Earthquake Geotechnical Engineering

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Lecture 18

Constitutive Relationships of Soils

I welcome you again for this NPTEL course on earthquake geotechnical engineering. And we are under the module 2, which is on dynamic soil properties. And this is lecture number 18, which is on dynamic soil properties. And in this lecture, we are going to talk about constitutive relationships of soils, which is very important. And this we will discuss not only in this lecture, another two lecture, lecture number 19 and 20. So, three lectures 18, 19 and 20, they are on this topic constitutive relationships of soils.

So, as we discussed, we are in the module 2 for dynamic soil properties, we already covered three topics, concept of stress and stress path, field test and laboratory test. And today let us see constitutive relationship of soils. That is our fourth chapter of this module 2. Under this fourth chapter, what we are going to cover first stress-strain behavior of cyclically loaded soils, then we are going to talk in detail equivalent linear model, which will consist of three components formulation for shear modulus and damping ratio and maximum shear modulus. So, up to this point, we are going to talk in this lecture that is this lecture number 18.

Then this modulus reduction damping ratio curves that is very important and we will have the next lecture on modulus reduction and damping ratio curves. The last one, third lecture for this chapter will be on cyclic non-linear models and advanced constitutive models. So, let us start from the first title that is stress-strain behavior of cyclically loaded soils. You know that when we talk about stress-strain relationship, you may be come across what we call the Hooke s law, where the Hooke s law says that stress is proportional to strain, and they are linearly elastic. So, that is a simple thing, but for dynamic soil properties with what happens the slope of the stress-strain curve is not constant like what you see in this Hooke s law.

In the Hooke s law, it is simple. You have stress versus strain. So, if I say this is stress on x-axis, strain on y-axis, you have stress and the relationship is simple that is they are proportional. This angle could be 45 degree or different, but it is linear relationship. So, this is for elastic material, elastic or what we call the linear material.

So, this is for elastic material for which Hooke s law is valid. If we have the normal stress, then sigma equal to E into epsilon. So, where you have on this axis normal strain and sigma normal stress. They are proportional and Young s modulus here is nothing, but sigma divided by epsilon. So, this is for the elastic materials.

However, particularly for dynamic cyclic loaded soils. Cyclic means it is part of a dynamic load or like it may not be exactly earthquake loading, but earthquake loading can consist of a number of cyclic loading and in this loading, the strain level changes. So, that how to deal with that we are going to discuss today. The mechanical behavior of soils can be quite complex even under static conditions, which is further complicated when we consider the dynamic loading including the cyclic loading. Engineers are challenged by the need to characterize the most important aspect of cyclic soil behavior by accurately as possible with simple and rational model.

So, if we can use the advanced models which is complicated, we will discuss that about, but the challenge is the model should be as far as possible simple and rational. The soil exhibit non-linear inelastic stress strain behavior under cyclic loading. So, normally as we said that linearity, it is not there. It is general behavior of the soil is non-linear and inelastic when you apply the cyclic loading, which is most dependent on the level of strain and as a result at low strain level, the stiffness of soil is greatest and the damping is lowest. If you reverse at high strain levels, the effect of non-linearity comes in picture.

As a result, it produces lower stiffness and greater damping in case of higher strain levels. So, that is the second one is like the normal characteristics of the soils that higher strain levels, its characteristics changes. Low strain level, we can simulate. The behavior is also dependent on the dynamic characteristics of the loading which is frequency dependence. So, that is the issue here.

So, one side the behavior is dependent on the strain level that is with the level of strain, the dynamic soil properties of the soil is changing and the second when the, for the dynamic loading, when you are changing the frequency of excitation with that frequency also it is varying. So, in general, for the stress strain behavior of cyclically loaded soils, there are three classes of soil models which are prevalent in use. First is equivalent linear model, which is simplest and most commonly used. The second is cyclic non-linear models. Third one is advanced constitutive models.

These three models are there, and we will discuss one by one. So, starting from the first one that is the equivalent linear model, but before that we will let us discuss more about the stress strain behavior. Equivalent linear model have limited ability to respond many aspects of behavior under cyclic loading conditions. So, the equivalent linear models why they are popular? They are popular because they are simple, but they are not accurate, but they are better than the linear models. So, certainly they are far better than the linear models.

So, that is why, but they have limitations. On the another end, you have advanced constitutive models which can represent many details of dynamic and cyclic loads behavior, but they are very complex and difficult to calibrate. So, whether like one end you have very simple model, which require the less calculation, but at the same time they are not so accurate. On the side you have very advanced constitutive model requiring lot of calculations. So, which one should select? There is a tradeoff between the simplicity and accuracy and that need to be decided project to project.

Now, the first one equivalent linear model we are going to talk about that. Treat soils as a linear viscoelastic materials. So, using these models we treat the soil is a linear viscoelastic material and non-linear behavior of the soil is accounted by the use of strain dependent stiffness and damping parameters. So, both parameters are like stiffness of the soil as well as damping, both are dependent on the strain which we will see later. So, they are strain dependent parameters that means they vary with the strain.

The stiffness of the soil is usually characterized by the maximum shear modulus which we will discuss in detail which is mobilized at lowest strain and a modulus reduction curve which shows that how the shear modulus decrease at larger strains. So, as for the first part stiffness is there, you have G normally we will discuss. Maximum shear modulus is basically will be represented by G max and the shear modulus will be simply G. So, basically you will have G by G max curves which is how the G by G max varies with the strain we will discuss that. While damping behavior is characterized by what is called the damping ratio which is normally represented by xi which increases with increasing strain amplitude.

So, G by G max decrease with the strain amplitude while damping ratio increases. The shape of the modulus reduction and damping curves are influenced by other parameters not only of course like they are dependent on strain that these both parameters are varying with the shear strain. But their shape depends on many factors including soil plasticity and for soils of very low plasticity by effective confining pressure. If you have the Pi, so first point will become on Pi, but if you have the sands then Pi becomes 0, in that case it depends on the effective confining pressure. So, these things we will discuss in the next slides.

So, for example here, this is one of the important things, like you know that to stress strain relationship. What you have in this figure you have a stress strain curve which is like how when you load the soil element with stress and how the relationship between the stress and strain varies. So, because this relationship for the linear material is constant, so which is represented by the center line for the linear. But for the soil like material, this is not a line rather relationship is curve. So, stress strain relationship is represented by a loop and this loop is called nothing, but it is called a hysteresis loop.

And in this case, hysteresis loop is relation between shear stress and shear strain. So, this actually what happens you load from here, it reaches to this point and after this point it will come here. The point is here when the you becomes the loading becomes 0 that means there is no shear stress, but still shear strain is not 0. In linear model when you remove in elastic material, when you remove stress, then strain will be 0. So, stress and strain become together 0. So, they start from 0, but after like when they reaches unloading time, there is you get some stress strain even when there is stress is 0 and then it goes on another side and then the reversal takes place and discontinue. So, in this figure, you have two modulus, one modulus which is called tangent modulus and tangent modulus is varying at each point and how you can determine the tangent modulus which is represented by g tan. So, g tan is asymptote you draw at for example, this is a point on this point draw an asymptote and this asymptote should be to this curve and this asymptote the slope of this line is g tan. Naturally, when you go along the curve then because for you will draw different asymptote and the value of g tan will vary. So, that means the value of g tan is continuously varying as you move from one point to another point on this curve.

However, average value of this g tan can be represented by the slope of the central line which is called g sec that is called secant modulus. So, like whatever we have discussed here, the stress can be described in two way. First by the actual path of the loop itself, and the second by parameter that is described as general shape. In the first case, if you want to have actual path of the loop itself, then you need to go point wise at each point it will vary.

So, it is difficult to deal. So, what is done? So, some parameters are taken to describe the general shape of this hysteresis loop. In general terms, two important characteristics of the shape of hysteresis loop are inclination and its width. So, this is a hysteresis loop and for this hysteresis loop, what will be the inclination of this loop and its width. So, its inclination will be represented by the slope of this central line that is g sec and then you have this width will be governed by the area. So, you have inclination as well as its width. The inclination of the loop depends on the stiffness of the soil which can be described any point during the loading process that is tangent shear modulus, but this tangent shear modulus will be varying point to point. So, as a result, this inclination is represented by an average value and this average value is called secant modulus and the secant modulus is given by the tau c and gamma c. So, if I go back to this loop, the tip of this hysteresis loop, the maximum value related to this tip is coordinates of these points are tau c and gamma c, where gamma c is representing the strain corresponding to this point and tau c is representing the shear stress corresponding to this point. So, the ratio of this tau c or gamma c will give you the secant modulus.

So, this secant modulus will give the average of the slope of this loop. So, rather than and this is like rather than varying at point to point. Here tau c and gamma c are the shear stress and shear strain amplitude respectively and the g sec describe the general inclination of the hysteresis loop. So, this is one of the parameter which is like depending, so this parameter defines inclination. Now, the second parameter which we say width for determining the width of the hysteresis loop is related to the area which is a measure of energy dissipation and it can be conveniently described by damping ratio, where damping ratio is defined Wd divided by 4 pi Ws.

$$\xi = \frac{W_D}{4\pi W_S} = \frac{1}{2\pi}\frac{A_{loop}}{G_{sec}\gamma_c^2}$$

What is Wd and Ws? It can be said here. In this loop, the area of this loop complete area of this Wd, while Ws the strain energy which is stored in this triangle, this triangle, the area of this triangle is Ws. So, the ratio of the work done divided by this area of this triangle divided by 4 pi will give you the damping ratio. If you solve then you can find out that this is 1 by 2 pi a log, what is a loop? A loop is nothing but the area of the loop, while g sec is already known, gamma c is already known. So, using this equation, the second equation, you can find the damping ratio. So, using these two parameters, one is damping ratio, another is secant modulus, these are called two parameters for equivalent linear modulus. So, these are the equivalent linear parameters. So, often referred to as equivalent linear material parameters. Continue with this, now let us go on the application part. For certain types of ground response analysis, these parameters are used directly to describe the soil behavior.

Other types requires the actual path of the hysteresis loop as described by cyclic non-linear or advanced constitutive model. So, here the issue is this one, for some material, this equivalent linear model is okay, but for some material we need to use, go more advanced and then we consider cyclic non-linear or non-linear advanced constitutive models, we will discuss later. Because some of the most commonly used methods of ground response analysis are based on the use of equivalent linear processes. So, I think ground response analysis which we called in the short, GRA, GRA stands for ground response analysis. And once this module completed, then in the next module we are going to talk about ground response analysis.

So, most of the GRA software, ground response analysis software are based on the equivalent linear models. And for example, which are the some of the software, you have the shake, use equivalent linear model, deep soil, program for ground response analysis based on the equivalent linear models. So, because many of the packages or algorithms for dealing with the ground response analysis, they are based on equivalent linear models. So, we need to examine the characteristic of G sec damping ratio for different soils in details, which we are going to do in the next. There are equivalent linear model, though it is good, they are simple, they are easy to understand, but they have their own limitations.

It is important to recognize that the equivalent linear model is only an approximation of the actual non-linear behavior of the soil. And the assumption of linearity is embedded in its use for important application when it is used for ground response analysis. So, basically in equivalent linear model, you are using linear model at a different stresses. So, for you, I mean, for as the level of strain increases, then you are updating the value of shear modulus and damping ratio, which we will discuss later. So, in continuation with this, with the limit, that means, these models cannot be used directly for problems, which will involve permanent deformations or failure.

If some problems is leading to the failure or permanent deformations, because in equivalent linear models, the strain will always return to 0 after cyclic loading. So, like even in the hysteresis loop, the strain becomes 0. So, where it becomes, here stress becomes 0, but this is the point, at this point, strain, stress become negative, and the strain becomes 0. Similarly, on this side, you have the stress becomes 0 and strain will be 0 here. So, there will be points in the hysteresis loop where strain becomes 0 and there is no permanent strain. So, as a result, linear material has no limiting strength. So, failure will not occur and the failure occurs in the other mode, like actual in the field, failure is observed. So, as a result, they have limitations that they will be assuming that there is no failure. Nevertheless, the assumption of linearity allows a very efficient class of computational models to be used for ground response analysis and it is commonly employed for that reason. So, this equivalent linear models are much used in GRA.

One of the most important used application of these equivalent linear models is ground response analysis, which we will discuss in the module third. Now, continue with the equivalent linear model. As we discussed, there are two parameters, one is shear modulus and another is damping ratio. So, we are going to discuss, let us discuss the shear modulus in detail. Laboratory tests have shown that soil stiffness is influenced by cyclic strain amplitude, void ratio, mean principal effective stress, plasticity index, over consolidation ratio and number of loading cycles.

So, there are so many parameters, it depends on cyclic strain, void ratio that is relative density, mean effective stress plus p i and so on. The secant shear modulus of an element of soil varies with cyclic shear strain amplitude. So, when you change the cyclic shear strain amplitude, the secant modulus will decrease. At the lower strain amplitude, the secant shear modulus is high, but it decreases as the strain amplitude increases. So, in fact, the secant modulus is maximum when you have the shear strain almost 0 negligible.

And the locus of points which correspond to tip of the hysteresis loop of various cyclic strain amplitude is called a backbone curve or a skeleton curve and its slope at the origin that is where the origin means at where 0 cyclic shear strain amplitude represent the largest value of the shear modulus that is nothing but G max. So, first of all, like here, this is the backbone curve. So, let me discuss it in brief. This is you have again shear versus shear strain, shear stress, tou and shear strain. This in the backbone curve is nothing but these are the points which are the coordinates of the hysteresis loop of the tips of the hysteresis loop.

So, what do we do in the from the hysteresis loop? We select the coordinates of the tip not of the whole loop. For example, one loop will be this where toe c and gamma c, you will have another loop where toe c may increase and the gamma c may also increase, you may have another third loop which may where toe c and gamma c is lower value. So, you just have the coordinates of the tips of this hysteresis loop and plot on the curve like this, which will give you the value of a backbone curve. So, this is how the backbone curve is created. And on this backbone curve, this the largest value if I draw the slope of this backbone curve on the at the origin, then this slope is nothing, but G max and this G max is represent the maximum shear modulus which is at the almost negligible shear strain. At greater cyclic shear strain amplitude, the modulus ratio which is G by G max will to value less than 1, where G is the same as the secant modulus. So, now onwards we will be using G only for simplicity rather than G sec. Here difference is there, the value of G or G sec vary with the strain level, they changes.

But the value of G max does not change because G max is constant irrespective of strain level, because G max is defined at a very low strain level, almost zero strain level. And value of G by G max will decrease, will become less than 1. The characterization of the stiffness of an element of soil therefore, require consideration of both G max and the manner in which modulus G by G max varies with cyclic strain amplitude and other parameters. So, how G max varies, how this ratio varies? As for G max is concerned, this is value at very low strain value. So, it does not change as with strain because you are defining it at a very low strain value.

While G varies, but G max varies with other parameters which we are going to discuss. So, what you have G max and G sec, then the variation of this modulus ratio with shear strain is described graphically what is called modulus reduction curve. And this is the modulus reduction curve. This is backbone curve, and this is modulus reduction curve. What is the difference between these two curves on the left and right hand side? On the left hand side, you have the stress strain relationship, while on the right hand side, you have G by G max and strain in terms of logarithmic scale.

So, y x axis is same, it was strain here, only the thing it is on the log scale. Now, on G by G max you have, how you determine the y axis of this one, your G will vary like that with the strain. So, what we do, G max is fixed. So, G by G max will be when the strain is 0, it will be 1 and then G start decreasing because the G is a slope, G is same as G sec and you see when we go, the slope decreases, slope is not increasing. So, the value of G will decrease and because G max is constant, the ratio of G by G max, initially it is 1 and then it start decreasing.

So, this is in the form of what we call the S curve, and this is modulus reduction curve. So, the modulus reduction curves represent the same information as with the backbone curve and one can get from one another. So, basically both the curves backbone curve and modulus reduction curves, they represent the same thing, but in different form. Here in backbone curve, it is in terms of strain and strain while on modulus reduction curve, it is G by G max versus strain.

So, shear modulus how it varies. So, but the first one will be called backbone curve or skeleton curve while the second one will be called modulus reduction curve. Now, continue with this one. Now, as you know that to draw this modulus reduction curve or otherwise you may require the value of G max which is the maximum shear modulus and this maximum shear modulus is nothing but the value of the shear modulus at very low strain level. And since most seismic geophysical test including your SASW test or cross borehole test, they induce shear strains lower than about 3 into 10 to power minus 4 percent. So, this is like less than this value and this is in terms of percentage. The measured shear velocity can be used to compute G max. So, whatever the shear velocity you measured from the geophysical test, seismic reflection, seismic refraction, MASW test, then shear velocity is known then G max can be found out simply rho into vs square where rho is nothing but mass density. This relationship we already discussed a lot of times. So, G max can be found using those tests. But sometime you may not have the test data for example, geophysical test data. So, the use of measured shear velocity is generally the most reliable means of letting the value of G max for a soil deposit.

$$G_{max} = \rho v_s^2$$

But suppose if you do not have the data from the field, then you can use the empirical relation. And this empirical relation, one of the relation which is given here G max is 625 Fe OCR Pa sigma m. This is given by Hardin and Brannwich and this relationship is very popular, most popular relationship here and this is okay for both sand and clay. So, this is for C phi soils, it is not only for the sand C phi soils.

$$G_{max} = 625F(e)(OCR)^k p_a^{(1-n)}(\sigma_m^/)^n$$

And in this relation Fe is a function of void ratio. So, when the void ratio varies Fe will vary and in fact, this Fe when the void ratio increases the Fe decreases. OCR is over consolidation ratio and k is its exponent, Pa is atmospheric pressure due to like atmospheric pressure and sigma m is mean effective like principal effective stress and sigma m in fact, because you have three principal stress, so using three principal stresses, the sigma m is given by here, sigma m is nothing but sigma 1 plus sigma 2 plus sigma 3 divided by 3, average of these three, where all three also effective, effective major principal stress, intermediate and the effective minor principal stress, sum of these three divided by three is your mean effective principal stress. Here the value of n is typically taken 0.5. So, you see the n, one thing is Pa atmospheric pressure and sigma m should be in the same unit.

So, whatever units you select, suppose you select kilo Pascal, then it will be kilo Pascal. So, the g max will come in the same unit. If it is in kilo Pascal, it will answer will be in kilo Pascal. If you select in feet per inch square, then it will be also in the same unit. So, g max unit will depends the unit of Pa and sigma m, but of course, Pa and sigma m in this formula should be kept in the same unit. k is exponent and this plus this depends on the plasticity index Pi. So, when the Pi which is given the table which is by Hardin and Ranwich. So, in this table, what you have, when the Pi equal to 0, that means for sand, the value of k is 0. And if I put k equal to 0, then what will happen this, this factor will become 1. So, this can be taken out of this formula for the case of sand. But as the Pi increases, the value of k increases from 0 to some value and then the effect of OCR comes.

So, for the clay or let us say for the cohesive soils, there will be effect of over consolidation ratio, but may not be for the sand. So, this was one of the formula. Then some formula developed for specifically for the sand as well as clay. For example, for maximum shear modulus for sand is often estimated using this relation, g max is 1000 into k 2 max into sigma m to the power 0.5. Sigma m is same as mean principal effective stress as before and which is here in pound per feet square. Because in the formula, this 1000 is taking some unit because if you see, the k 2 max is a ratio, it does not have any unit. So, the dimensionally there is a unit for 1000, otherwise there will be imbalance of units. So, as a result, this formula is unit specific and in this formula, sigma m should be kept in pound per feet square and your answer of g max will be also coming in the same unit. k 2 max depends on the weight ratio or relative density and its value is in the range of 30 to 74 for sands.

So, for the sands, the value of k 2 max is here. But for it have been found from the filters, the shear velocity of gravels are significantly higher than those of sands. Therefore, indicate that g max of gravel is higher than that of sands and k 2 max values for gravels are typically in the range of 80 to 180. So, the value of k 2 max is higher for gravels, less for sand which is like as expected because g 2 max g max value, maximum shear modulus is higher for the gravels than the sands. Similarly, for fine grained soils including clay, preliminary estimates of the maximum shear modulus can be obtained from PI, plasticity index, over consolidation ratio and undrained strength SU. So, these tables give you the ratio of g max by SU and this table depends on two parameters.

One is plasticity index which is on the left hand side, first column. So, you select the value of PI and then select the value as the PI increases, you see this ratio decreases. And when the OCR increases, again this ratio decreases. As much because in this table, you have this ratio, maximum g max by SU will be maximum for OCR 1 and PI equal to and the minimum when the OCR equal to 3 and the PI is in the range of 35 to 42. So, using this table, you find the value of g max by SU where SU is nothing but SU's undrained strength and once this ratio is known, multiply by this number with SU, you get the maximum shear modulus.

Maximum shear can also be found from in-situ test parameters. A number of empirical relationship between g max and various in-situ test parameters have been developed. The inherent difficulty of correlating a small strength parameter such as g max with penetration parameters that relate to much larger strength evident from the scatter in the data which is available. So, there are like you know that these in-situ relationship for example, SPT data or CPT data, so they may not give you the accurate value. So, with this, I thank you very much for your kind attention and we will continue in the next lecture for modulus reduction and damping ratio curves. Thank you.