Earthquake Geotechnical Engineering

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Lecture 22

One-Dimensional (Conti.)

I welcome you again for this NPTEL course on earthquake geotechnical engineering. And now we are going to discuss lecture number 22 which is in continuation of lecture number 21. We are under the module 3 and module 3 consists of 4 chapters, 3 chapters as we discussed. The first one is on ground response analysis and we continue with that. So, we were discussing in last lecture on ground response analysis for 1D ground response analysis. So, we continue and in 1D ground response analysis already we have talked for undamped case, a uniform undamped soil on rigid rock. Today in this lecture we are going to talk to uniform damped soil on rigid rock and uniform damped soil on elastic rock. So, now damping will be considered, 2 cases will be considered with the damping. One is when this layer is on rigid case rock, or another case is it is on elastic rock.

The previous analysis which we have considered in the last lecture, no dissipation of energy or damping was considered. So, damping was neglected, but since damping is present in all materials more elastic results can be obtained by repeating the analysis with the damping. In fact, you will see when the damping is considered the amplification or ground response analysis is completely changed what was the case for undamped system or undamped case. Here in this case, assuming the soil to have the shearing characteristic of what Kelvin Voigt solid. Kelvin Voigt solid I think we have discussed it is nothing, but it consists of a spring and dashboard. So, if I consider single degree of freedom system, one spring and another dashboard and this is connected with some mass m, this is k, this is c. So, in this case this k and c this is called Kelvin Voigt solid. So, this will be called Kelvin Voigt solid. In this case the wave equation can be written as given equation number 6 or in this equation number 6 you have on the left hand side what is called the inertial force rho into del u square by delta t square while on the right hand side you have two term. One is due to elasticity shear modulus G and another is called viscosity.

$$\rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} + \eta \frac{\partial^3 u}{\partial z^2 \partial t} \qquad (6)$$

So, eta and viscosity the equation number 6 we already discussed when we discussed the wave propagation that is both it is viscoelastic material. Similarly equation 6 will be applicable when viscosity as well as elasticity of the material is considered. And as shown this equation the solution of the wave equation will be in the form which is given by this relation. Here in this relation k star is a complex wave number which will have k star will have two part one is real part k 1 and another is what we call the imaginary part k 2. So, repeating the previous algebraic manipulations with the complex wave number the transfer function for the case of damped soil or rich rock can be expressed using this relation.

$$u(z,t) = Ae^{i(\omega t + k^* z)} + Be^{i(\omega t - k^* z)}$$

$$F_2(\omega) = \frac{1}{\cos k^* H} = \frac{1}{\cos(\omega H / v_s^*)} \qquad \textbf{(7)}$$

So, you have this relation here and, in this relation, you have transfer function. The transfer function is similar to what we have discussed earlier for undamped case, but with one difference now k is converted into k star and vs is converted into vs star and what is star is denoting this is not a real number but a complex number. Now, to simplify it what happens G let us have the G star, G star is G. So, let me put like this here this is G star. So, this G star equal to G, 1 plus iota into 2 xi where what is xi is static damping ratio.

$$v_s^* = \sqrt{\frac{G^*}{\rho}} = \sqrt{\frac{G(1+i2\xi)}{\rho}} = \sqrt{\frac{G}{\rho}}\,(1+i\xi) = v_s(1+i\xi) \qquad \textbf{(8a)}$$

So, using this we simplify. So, shear velocity vs star G star over rho G is written 1 plus iota 2 xi xi G by rho this is how this have come this is simplification is done for the small damping this from here to here assumption is here the damping is small and why this assumption is damping is small, the small damping. This can be go for a small damping only why because root 1 plus iota 2 xi can be written as 1 plus iota 2 xi to the power 1 by 2 then you can open the series 1 plus iota xi plus iota xi whole square and then some root and then series this will go. But because the damping is very small so the terms with square and cube may be cancelled out so that is why this has been 1 plus iota xi even it is root then it will be still there. So, that is the assumption here G by rho is vs so this is approximate value when the damping is small and normally damping is small less than 20 percent.

So, even if I consider 20 percent then because if I consider xi equal to 0.2 then xi square will be 0.04 so it will be very much reduced it will not 0.4 it will be 0.04. So for the lower damping for a small damping this approximation can be done. So K star will be omega over vs star where I put vs 1 plus iota xi and then I get either this equation in the form omega over vs 1 minus iota xi or maybe you can write k 1 minus iota xi. So ultimately you end up f 2 omega which is given by this relation f 2 omega was given from equation number 7. Now you can write in this form cos k 1 minus iota xi H and this can be where k is omega H over vs and then this is written as vs 1 plus iota xi. Here this should not get confused like this term is omega H over vs into 1 plus iota xi.

$$k^* = \frac{\omega}{v_s^*} = \frac{\omega}{v_s(1+i\xi)} = \frac{\omega}{v_s}(1-i\xi) = k(1-i\xi) \qquad \textbf{(8b)}$$

and finally, the transfer function, as

$$F_2(\omega) = \frac{1}{\cos k(1-i\xi)H} = \frac{1}{\cos(\omega H/v_s(1+i\xi))} \qquad \textbf{(9a)}$$

So, this is this term this term downturn here. So here 1 plus iota xi is the denominator while here 1 minus iota xi is in the numerator of this like here on the top and this is written here. Now what we do in f 2 omega we can find out its what we call the magnitude or absolute value, and the absolute value can be find out using what using an identity if you have cos x plus iota y and you need to find out the absolute value of this number then square root of cos square x plus sin h square y. Here it is hyperbolic function it is not normal sin y sin h. So, you have sin theta and then you have another thing sin h theta sin h theta is different than the sin theta.

$$|F_2(\omega)| = \frac{1}{\sqrt{\cos^2 kH + (\xi kH)^2}} = \frac{1}{\sqrt{\cos^2(\omega H/v_s) + [\xi(\omega H/v_s)]^2}} \qquad \textbf{(9c)}$$

So, when you have sin h y and for small y when y is small then you can assume that sin h y will be simply y and this with this approximation equation number 9b is simplify as a sin h square y will simply become y square and then the simplified equation is written cos square omega h over vs plus xi omega h over vs whole square. Here because you are finding the absolute value f 2 omega as a result if I am finding the absolute value then in this number you will get only the real number on the right hand side. On the right hand side you do not have the complex number because absolute number is found out. For a small damping ratios equation 9c indicates that the amplification by damp soil layer also varies with frequency. As was the case for undamped case the transfer function was varying with frequency here also the transfer function varies with frequency.

And then what we do we plot this transfer function and before that let us analyze here using this equation 9c which is the final equation. The amplification will reach a local maximum when k h or this term is still the same as earlier pi by 2, plus n pi. So, still the maximum for maxima the same equation is applicable as was for undamped case. But there is a one difference when k h equal to pi by 2 plus n pi for any value of n then this term will 0 cos square omega h or v s will be 0 this term will become 0. But even when this term becomes 0 then this because due to the presence of the damping in the system xi is not 0 is still the value the maximum value does not reach to infinity that means local maxima is not going to reach to infinity value.

So, it will never reach a value of infinity since for xi equal to greater than 0 the denominator will always be greater than 0 it will not be 0. So, if it is not 0 it is always greater than 0 in that case you will not get a infinity peak. The frequency that correspond is the local maxima are the natural frequency of the soil deposit. So, these natural frequency are the same exactly as was the case for undamped case. So, damped or undamped case the natural frequency is find out using this relation k h equal to pi by 2 plus n pi as before.

The variation of amplifying factor with frequency is shown for different levels of damping and which is shown in this figure. This figure is same for undamped case and this is was the case for damped case. In this figure this on x axis you have what we will call k h which is nothing but k is omega over vs omega over vs into h. So, this is k h. So, influence of frequency on steady state response of damped linear elastic soil. So, there are three curves the top curve is for 5 percent damping, the second curve is for 10 percent and third one is for 20 percent damping. So, you have 5 percent, 10 percent and 20 percent damping and on y axis you have amplification factor. So, one thing you see is still because this x axis is representing frequency. So, when there is a change in frequency when the frequency is changing, so your amplification factor is continuously changing with the frequency. And the peak values if I put in terms of like pi, so still this peak this is pi by 2, this values will be 3 pi by 2, this is 7 pi by 2, this is 9 pi by 2 and so on.

So, and this is value 1. So, all the curves for all the damped case they will start from 1, but they will never reach to the infinity value rather they will reach to finite value due to the presence of damping. So, one more important thing when the value of damping is increased let us say it was for 5 percent, when it is increased to 10 percent then this peak becomes half of the 5 percent. If I increase further 20 percent then it becomes, so it is roughly let us say 12 point something, it becomes 6 point something and then it becomes 3 point something. And this was for the first peak, but for the second peak you will get this becomes one third of the first peak. So, this will this value if I say it is 12 then it will be 4 point something like this and so on.

And this will become one fifth, one seventh or so the peaks and that can be easily proved from this equation which was here equation number 9c. So, the natural frequency and now the amplitude factor is also equal to the ratio of free surface motion amplitude the bedrock which is we already defined. Using the previous figure shows that damping affects the response at high frequency more than at lower frequencies. So, when you have damping 0 this was the case un-damped case, this is what we already discussed in the last lecture. So, what you have you get at the peak values always the infinity value and for one cycle you get the repetition it was 1, 1, 1, 1 here. But for the first peak what happens due to the presence of the damping your infinity peak comes down to a finite value. But when for the second peak it is not only coming at finite value, but it decreases also. So, that means the effect of damping keep increasing at higher frequency. This was effect, but here it was more here more more more and then continuously it become flat. So, we can say that the effect of damping is more at higher frequency compared to lower frequency.

Now, the Nth natural frequency of the soil deposit is given by this relation omega n equal to Vs by h pi by 2 plus n pi where value of n is 0, 1, 2 and so on infinity. So, if I take the minimum value n equal to 0 then I will get omega n equal to Vs h into pi by 2. Since the peak amplification factor decreases with increasing natural frequency the greatest amplification factor will occur approximately at the lowest natural frequency also known as the fundamental frequency. So, fundamental frequency or the lowest natural frequency. So, the peak amplification factor is decreasing with natural frequency which you could see here, and this is happening because n is increasing as a result when the here for example, n equal to 1 for this n equal to 0 for this peak.

$$\omega_n = \frac{v_s}{H}\left(\frac{\pi}{2} + n\pi\right) \qquad n = 0, 1, 2, \ldots\ldots, \infty$$

So, n equal to 0 for this peak n equal to 1 and this is 2 and this is for n equal to 3 so and so on. So, this will keep decreasing as we discussed. And for the first peak which is the fundamental peak omega naught is n equal to 0 and you get Vs pi divided by h into pi by 2. In another form we can write like you know the f naught which is the natural frequency is omega over 2 pi and you get Vs by 4 h which is the same as we discussed in the last lecture. The period of vibration corresponding to the fundamental frequency is called the characteristic site period and that is given by this relation which will be the opposite of this one 4 h or Vs so that can be find out.

$$\omega_0 = \frac{\pi v_s}{2H}$$

$$T_s = \frac{2\pi}{\omega_0} = \frac{4H}{v_s}$$

The characteristic site period, which depends only on thickness and shear velocity of the soil provides a very useful indication of the period of vibration which is the most significant amplitude that can be expected. Naturally the fundamental frequency or let us say period of vibration that depends only in two factor one is thickness of soil layer another is shear velocity because the frequency it is already said that it will be at the case when omega h over Vs equal to pi by 2. And this will provide a very useful indication at which the most significant amplitude can be expected. These analysis provide amplitude for soils overlying a rigid bedrock. Now before we talk about elastic rock, I would like to discuss one simplification of this equation.

In this equation number 9c which gives the transfer function for damped case if omega h in equation 9c if I put omega h over Vs equal to pi by 2 plus n pi whatever value of n then what will happen? This term first term will be 0 and in the second term it will be square and the root so it will be simplified so what you will get f2 omega is very much simplified for this case and for this case f2 omega will be simply xi into omega h over Vs is nothing but pi by 2 plus n pi. So this will be your answer. That means in this case f2 omega depends only for example when n equal to 0 this becomes f2 omega is simply becomes 1 over xi into pi by 2. That means for even for damped case the peak value the value of the peak depends only on the damping ratio. It does not depend on h; it does not depend on Vs here velocity. That means and even these values these factors because xi into pi by 2 so this xi into pi by 2 can be calculated for example for xi equal to 5 percent what you get? You are getting f2 omega the formula is here ready 1 over xi into pi by 2. So if I put 5 percent then I get 1.05 into pi by 2. So this is about 40 divided by pi which is 12.67. So this value is 12.67. This is one half of that 6 point and this is. Similarly for another peak you will have here it will be n plus 1 so it will be divided by 3 pi by 2 instead of pi by 2 one third of that. So the peak values can be depends only on the damping ratio. It does not depend on the thickness of the soil layer or it does not depend on the shear velocity. So, this was all about the case when you are considering a damped soil on rigid rock.

But it may not always be possible that you have a soil layer on ridged rock only. Normally you may have an elastic rock that means below your soil layer bedrock is not there rather you have some elastic rock. So, these two analyses provide amplifying factors for soil overlying ridged bedrock. If the bedrock is this its motion will be unaffected by motions in the overlying soil. It and this will act like a fixed boundary condition. So, in this case when we consider elastic rock any downward travelling waves in the soil will be completely reflected back toward the ground. So ground surface by rigid layer thereby trapping all of the elastic wave energy within the soil layer. So what is said here that suppose if you have here one soil layer at the base of the soil layer you have fixed soil layer. And in this case whenever any waves is striking this boundary like suppose it is coming from the up or maybe so when it is coming so when it is strike then it will reflect back here. Because there is a bedrock it will not go beyond the bedrock.

So, all the energy is trapped inside the soil layer only. So in this case it will not go beyond the bedrock it will come back to the same soil layer. So thereby trapping all of the elastic wave energy within the soil layer. But instead of this if I consider an elastic layer here where you have elastic rock here rather than and so when the any wave strike it so some part is reflected and some part is refracted. So, some the part of the energy some energy get reflected from the interface between these two layers and some continue travel.

And once it is continued travel then it may possible that it may not come back or even if it come back then its amplitude may diminish. So as a result what happens the part of the elastic energy of these waves are effectively removed from your system from the soil layer.

And this is called this type of effect is called what we call the radiation damping. We already discussed when we discussed radiation damping or geometrical damping which is due to the spread of the waves, and it causes the free surface motion amplitude to be smaller than the case of the rigid bedrock. So, when the amplitude at the free surface in this case is expected to be less than the amplitude in case of bedrock and you will see that this is true, we will mathematically prove it.

Earlier this is the case which we are discussing you have uniform damped soil on elastic rock. Now one major difference have come earlier you have only one layer still you have one layer with thickness h but below this layer you do not have the bedrock rather than you have another layer of rock with property as a shear modulus gr and rho r. So now the properties of this rock material influence the response of the soil layer. When you consider the bedrock then whatever we say the bedrock or then the properties of the bedrock is not considered that is considered fixed at the base. But in this case, you will have in the one layer you have in soil layer the amplitude of the wave is As and Bs.

As is the wave going upward Bs is downward. Similarly in the rock it is Ar and Br. So simply S subscript is denoting soil while R subscript is denoting rock. So that is the nominal creature for the case of soil layer overlying F. Now to deal with this case rather than going in the derivation I give you directly the case answer that the transfer function that means the amplitude of the motion at the free surface divided by the amplitude of the motion at the base that is the interface between soil and rock. And finding this transfer function for a damped case is very little complicated.

As a result, for the simplicity, we are considering a case when the damping in the system is 0 and when you consider the damping is 0 in this case then this transfer function is simplified using this equation where ks is again wave number but this ks in this equation which is coming is related to soil omega over vs where vs is the shear velocity of the soil. And what is alpha? Alpha z is called an impedance ratio. Alpha z is an impedance ratio. So, this should be treated as alpha z here is the complex impedance ratio and this is given by this relation gs into ks divided by gr into kr where s subscript is for the soil and r subscript is for the rock. This can be simplified rho s vss, rho s and rho r the mass density of the soil and rock respectively and vs this denote the shear velocity of the soil while this is the inventory shear velocity of the rock and star is saying you that you may consider the damped case also for this case.

$$|F_3(\omega, \xi = 0)| = \frac{1}{\sqrt{cos^2 k_s H + \propto_z^* sin^2 k_s H}} \qquad \textbf{(10a)}$$

$$\propto_z^2 = \frac{G_s k_s^*}{G_r k_r^*} = \frac{\rho_s v_{ss}^*}{\rho_r v_{sr}^*} \qquad \textbf{(10b)}$$

So, when you consider the damping. So, vs star and vss star are the complex shear velocity of the soil and rock respectively as we discussed. Once this is known then you can find out the transfer function. Alpha z can be substituted. Now suppose if you consider the bed rock, bed rock is a specific case. In that case bed rock, the property is shear velocity is going to be quite high compared to the alpha z.

As a result, vsr is going to be quite high and alpha z will be in that case 0. So if I put alpha z equal to 0 in this equation number 10a then this becomes same as was the case for bed rock but with undamped case. Now this was the case when you consider single layer. The effect of damping on the bed rock stiffness as reflected by the impedance ratio.

So, it will depends on the value of alpha z. Naturally alpha z is coming in the denominator. It will influence your results. The resonance cannot occur. In this case also resonance will not occur because why when you have even cos the first term equal to 0 due to the presence of elastic rock alpha z will not be 0. As a result, if alpha z is not 0 even ksh equal to pi by 2 then still it will not be 0.

So, the denominator will not be 0. The denominator is always rather than greater than 0 even when the soil is undamped case. The effect of the bed rock stiffness as reflected by the impedance ratio on amplifiers is illustrated here. So, in this figure what is shown in this figure. So, when the impedance how the value of F3 changes when you have impedance ratio from 0, 3 impedance ratio this curve are for 0, 0.1, 0.5. So when for the 0 curve is same as for the case because it is undamped case that means this will be the same value as F1 omega. So here this is going to infinity. This is going to infinity in this case also. So, for impedance ratio equal to 0 then the case is similar as we discussed for undamped case on bed rock that is F1 omega.

But when you consider impedance ratio equal to 0.1 then you see that there is a change the peak value which was infinity have come to the finite value and then further it is going the same. But if I increase impedance ratio to 0.5 instead of 0.1 then again what will happen then this will further decrease, and you get the peak values here. So, effect of on amplifying factor for case of undamped soil. So this was the case. Here in all the cases we consider that this single layer. So, one thing is important to notice here there is a similarity between the effect of soil damping and second the bed rock elasticity and this is quite similar. The elasticity of rock affects simple amplifying factor similarly to the damping ratio. So, when you consider damping the system with you consider damping with the bed rock or without damping but then instead of bed rock you consider elastic rock.

So in both the cases the response will not go to infinity and you get a finite value. The radiation damping effect has significant practical importance. So all these cases were for the single layer. But suppose in real practical case you do not have a one single layer in the practice rather than you have a number of layers and when you have a number of layers then how to deal then we need to use either some software. So, for example here layer 1, layer 2, layer 3 so you have thickness of different layer h1, h2, hm, hm minus 1, hm and then you have this property shear modulus g1, g2, gm on damping ratio and the mass density. So, the first case z1, z2 and u1 and u2, um these are the displacements here. So, for solution because you need to find the transfer function for one layer then another layer and the top one will be influenced by the down layer. So let us say the transfer function between two layers which is i th and j th layer is can be simply given by this relation where fij omega is a transfer function that means you have i th layer and j th layer. So, I have this layer, two layer so this is i th and this is j th layer and below there are more layers on the top also you layer. So, this transfer function says going from j to i if wave is coming then how this will be transferred.

$$F_{ij}(\omega) = \frac{a_i(\omega)+bi(\omega)}{a_j(\omega)+b_j(\omega)}$$

So, this will be ai and this should be in fact this is b subscript i, this is not bi so this is b subscript i and as was the case earlier even for single layer this transfer function was the frequency dependent, they was function of omega here also they will be function of omega that means they are highly dependent on frequency of excitation, and they will keep changing. So with this thank you very much and this is the second lecture on the module 3 is completed. So in fact we are like completed in the 1D ground response analysis we have considered all the three cases but we will continue and this was for the linear case. In the next lecture we will continue with 1D ground response analysis but for with the non-linear case. Thank you very much for your kind attention.