Earthquake Geotechnical Engineering

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Lecture 09

Wave Propagation

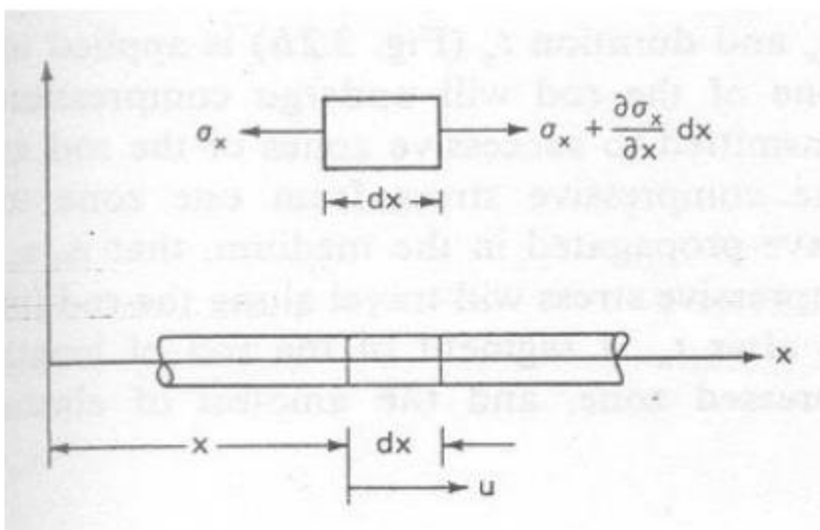
Welcome you again for this NPTEL course on earthquake geotechnical engineering. And today we are continuing with our module 1 and chapter number 4 of the module. Today we are going to start a new chapter of this module that is on wave propagation and this chapter will be also in two lectures, lecture 9 and lecture 10. So, let us see what is this wave propagation, what we are going to talk in this or like you know that module 1. So, if we recall module 1, already four sections have been covered geotechnical issues during earthquakes, engineering seismology, strong round motion, seismic hazard analysis. So, that is fourth chapter is already over and today we are going to talk about last chapter of this module that is on wave propagation.

So, what we are going to cover in this topic today, introduction to wave propagation, then we are going to talk one dimensional wave propagation in an elastic road and three-dimensional wave propagation in elastic infinite medium and finally, wave propagation in a semi-infinite elastic half space. So, let us start from the introduction to wave propagation, it is similar to like you can consider what is wave propagation. Suppose you have a pond of water and if you put some small stone or pebbles inside the water, then what will happen the waves will generate and they will move. So, you can in that case you can see the waves and they are traveling like away from the source of disturbance.

Similarly, earthquake wave is similar to that only. However, you can see water waves, waves which is generated due to water, but for earthquake waves which is elastic waves you can feel their effect, but you cannot see those waves. So, in general waves are generated in a continuous medium due to a disturbance in the medium. So, some disturbance is required to generate these waves and disturbance here in our case in seismology will be related to earthquake, earthquake will be the cause of a disturbance. Now, when the waves are generated due to some earthquake and then they may propagate, then wave velocity through the medium which is a propagate that will depends on the property of the medium.

So, that is important and I think we discussed this issue earlier also. The part closest to the source of disturbance is naturally going to be affected first and then deformation which produces due to this disturbance are subsequently spread throughout the body in the form of stress waves. So, these are the stress waves or elastic waves which are generated. Now, coming to 1D wave propagation in elastic road, for the simplicity in real medium in the ELR you have 3 dimensional case. So, we will discuss later, but for time being let us say consider you have an elastic road which is in 1 dimensional that means, it is long like I could show you the that here this is you could see this elastic road is there which is going to length is going to very big.

In this case you could have 3 types of motions one is longitudinal vibrations that is along the axis longitudinal axis of the road and this could be either extension or compression. So, you could stress tension or compression could be there, but let us say if this is a road or let us assume this is elastic road. So, this will be longitudinal, but there could be torsion in case of torsion what you have this will rotate about its axis. So, it could be a torsional vibrations or suppose if you move around the in the lateral directions like here then it will be bending or flexural vibration, but we consider at a time only one of the component rather than 3 all are together. So, now continue with this elastic road let us consider the first one that is longitudinal vibrations and do the analysis here.



So, this is the here you have a road which is length is going in front, but what we are considering a small element which is shown here as a in this case which is dx is the length of this element and area of this element is a which is same as the cross-sectional area of the road. Now, what happens like you have the stress on this element one side sigma x and on another side of this element stress is little bit higher this is sigma x plus del sigma x del x into dx. So, this element is subjected to this stress condition and in this stress condition we need to we need to find out the equilibrium of this element with the different forces and that equilibrium will give you the wave velocity in this road. In fact, we write the force equilibrium that is nothing, but equation of motion. So, forces are will be simply what will be the force? Force will be sigma x into area cross sectional area on one side on another side also. If this force would be like you know because this force is like one side it is more on side another side is less. So, that is imbalance. So, what will happen as a result your road will start moving in one direction, but it does not happen why? Because inertial forces of the road will resist this and as a result you get these equilibrium equations. So, the first one is due to the equation number 1 which we already discussed a summation of is simply del sigma x del x dx into A that is simply giving you the force equilibrium due to stress. But as I mentioned the inertia will try to resist the inertia force can be volume multiplied by the acceleration.

$$\sum F_x = -\sigma_x . A + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} . dx\right) . A$$

$$\sum F_x = \frac{\partial \sigma_x}{\partial x} dx\, A \qquad (1)$$

So, dx into A is the volume of the road and this gamma by g which is gamma by g is nothing, but rho which is mass density. Mass density multiplied by del u del u by del t square is your acceleration. So, dx into A simply and if I multiply by rho it will give you the mass. So, mass multiplied by acceleration give you the force. So, now what we do? We equate because left hand side is same equation number 1 and 2 and find out.

$$\sum F_x = d_x . A \, \frac{\gamma}{g} \frac{\partial^2 u}{\partial t^2} \qquad (2)$$

So, once you equate then you end up in the equation number 3. So, in the equation number 3 is give you the equilibrium what is u in this equation? u is nothing, but displacement in x direction. Sigma x which is a stress can be represented simply Young's modulus into strain that is so e into del u by del x. So, we replace sigma x and finally, you end up in this equation which is equation number or like you know this equation. We analyze this equation further and we write in another form this equation where del u by del t square is e by rho and here this is ultimately you end up v r square equal to e by rho. What is v r? v r is nothing, but it is called longitudinal wave propagation velocity in the road which is written here and e is Young's modulus and rho is mass density.

$$\frac{\partial \sigma_x}{\partial x} = \frac{\gamma}{g} \frac{\partial^2 u}{\partial t^2} \qquad (3)$$

- **As stress in x direction**

$$\sigma_x = \text{E x Strain} = \text{E} \frac{\partial u}{\partial x} \qquad (4)$$

- **Substituting in Eq. 3**

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

So, as expected the velocity which is will be travelling in this road will depend on its material property and what is the two material properties one is Young's modulus and another is mass density. So, Young's modulus and mass density will govern and you can find out the you know the longitudinal wave velocity.

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = v_r^2\frac{\partial^2 u}{\partial x^2} \qquad (5)$$

- **Eq. 5 is called one-dimensional wave equation.**

$$v_r^2 = \frac{E}{\rho} \qquad (6)$$

Continue with this. So, this was the like you know about one dimensional wave propagation, but our earth is not 1D case our mother earth is a 3 dimensional and when we talk about 3 dimensional it is kind of a you know basically a sphere. So, if I see in the elevation or in the plan then it will look like a circle right. So, all together you have and all three directions are there if I say in the coordinates. So, you have two horizontal on one vertical or I have two radial directions and then suppose this is plan. So, then I have this is my plan then in this plan you have a circle and on this is radius r this side and this side and then on the vertically or like perpendicular to this board you have the third dimension. So, this is a 3D case now the way when the earthquake wave or elastic wave or a stress waves propagate through this medium how they respond.

So, there we are going to discuss that what is done here it is assumed that the infinite medium through which we waves are propagated is homogeneous and isotropic. So, our first assumption though again your medium may not be in really homogeneous and isotropic, but for simplicity we are assuming that the medium is homogeneous and isotropic. In this case considering a small element of dimension dx dy and dz and those are in x y and z directions respectively and the displacements u v w, u is in x direction v in y direction and w is in z direction we continue and in 3D case how it looks. So, this is a 3D medium. So, it is a kind of a cuboid where you have a like let us say dx dy dz are the dimension small dimension along the x direction y direction and z direction.

Now, stress conditions stresses in this case will act all along you know x direction y direction z direction. If we consider a one phase let us say a phase which is like this phase we are considering in front a perpendicular to this is in x direction positive x direction positive x direction is this direction negative is an opposite side. Similarly, y is positive this direction z is positive upward. So, on each phase one normal stress and two shear stresses act. For example, on phase on which you have x axis is perpendicular sigma x will be the normal stress, tau xy and tau yz will be the that like tau xy and tau xz will be the shear stresses.

In this case notations are that particular for the shear stress notations for example, let me talk about tau xy. What is meaning of tau xy? First of all tau xy will be shear stress not a normal stress. Tau xy meaning is here a shear stress which is acting in x direction. So, first subscript give you the direction and y is it is acting on a phase normal to which is in y direction. So, tau xy is acting here this stress will be tau xy and this is acting in x direction and on a phase normal to which is in y direction.

Similarly, you have tau xz which is acting again in x direction and if on a phase normal to which is in z direction. So, let us consider in x direction three stresses one is sigma x here. As for sigma x is concerned being normal stress it act in x direction and it act on a plane perpendicular to which is sigma x. In fact, you can say that in the short sigma xx is same as sigma x here. Similarly, we are assuming sigma yy equal to sigma y and sigma zz equal to sigma z. So, that is there. So, for the normal stress we use only one subscript rather than two subscript that may be you already aware about that. So, now let us say we consider the equilibrium of forces in x direction first then yz direction z direction. So, in the x direction three stresses are acting and if you want to get the force then what you need to do? You need to multiply by that stress with the area phase area. For example, sigma x if I multiply by the area of this phase which is will be simply what you have here this is dy and dz. So, dy into dz will give you multiplication.

Similarly, for other cases so that is work out here let me here. So, what you have and then what you need to understand that one of the like sigma x plus delta z similar in the road on another direction sigma x will be there. So, on this phase this stress is left and this stress need to be multiplied to get the force you need to multiply by dy into dz. What is dy dz dy dz will be the area of the phase here this phase front phase. Similarly, this stress which is acting on this phase and ultimately this component left out.

This component need to be multiplied by the area which is dx into dz. Similarly, this component is left out and because this will be negative direction. So, you will have this phase dx into dy. So, as a result you get dx dy dz in all the three components which are acting along x axis and then are the remaining the left out. So, you get this equation for the total force equation number 1 where dx dy dz will be common in all the three cases which is nothing but basically volume of this cuboid.

So, here before considering the variation in opposite phases of this element as so negatives are will be cancelled out sigma x minus sigma x will cancel out. The stresses on each phase of this element are represented by sets of orthogonal vectors. Transitional equilibrium this element can be expressed by writing the sum of forces acting parallel to each axis. So, that is done here we already explained it in equation number 1. Now equation 1 give you the force due to the stresses which is generated.

$$\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dx\, dy\, dz = F_x \qquad \textbf{(1)}$$

$$\sum F_x = \rho(dx\, dy\, dz)\frac{\partial^2 u}{\partial t^2} \qquad \textbf{(2)}$$

But again, if due to this force this element should start moving but it will not move because the initial force will try to resist it. And inertial force is again rho into dx dy dz this is nothing but your m mass and what is this quantity? This quantity is nothing but acceleration which could be a. So, mass multiplied by acceleration gives you the force. Now what we do like earlier we equate equation number 1 and 2 because both are equal to fx and as a result in both the cases dx dy dz is coming. So, dx dy dz will be cancelled out and you end up in equation number 3a.

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \qquad \textbf{(3a)}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \qquad \textbf{(3b)}$$

$$\rho\frac{\partial^2 w}{\partial t^2} = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \qquad \textbf{(3c)}$$

So, 3a equation is coming from equation number 1 and 2. This was the process which we followed in x direction. Similarly, we can follow this is the end product we follow do the same exercise in y direction z direction you get 3b and 3c. So, these equations are equations of force equilibrium and this is called in dynamics equations of motion. So, this is equation of motion is nothing but you can say that equation of motion is nothing but force equilibrium.

In fact, I will like to extend definition if you are considering the moment then it could be moment equilibrium forces and moments we say together as the same thing. So, here we are considering the force equation. What is u v w in the left hand side of these equations? Displacement in x y and z direction. So, that is our unknown. Now solving these 3 equations a b c you need to find out the value of u v w 3 unknowns are there 3 equations are there you are fine.

But the issue is that on the right-hand side you have sigma x tau x y they need to be also represented in terms of u v w then only you can solve these equations and for that we need to use some other relationship. So, what we do? We need to express the stresses in terms of displacement. So, for that you require stress strain and strain displacement relationship. So, stress should be represented in terms of strain and strain should be represented in the displacement. If you do this then as a result on the right-hand side also you get u v w in all 3 equations and we can solve.

$$\sigma_x = \lambda \in_v + 2\mu\epsilon_x \quad \text{(4a)}$$
$$\sigma_y = \lambda \in_v + 2\mu\epsilon_y \quad \text{(4b)}$$
$$\sigma_z = \lambda \in_v + 2\mu\epsilon_z \quad \text{(4c)}$$
$$\tau_{xy} = \tau_{yx} = \mu\gamma_{xy} \quad \text{(4d)}$$
$$\tau_{yz} = \tau_{zy} = \mu\gamma_{yz} \quad \text{(4e)}$$
$$\tau_{zx} = \tau_{xz} = \mu\gamma_{zx} \quad \text{(4f)}$$

So, stress strain relationship for 3D case is given here 6 relations are there first 3 relations sigma x sigma y sigma z gives you the normal stresses and last 3 equations are for shear stresses. So, for shear stresses equation relationship rather simple that you are just tau x y is multiplied by mu into x y and while here both lambda and mu are involved. What is lambda and mu here in these equations? Lambda and mu are nothing but called Lame's constant it is like you know lambda and mu are the Lame's constants which is used in the both 6 equations and these Lame's constant can be found out using Young's modulus and Poisson's ratio. Lambda can be represented by nue 1 plus nu 1 minus twice nu similarly mu is represented by e divided by twice 1 plus nu. So, here you have nu, nu will be using for Poisson's ratio because we will in this throughout this course we will be using nu for Poisson's ratio, nu is what is this? It is written here nu is Poisson's ratio.

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{(5a)}$$

$$\mu = \frac{E}{2(1+\nu)} \quad \text{(5b)}$$

So, rather than using mu sometimes sometimes we use the mu because mu we are using mu notations here, we are using for the Lame's constant and mu is nothing but in fact it is simple same as a shear modulus g. So, that you need to understand. Then we define before we solve these equations, equation 3a, 3b, 3c we need to also define volumetric strain and volumetric strain is nothing but which is given by epsilon v epsilon x epsilon y plus epsilon z, 3 strains are there. Where epsilon x is nothing but what is epsilon x? Epsilon x is del u by del x. Similarly, epsilon y is del v by del y, epsilon z is del w over del z.

$$\in_v = \in_x + \in_y + \in_z \quad \text{(5c)}$$

So, which is already defined. So, the combination of all three give you volumetric strain, this is volumetric strain. So, this was relation between the stress and strain. Now, strain need to be also represented in terms of displacement which is like it is already here. So, linear this is which I write in the last slide del u by del x and shear strain for the shear strain xy you will have xy so down you will have xy here, yz yz here, zx like this.

So, it goes in the cycle xy yz zx like this. So, it is easy to remember. Once you remember one you can remember another. So, what do you have here? Because it is shear strain there will be cross. So, del v by del x and y you will get u. Similarly, del w by del y and del v by del z same thing is for 6.

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \textbf{(6a)}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \textbf{(6b)}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial y}{\partial z} + \frac{\partial w}{\partial x} \qquad \textbf{(6c)}$$

So, what we do? Now, equation we need our objective is to solve these 3a, 3b, 3c. So, what we do? In this equation we use equation 4, 4a to f and then fifth and then sixth. Using 4, 5, 6 in these equations, equation 3 can be written as like this. So, a, b, c and this is 3a, 3b, 3c. So, what we have done? We have represented left hand side is same as before LHS is same. On the right-hand side, you have converted stresses in terms of strain and strain in terms of displacement. So, basically you have u, v, w and this is volumetric strain also and lambda and mu are known to you. So, and rho is also known to you. So, you have now 3 equations 7a, 7b, 7c and all 3 equations have only 3 unknowns which is 3 equations which is u, v, w.

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \epsilon_v}{\partial x} + \mu \nabla^2 u \qquad \textbf{(7a)}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial \epsilon_v}{\partial y} + \mu \nabla^2 v \qquad \textbf{(7b)}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + \mu) \frac{\partial \epsilon_v}{\partial z} + \mu \nabla^2 w \qquad \textbf{(7c)}$$

- **Where the Laplacian operator ∇^2 is**

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \qquad \textbf{(8)}$$

We solve and find out the u, v, w. Then once you are done then you are. So, for the solution to proceed the solution a Laplacian operator del omega square is used which is given in equation number 8 which is the second derivative of del square del x square plus del square del y square del square del z square. The equation 7 are the equation of motion of an infinite homogeneous isotropic and elastic medium. So, this is the condition. This is medium consider is homogeneous isotropic and elastic.

So, equation on the basis of assumptions this have come. Now, to solve this equation we need to do some manipulation. What manipulation is done? You you multi differentiate each of 7a, 7b, 7c by first equation by del x, second by del y, del y and the third one del by del z and then add which is given here. So, for 2 solutions for the first solution which will be describe propagation of a rotational wave and the second will describe a pure rotation. So, to obtain the first solution differentiate equations 7a, 7b and 7c with respect to x, y and z respectively and add the together.

So, once you add after differentiating so what you will get? You will end up in equation number 9.

$$\frac{\partial^2 \varepsilon_v}{\partial t^2} = v_c^2 \nabla^2 \varepsilon_v \qquad (9)$$

In this equation 9 epsilon v is already defined epsilon v is nothing but your volumetric strain and v c you will end up lambda plus 2 mu over rho v c square. If I put lambda and mu value in terms of e and rho then you get e by this equation and this can be also written k where what is k? k is a bulk modulus. So, basically what you get v c in this case v c is simply you can say square root of k by rho. So, instead of like in the case of elastic road you got v r and v r was nothing but e by rho.

So, instead of e which was Young's modulus it is replaced by k which is bulk modulus and you know the bulk modulus value is higher than the like Young's modulus value for a new value if nu is not 0. So, if nu is 0 then in that case your k will be same as e because if I put in this equation so k will be same as e there will be no difference if Poisson's ratio is 0 then in fact, there will be no difference between e and k and e, but then we lose the meaning of the Poisson's ratio. In fact, all the constant becomes same k, e and g for the when the Poisson's ratio are same. So, what do you have in this case if nu equal to 0 then your v c what is v c here? v c is the velocity of compression waves in the not in the road in 3D case. So, c subscript is stand for the compression wave it is velocity of compression wave and this is given by this relation.

$$v_c^2 = \frac{\lambda + 2\mu}{\rho} = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)} = \frac{K}{\rho} \qquad (10)$$

If nu equal to 0 then you get this one, but if your nu is not 0 in that case v c will be always greater than v r. So, for the real material where nu is not 0 we can expect that the wave velocity in 3D medium will be higher than the compression wave velocity in the elastic road or one dimensional case. So, that is the outcome here. Continue with this for another solution as we discussed that these equations this 7a, 7b, 7c have two solutions, one solution we already find out. For the second solution you need to differentiate opposite like for example, x needs to be differentiated with y and the second y should be z and then subscript rather than adding.

So, here for another solution for another solution equation can be obtained by differentiating 7b with z and equation 7c with y. Then eliminate by substituting so what you do with this operation. So, you differentiate equation 7b by z and 7c with y and subtract with from one another then you end up in equation number 11.

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \mu \nabla^2 \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \textbf{(11)}$$

Now, in equation number 11 this quantity del w, del y, del z this is same on both sides and this in the compact form can be written as this form. So, what is omega x bar? This quantity we are writing which is rotation basically omega x and the bar which is nothing but rotational component.

$$\rho \frac{\partial^2 \bar{\omega}_x}{\partial t^2} = \mu \nabla^2 \bar{\omega}_x \qquad \textbf{(12a)}$$

$$\frac{\partial^2 \bar{\omega}_x}{\partial t^2} = v_s^2 \nabla^2 \bar{\omega}_x \qquad \textbf{(12a)}$$

This is denoting rotational component in x direction and we write equation number 12a is coming from 11 which is again written in the compact form vs square. What is vs square? vs square in this case is nothing but your mu by rho which is given in the next slide. So, here similar expression first of all obtained for y and z direction and vs square is represented by this so which represent your this and on the if you go on the right hand side then you get vs square g by rho. So, this could be also it says vs equal to g by rho and this is very very important relationship in geotechnical earthquake engineering. Shear wave velocity gives you shear velocity can be find out if you know the shear modulus and mass density.

$$v_s^2 = \frac{\mu}{\rho} = \frac{E}{\rho} \cdot \frac{1}{2(1+\nu)} \qquad \textbf{(13)}$$

So, this is the relation for the shear velocity we obtained. So, this was there continue with this. Now suppose we find the ratio of two wave velocity one is compression wave velocity and it is typically called P wave velocity and another is called shear wave velocity. The ratio of two wave velocity depends only one factor which is Poisson's ratio and Poisson’s ratio given here and we will plot like how this varies on this case this relationship. So, we will continue use this relationship in the next lecture also. So, you remember this relation that is 2, 1 minus nu divided by 1 minus twice nu.

$$\frac{v_c}{v_s} = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}} \qquad \textbf{(14)}$$

So, this relationship will be used further. So, two kinds of wave was there. Now continue with this. So, what is our conclusion for the three dimensional case? For 3D wave propagation in elastic infinite medium there are two types of wave one is called compression wave and the travel velocity of this is v c it is also called primary wave P wave or irrotational wave. So, there are four names for the same wave. In general compression wave, primary wave, P wave and irrotational wave and the wave velocity will be v c in 3D case or the same will become v r for 1D case v r for 1D one dimensional case.

So, then shear another velocity shear wave velocity that is also called secondary wave S wave, distortion wave and voluminal wave. So, these two waves which represent different types of body motions travel at different velocities and these are basically both of these I think you would have in the second lecture second chapter we have discussed they are nothing but in short both called body wave. So, this was body waves this is no out of the both are body wave nothing surface wave. So, when the waves are traveling inside the earth, they are not coming on the surface they are not touching your earth ground until they are not coming on the ground then they will be as a body wave only they will not there will not be any surface wave.

Now, continue with this in the rod we already discussed E by rho. So, this is a summary only. So, all the things we on the slide is already discussed. So, v r will be E by rho and v c will be lambda plus twice mu by rho and shear velocity another wave is given by E by rho. So, this is in general and when mu equal to 0 this becomes same as E by rho. So, now you may have that what is the typically wave velocity for different mediums.

$$V_c = \sqrt{\frac{\lambda+2\mu}{\rho}} \qquad V_r = \sqrt{\frac{E}{\rho}} \qquad V_s = \sqrt{\frac{G}{\rho}}$$

So, it will depend on the property of your medium whether it is hard, whether it is rock, whether it is soil depending on the property of the medium you can find the what is called the compression wave velocity or shear wave velocity. So, for example, for different material here mass density is listed which is in kg per meter cube and you have this compression wave velocity v c and the shear velocity in both are in meter per second. Naturally v c is going to be higher than v s and minimum value if I put even rho equal to 0 in this last in this case even for mu equal to 0 what do you get from this equation for mu equal to 0 you get v c over v s equal to root 2. So, minimum this ratio is root 2 for mu equal to if I say 0.5 what will happen this will go to infinity this ratio will be tending to infinity.

So, that means, the minimum this ratio should be root 2 and it will exponentially increase as the Poisson's ratio increases. So, the difference in this column v c and v s is large when the Poisson's ratio is large and this difference will be minimum when the Poisson's ratio is 0, but for the rock or for soil Poisson's ratio is not going to be 0. So, as a result this ratio will you will never get root 2 it will be always you see more than 2 times or 3 times or 4 times even 10 times in the first case or like this. So, these are shear compression wave which is ranging from minimum 300 meter per second to 1500 while here minimum 110 to going to 260 meter per second. So, this was all story about 3 d k s, but what happens like if you have the water then there is some issue the measurement of the velocity of compression in water saturated soil will not be a representative velocity why because the water has water has shear and water do not have shear strength and 0 value in shear modulus.

So, water shear modulus is 0 and it does not have any shear strength. Water has it should be water has 0 shear strength. So, 0 is missing here water has 0 shear strength and has 0 value in shear modulus. So, once we are done with the with 3 d k s now the issue comes here in the real scenario you do not have 3 d in the actually what happens you have on the ground what we called elastic half space and how elastic half space looks elastic half space is like this. So, this is medium and in this medium that is medium is called elastic half space and this is in fact, what we call this is going to infinity. So, we call it semi-infinite medium. Semi infinite medium. So, our earth is semi infinite medium it is not infinity means because you have one free surface here. So, it is ground level or surface level. So, when the waves travel from the focus when they are travelling inside the earth they can go inside and like you know travelling, but when they come towards the earth then they will hit free surface and what happens when the waves travels on elastic half space this is called a half space this medium is nothing, but half space this is half space. So, when the waves travel through the half space what will happen in an elastic infinite medium there are two types of waves, but when you consider the semi infinite medium a third wave is also generated which is basically the Lf wave or surface wave which we are going to discuss how it is generated. So, how half space is defined in xy plane in with z assume positive in the downward direction and for this case what happens because you have let us say in one direction if I say this is my x and this is z direction.

So, I no need to consider y. So, as a result displacement in y direction v equal to 0 when we consider semi infinite medium and the strain this is a plane strain case. So, as a result del u by del y equal to del w or del y also 0. So, this is plane strain case. So, this is these conditions.

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} = 0$$

So, we are working with these condition v should be 0 as well as strain should be 0. So, plane strain case strain in y like you know y directions. So, now coming to this so as a result instead of three equations when you consider semi infinite half space then you end up in two equations only one in x direction another in z direction. So, these are the two equations which is equation of motion it is quite similar to what we discussed earlier as equation 3a 3b 3c. Now to solve these equations in terms of u and v we use some potential functions which is phi and psi is given here.

$$(\boldsymbol{\lambda}+\boldsymbol{\mu})\frac{\partial \epsilon_v}{\partial x}+\boldsymbol{\mu}\nabla^2\boldsymbol{u} = \boldsymbol{\rho}\frac{\partial^2 u}{\partial t^2} \quad \textbf{(2a)}$$

$$(\boldsymbol{\lambda}+\boldsymbol{\mu})\frac{\partial \epsilon_v}{\partial z}+\boldsymbol{\mu}\nabla^2\boldsymbol{w} = \boldsymbol{\rho}\frac{\partial^2 w}{\partial t^2} \quad \textbf{(2b)}$$

- **A general solution of these equations may be written**

$$\boldsymbol{u} = \frac{\partial \emptyset^*}{\partial x}+\frac{\partial \psi^*}{\partial z} \quad \textbf{(3a)}$$

$$\boldsymbol{w} = \frac{\partial \emptyset^*}{\partial z}-\frac{\partial \psi^*}{\partial x} \quad \textbf{(3b)}$$

So, like phi is a function of x and z and psi is also x and z. Beside that for the dynamic case they will also vary with time where what is omega in this case omega in this case this omega is not w this is of angular frequency.

In which $\emptyset^*$ and ψ^* are analytical functions which may be

$$\emptyset^* = e^{i\omega t}\emptyset(x,z) \quad \textbf{(4a)}$$

$$\psi^* = e^{i\omega t}\psi(x,z) \quad \textbf{(4b)}$$

Then the equation that should be satisfied by the function $\emptyset$ and ψ are as follows

$$(\nabla^2+h^2)\emptyset = 0 \quad \textbf{(5a)}$$

$$(\nabla^2+k^2)\psi = 0 \quad \textbf{(5b)}$$

So, this represent your frequency of excitation and if we solve for the solution you need to have these equations are satisfied if we del square plus h square del square is same as a Laplacian operator as earlier and h square and k and phi equal to what is h and k h and k are nothing but they are called wave numbers in which wave numbers h is simply this is not w actually this is wrongly written this is omega and this is also omega. So, this is omega vc and this is omega vs. So, omega divided by vc and omega divided by vs both are called wave numbers.

In which wave numbers $h = \frac{w}{v_c}$ and $k = \frac{w}{v_s}$ (6a)

Since $\omega = \frac{2\pi}{T}$ $h = \frac{2\pi}{v_c T}$ and $k = \frac{2\pi}{v_s T}$ (6b)

Since $v_c > v_s$ hence k > h $\omega = k v_s$

For Rayleigh wave $\omega = v_R \lambda = k v_s$

where λ is not wavelength (unit is 1/m)

$$v_R = \frac{k}{\lambda} v_s = c_1 v_s \quad \text{Where } c_1 < 1$$

Now, omega here it is correctly written here omega is replaced by 2 pi t here. So, if I replace by this then I get because vc is greater than vs as a result your k will be greater than h and using these equations what we can have omega, omega can be written from the top equations into k into vs or it can be written as h into vs. So, if I use k into vs now omega is also vr into lambda what is vr L of velocity and lambda is some number it is not wavelength here lambda is not a wavelength and it is not lames constant it is something else. So, what is here this is a constant only which is unit is over 1 meter 1 divided by meter same as a like opposite of wave number. So, here what you see if I work on this equation then vr k by lambda and ultimately you get c1 into vs.

So, that c1 is a constant c1 is a ratio which is less than 1. So, what has been observed that vr is a function of we can represent real wave in terms of shear velocity and normally it has been seen that depending on the Poisson's ratio if your Poisson's ratio is 0.5 which is maximum value then vr end up 0. 99533 vs that means around 0.96 vs shear velocity. If you consider 0.25 then this decrease to 0.9194. So, in general the Rayleigh wave velocity is almost similar to shear velocity but always less than vs. So, vr will be always less than vs not equal or greater than vs.

So, this is a for the case here. So, these wave have different velocity of propagation and knowing their velocity it is easy to predict in which order the wave will travel. So, this was about in the elastic half space. So, this with this I end up this lecture here which was little longer and I think we will continue in the second lecture on wave propagation. Thank you very much for your kind attention.