigma_r^{''}) = \frac{1}{r}[2c\cos\emptyset + \left(\sigma_\theta^{''} + \sigma_r^{''}\right)\sin\emptyset]$$

Now, from this equation 1b we can write

$$\sigma_\theta^{''}(1 - \sin\emptyset) = 2c\cos\emptyset + \sigma_r^{''}(1 + \sin\emptyset)$$

Or we can write

$$\sigma_\theta^{''} = \frac{2c\cos\emptyset}{1 - \sin\emptyset} + \sigma_r^{''}\{\frac{1 + \sin\emptyset}{1 - \sin\emptyset}\}$$

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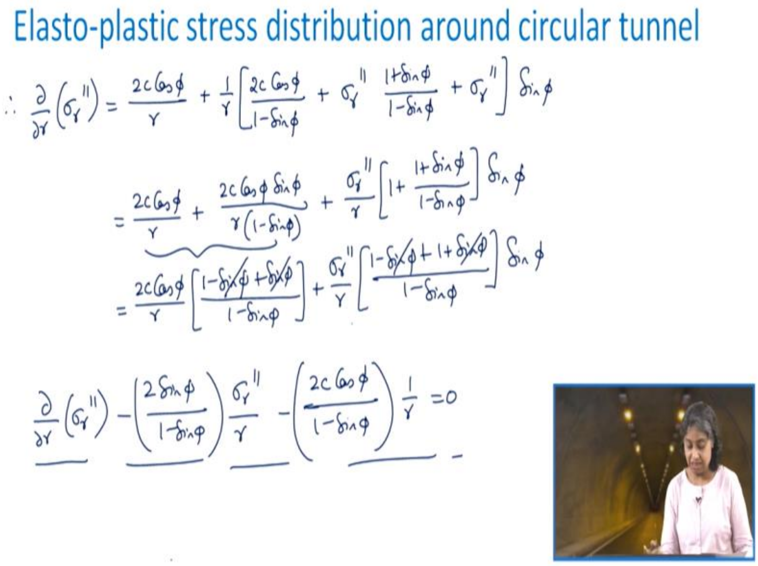

So, what we have is

$$\frac{\partial}{\partial r}(\sigma_r^{''}) = \frac{2c\ cos\emptyset}{r} + \frac{1}{r}\left[\frac{2c\ cos\emptyset}{1 - sin\emptyset} + \sigma_r^{''}\ \frac{1 + sin\emptyset}{1 - sin\emptyset} + \sigma_r^{''}\right] sin\emptyset$$

Or we can write

$$= \frac{2c\ cos\emptyset}{r} + \frac{2c\ cos\emptyset sin\emptyset}{r(1 - sin\emptyset)} + \frac{\sigma_r^{''}}{r} + \left[1 + \frac{1 + sin\emptyset}{1 - sin\emptyset}\right] sin\emptyset$$

and, you solve this further.

So, I take

$$= \frac{2c\ cos\emptyset}{r} + \left[\frac{1 - sin\emptyset + sin\emptyset}{1 - sin\emptyset}\right] + \frac{\sigma_r^{''}}{r}\left[\frac{1 - sin\emptyset + sin\emptyset}{1 - sin\emptyset}\right] sin\emptyset$$

So, this will get canceled here also, so, ultimately what we have is,

$$\frac{\partial}{\partial r}(\sigma_r^{''}) - \left(\frac{2sin\emptyset}{1 - sin\emptyset}\right)\frac{\sigma_r^{''}}{r} - \left(\frac{2ccos\emptyset}{1 - sin\emptyset}\right)\frac{1}{r} = 0$$

Now, you see that this is the linear differential equation, so, you may not be remembering how to solve this.

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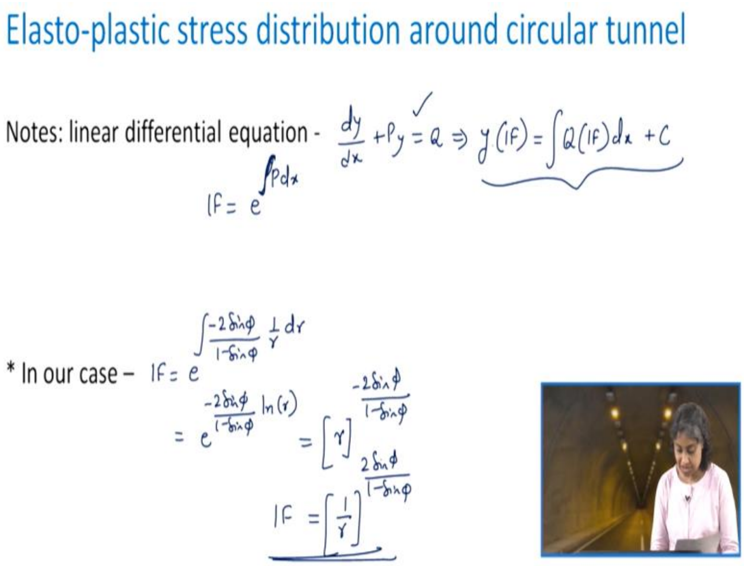

So, I will just take a moment to explain you that, what exactly is that? So, this is just a apart from the what we are discussing, so, linear differential equations we have seen that it is in the form of

$$\frac{dy}{dx} + p_y = Q \Longrightarrow y(IF) = \int Q(IF)dx + c$$

And, this factor is defined as

$$IF = e^{\int pdx}$$

Now, what is happening in our case? So, if you compare the equation in the previous slide with this equation. So, in our case this factor

$$IF = e^{\int \frac{-2sin\emptyset}{1-sin\emptyset}\frac{1}{r}dr}$$

So, this is going to be equal to just integrate it, so, this is going to be

$$IF = e^{\frac{-2sin\emptyset}{1-sin\emptyset}\ln(r)} = [r]^{\frac{-2sin\emptyset}{1-sin\emptyset}}$$

or we can write this as

$$IF = [\frac{1}{r}]^{\frac{2sin\emptyset}{1-sin\emptyset}}$$

this is what is the factor IF in our case. So, let us try to find out the solution, following this so let us come back to our Mohr-Coulomb criterion

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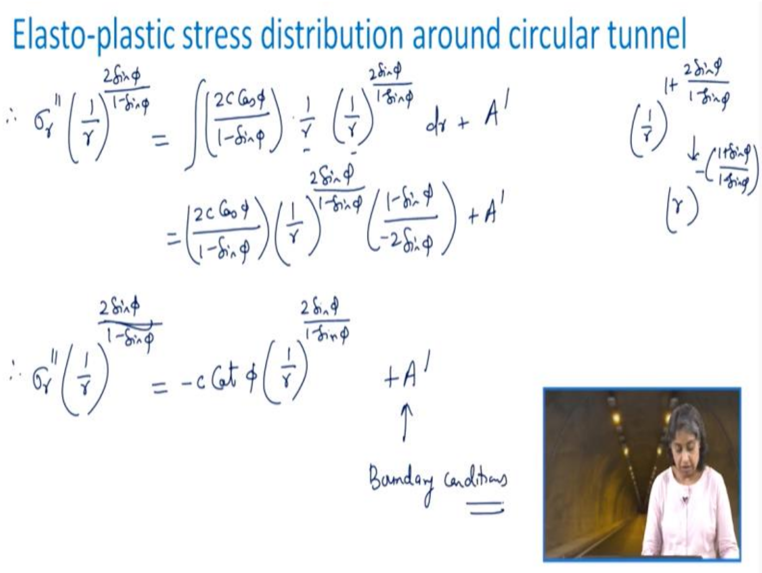

So, let us come back to our Mohr-Coulomb criterion, so what we have here is

$$\sigma_r^{''}(\frac{1}{r})^{\frac{2sin\emptyset}{1-sin\emptyset}} = \int \frac{2ccos\emptyset}{1-sin\emptyset}\frac{1}{r}(\frac{1}{r})^{\frac{2sin\emptyset}{1-sin\emptyset}}dr + A'$$

So, this part will come out of the integration, sin because it is independent of r and here it is the

$$(\frac{1}{r})^{\frac{2sin\emptyset}{1-sin\emptyset}}$$

So, if you just take or combine these 2, so, what you will get is

$$(\frac{1}{r})^{1+\frac{2sin\emptyset}{1-sin\emptyset}}$$

So, ultimately this will become

$$(r)^{-\{\frac{1+sin\emptyset}{1-sin\emptyset}\}}$$

this is what it is going to become. So, the integration of this is going to be

$$= \left(\frac{2c\cos\emptyset}{1-\sin\emptyset}\right)\left(\frac{1}{r}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}}\left(\frac{1-\sin\emptyset}{-2\sin\emptyset}\right) + A'$$

So, what we have here is

$$\sigma_r''\left(\frac{1}{r}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}} = -c\cot\emptyset\left(\frac{1}{r}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}} + A'$$

Now, how to determine this A prime, we need to apply boundary conditions

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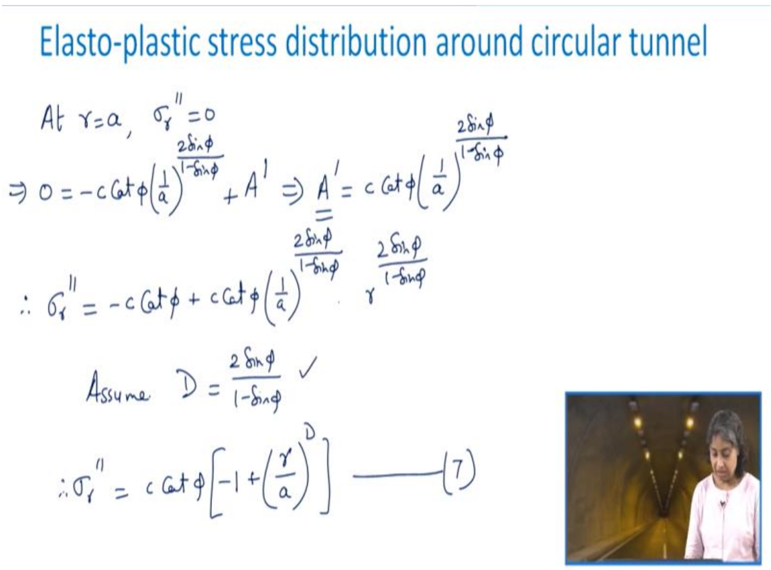

So, we have

$$At\ r = a, \sigma_r'' = 0$$

so, just substitute it there what we get is

$$0 = -c\cot\emptyset\left(\frac{1}{a}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}} + A'.$$

And, from here I can determine this

$$A' = c\cot\emptyset\left(\frac{1}{a}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}}$$

So, substitute this A′ in the expression of σ_r double prime, so, therefore we get

$$\sigma_r'' = -c\cot\emptyset + c\cot\emptyset\left(\frac{1}{a}\right)^{\frac{2\sin\emptyset}{1-\sin\emptyset}} r^{\frac{2\sin\emptyset}{1-\sin\emptyset}}$$

Now, for this simplification in this further analysis, I just

$$Assume\ D = \frac{2\sin\emptyset}{1-\sin\emptyset}$$

This is just for my convenience, it will help us in understanding. So, this is

$$\sigma_r'' = c\cot\emptyset\left[-1 + \left(\frac{r}{a}\right)^D\right]$$

So, will have here this as equation number 7 so, when we are done with the analysis, we can always substitute back this expression for D.

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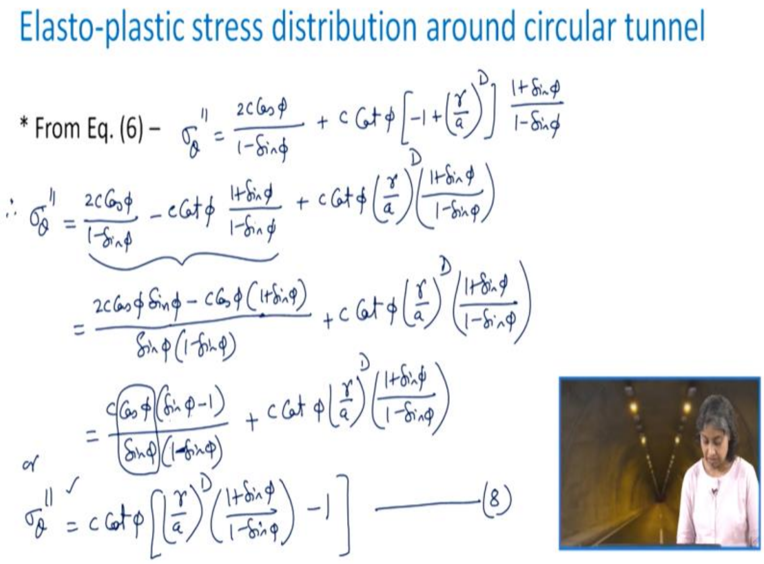

Now, this from this equation number 6 that we had earlier with respect to

$$\sigma_\theta^{''} = \frac{2c\cos\emptyset}{1-\sin\emptyset} + c\cot\emptyset[-1+(\frac{r}{a})^D]\frac{1+\sin\emptyset}{1-\sin\emptyset}$$

so, just further simplify sigma theta double prime is going to be equal to I will keep this first term as it is.

Then, I will split it up

$$\sigma_\theta^{''} = \frac{2c\cos\emptyset}{1-\sin\emptyset} - c\cot\emptyset\frac{1+\sin\emptyset}{1-\sin\emptyset} + c\cot\emptyset(\frac{r}{a})^D(\frac{1+\sin\emptyset}{1-\sin\emptyset})$$

So, if I just further simplify this, I am going to get

$$= \frac{2c\cos\emptyset\sin\emptyset - c\cos\emptyset(1-\sin\emptyset)}{\sin\emptyset(1-\sin\emptyset)} + c\cot\emptyset(\frac{r}{a})^D(\frac{1+\sin\emptyset}{1-\sin\emptyset})$$

Or we can write

$$= \frac{c\cos\emptyset(\sin\emptyset-1)}{\sin\emptyset(1-\sin\emptyset)} + c\cot\emptyset(\frac{r}{a})^D(\frac{1+\sin\emptyset}{1-\sin\emptyset})$$

And, then this

$$\sigma_\theta^{''} = c\cot\emptyset[(\frac{r}{a})^D\left(\frac{1+\sin\emptyset}{1-\sin\emptyset}\right)-1]$$

I am marking this as equation number 8,

so, this is how we can derive the expression for sigma theta prime, it may be bit lengthy, but it is not difficult. So, you need to be systematic so that you do not make mistake here.

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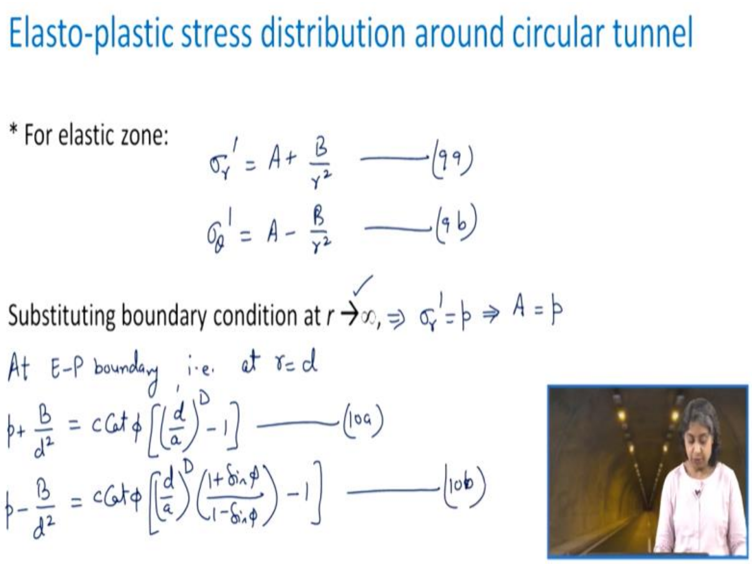

Now, if I just substitute the boundary condition, so before that, I mean substituting the other boundary condition. But, before that what happens in the elastic zone, till now we saw sigma r double prime and sigma theta double which were the stresses in the plastic zone. So, what happens in case of the elastic zone? We know that these stresses are of this form that is

$$\sigma_r' = A + \frac{B}{r^2}$$

which is 9a equation and

$$\sigma_\theta' = A - \frac{B}{r^2}$$

this is say 9b.

So, I substitute the boundary condition when

$$r \to \infty \Rightarrow \sigma_r^1 = p \Rightarrow A = p$$

Then, at the elastoplastic boundary, that is at r = d what will happen? The stresses in the elastic zone will be equal to the stresses in the plastic zone. So, that is what we are going to write here that is

$$p + \frac{B}{d^2} = c\ cot\emptyset[(\frac{d}{a})^D - 1]$$

And, we have

$$p - \frac{B}{d^2} = c\ cot\emptyset[(\frac{d}{a})^D(\frac{1 + sin\emptyset}{1 - sin\emptyset}) - 1]$$

So, these are the equations that we are going to get now.

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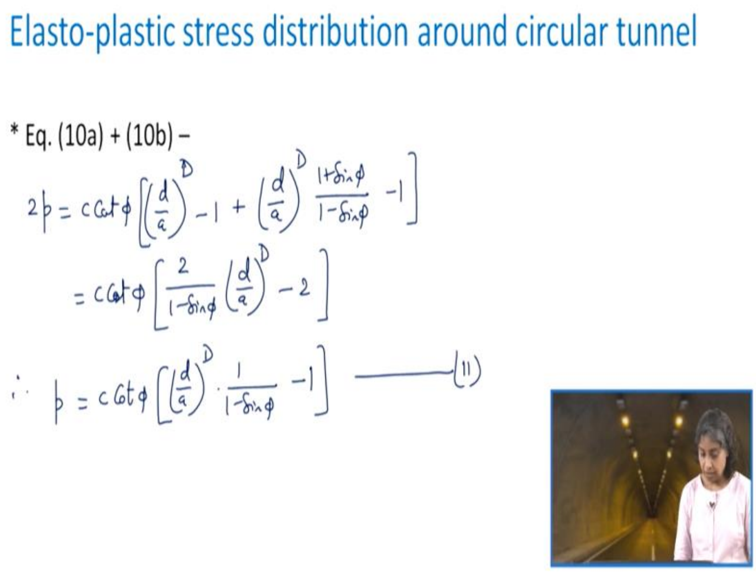

Let us solve these so what we do is we first add equations 10a, and 10b. So, what we will get here as

$$2p = ccot\emptyset[\Big((\frac{d}{a})^D - 1 + (\frac{d}{a})^D \frac{1}{1-sin\emptyset} - 1\Big]$$

Or we can write it to be

$$= ccot\emptyset[\frac{2}{1-sin}(\frac{d}{a})^D - 2]$$

Or divide this whole equation by 2, so, ultimately we get

$$p = ccot\emptyset[\Big((\frac{d}{a})^D \frac{1}{1-sin\emptyset} - 1\Big]$$

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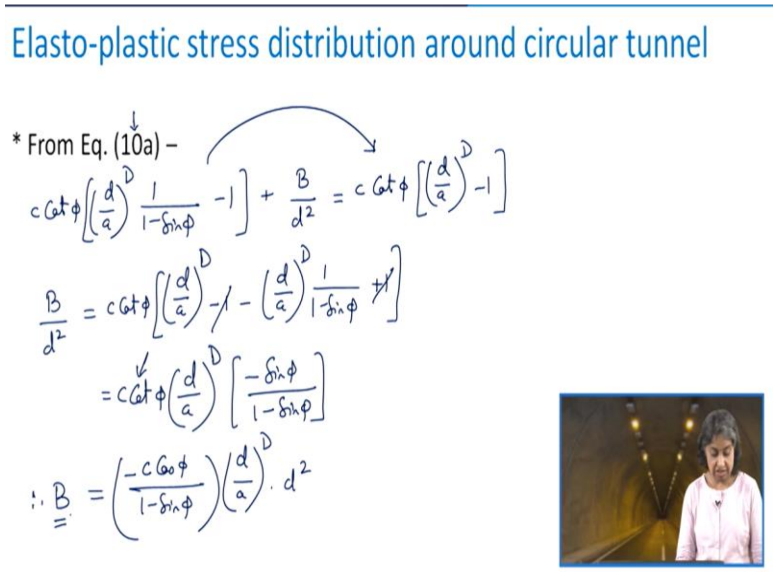

So, I just substitute this into equation 10 a so what I will get is? This that is

$$ccot\emptyset[\left((\frac{d}{a})^D \frac{1}{1-sin\emptyset} - 1\right] + \frac{B}{d^2} = ccot[(\frac{d}{a})^D - 1]$$

Now, you just transfer this term on the other side of the equality sign. So, what we will get is

$$\frac{B}{d^2} = ccot\emptyset[(\frac{d}{a})^D - 1 - (\frac{d}{a})^D - \frac{1}{1-sin\emptyset} + 1]$$

So, this will get cancel so ultimately what you get is

$$= ccot\emptyset[(\frac{d}{a})^D [\frac{-sin\emptyset}{1-sin\emptyset}]$$

so that way what you will get is

$$B = (\frac{-ccot\emptyset}{1-sin\emptyset})(\frac{d}{a})^d . d^2$$

so this is how you get the expression of this other constant B.

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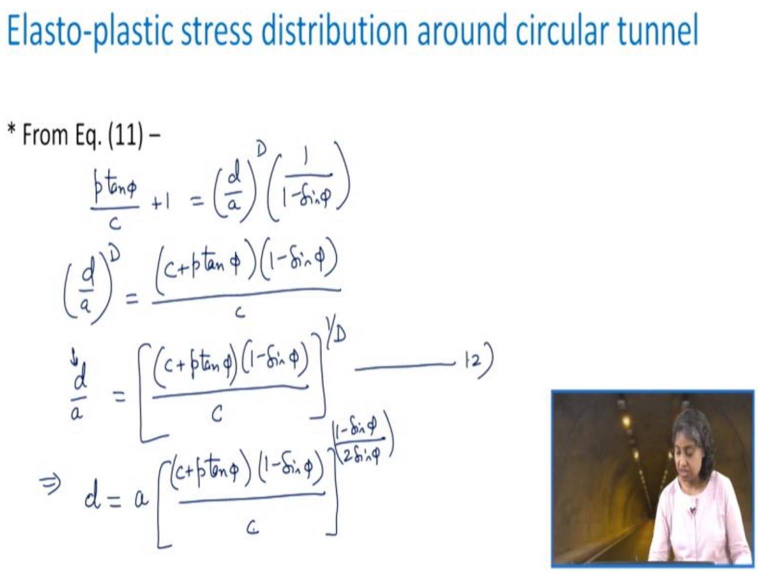

Now, just substitute all these in equation number 11, so what we get here as

$$\frac{ptan\emptyset}{c} + 1 = (\frac{d}{a})^D (\frac{1}{1-sin\emptyset})$$

Or, we can have

$$(\frac{d}{a})^D = \frac{(c+tan\emptyset)(1-sin\emptyset)}{c}$$

Or, we can write

$$\frac{d}{a} = [\frac{(c+ptan\emptyset)(1-sin\emptyset)}{c}]^{1/D}$$

Because ultimately, we want to find out the radius of the elastoplastic boundary which is D. So, that is why I am doing all this exercise, so from here you will be able to get

$$d = a[\frac{(c + ptan\emptyset)(1 - sin\emptyset)}{c}]^{(\frac{1 - sin\emptyset}{2sin\emptyset})}$$

Incidentally, I will write this equation as equation number 12.

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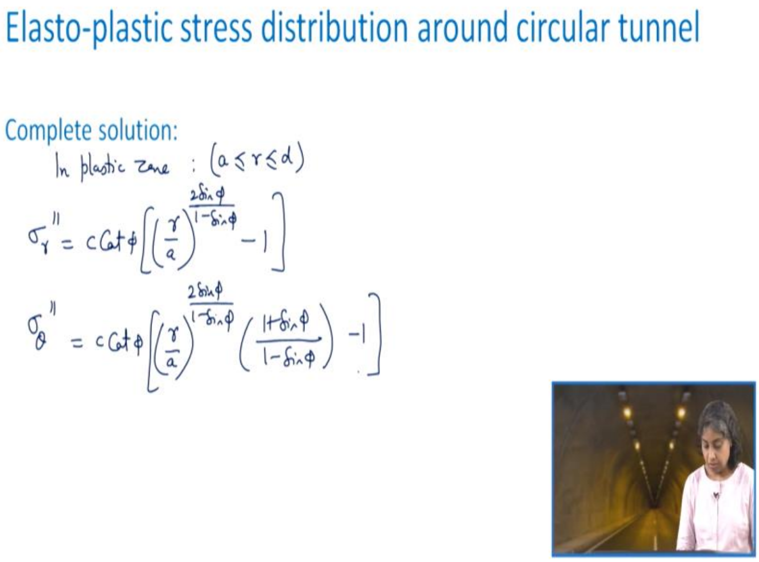

So, let us write the complete solution in this case so what will happen? What all that we have derived? So, in the plastic zone, this means that r is varying between a and d what we had?

$$\sigma_r^{''} = ccot\emptyset[(\frac{r}{a})^{\frac{2sin\emptyset}{1-sin\emptyset}} - 1]$$

Then

$$\sigma_\theta^{''} = ccot\emptyset[(\frac{r}{a})^{\frac{2sin\emptyset}{1-sin\emptyset}}(\frac{1 + sin\emptyset}{1 - sin\emptyset}) - 1]$$

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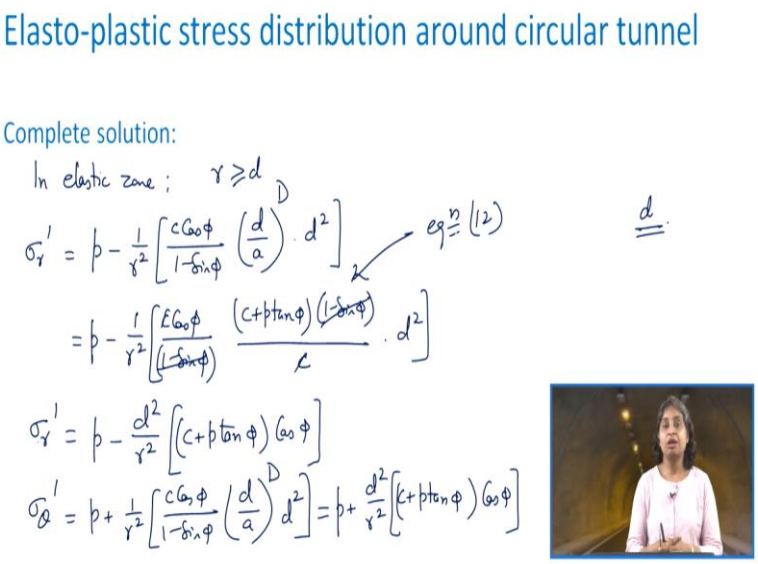

What happens in the elastic zone,

we have
In elastic zone: r ≥ d

so, we just derived

$$\sigma_r' = p - \frac{1}{r^2}[\frac{c\ cos\emptyset}{1 - sin\emptyset}(\frac{d}{a})^D . d^2]$$

Now, from the equation number 12 just substitute the expression that this going to give you

$$= p - \frac{1}{r^2}\left[\frac{c\ cos\emptyset}{1 - sin\emptyset}\frac{(c + ptan\emptyset)(1 - sin\emptyset)}{c} . d^2\right]$$

that is what you will get this from equation number 12.

So, this

$$\sigma_r' = p - \frac{d^2}{r^2}[(c + ptan\emptyset)cos\emptyset]$$

So, this is what is the expression for the radial stress in the elastic zone. Coming to the tangential stress, that is

$$\sigma_\theta' = p + \frac{1}{r^2}[\frac{c\ cos\emptyset}{1 - sin\emptyset}\left((\frac{d}{a})^D d^2\right]$$

Or, we can write

$$= p + \frac{d^2}{r^2}[(c + ptan\emptyset)cos\emptyset]$$

So, this is how we can determine the complete solution in the elastic zone as well as a plastic zone. Now, here you know the expression for the elastoplastic boundary radius, d just find out for various values what are going to be the values, of the once you know that, then corresponding to different values of phi we can determine the radial as well as tangential stresses in the elastic as well as the plastic zone.

So, this finishes our discussion on the elastoplastic stress distribution around circular tunnel. So, we in the next class, we will start fresh topic till then just take care of yourself, thank you so much.