Time Series Modelling and Forecasting with Applications in R

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Week 07

Lecture 31: Multivariate Time Series Analysis: Examples and Motivation

Hello all, welcome to this course on Time Series Modeling and Forecasting using R. Now we are entering a new week with a slightly different idea, and that idea is in front of all of you. So, we are going to analyze or try to find out what one means by multivariate time series or multivariate time series analysis. So far, whatever we have covered would be simply univariate, isn't it? I mean, we would be analyzing, let's say, a single data.

So, be it stock price, temperature data, rainfall data, or whatever, but then you have y_t and its iterations, or something like y_t, y_t+1, y_t+2, etc. But then the whole idea is that whatever models we studied so far—AR, MA, ARMA, ARIMA, even ARFIMA—all those models were applied to a single series. So, whenever you want to analyze or, let's say, model and forecast a single series, that idea is called univariate time series analysis. And if you slowly transition into multiple series, then that idea is called either bivariate, trivariate, or, in general, multivariate time series analysis, right? So, this week, we will spend some time understanding the idea of multivariate time series analysis.

Why is it important? What aspects do we have to consider when we want to analyze multivariate time series, right? What are the interrelationships between, let's say, any two series, and so on and so forth, right? And, of course, down the line, we will have a half-hour session that would be practical, right? So, first, motivation.

So, why should we study or why should there at all be an idea of understanding or analyzing multivariate processes? So, a couple of points here. So, multivariate processes are also called vector processes. So, you have two names. So, either multivariate series or vector series emerge when several related time series processes are observed simultaneously over time.

So, maybe I will pause here for a second just to give you a very quick idea or example. Of course, you will see similar examples later on, but just to start the motivation idea. So, let us say you have y t and at the same time you have x t, right? And you assume that there should be some relationship between x t and y t, but then x t and y t are completely different series, right? In what terms? So, you have y t, then y t plus 1, y t plus 2, etc. in the future. Similarly, you could have, let us say, x t, then x t plus 1, x t plus 2, etc. in the future, right?

Now, how do you analyze and understand the interrelationship between two series? So, let us say x t and y t, right? This is the idea of putting forward the multivariate processes idea. So, in short or in other words, how do you model these two series simultaneously or, let us say, together is the whole idea. And then, the second point in the motivation slide is that one might be interested in investigating the cross-relationships between the series.

So, for example, how is, let us say, Xt plus 1 related to, let us say, Yt plus 2, or how, let us say, Yt plus 1 might be related to, let us say, Xt plus 2, or Yt plus 1 might be related to simply Xt plus 1, and so on. So, all these are cross-relationships among the lags also. So, if such a thing exists, it will be very suitable to study them together or put forward a model together, which is sort of combined on both Xt and Yt. So, this is exactly where the idea of multivariate time series analysis or multivariate time series processes emerges. Now, the objectives.

The objectives for jointly analyzing and modeling the series are as follows. So, you have two objectives here, and of course, there might be many more, but just the important ones may have been listed down. So, the first one is to understand the dynamic relationships of the series over time. Now, again, a couple of points to note here is that dynamic relationship. So, dynamic relationship means how the relationships evolve over time for both the series.

So, let us say yt, yt plus 1, yt plus 2, yt plus 3, and at the same time, xt, xt plus 1, xt plus 2, xt plus 3, etc. So, this sort of relationship is called the dynamic relationships of the series over time. And the second point is to improve the accuracy of forecasts for individual series by utilizing the extra information or the additional information available from the related series. Now, again, the second point is also important: let us say your goal is to forecast YT down the line. So, YT is the series in focus, and then your goal is to forecast yt, let us say yt plus 10 or yt plus 11, yt plus 12, whatever, of course, depending on all the historical time points right up to, let us say, yt plus 9.

But let us say the focus is to forecast yt, and then you have another similar series, let us say xt. So, how can you make use of xt and its values to forecast this yt? So, such cross-relationships, when it comes to the lags or even for the forecast of individual series, then we have to study some multivariate time series processes. So, we will put forward some examples right away. So, the first example could be in finance. So, let us say price movements in one market can spread easily and instantly to another market.

So, let us say you have two markets, both of them are separate, and then whatever movements you observe in the first market, those could spread or could impact the movements in the other market. For this reason, financial markets are highly dependent on each other. So, I'll give you a very simple example. Let's say these days people trade in cryptocurrency. Again, I'm not saying that one should trade in cryptocurrency.

A small disclaimer: I'm not for or against trading in cryptocurrency, but just a small example that the trading activity in general has risen. And if you are, let's say, trading in two cryptocurrencies—Bitcoin and Ethereum—one can actually assume that there exists some correlation between Bitcoin and Ethereum. So, whenever the Bitcoin price rises, the Ethereum price also rises, and vice versa. So, one of the assets there, which is, let's say, Bitcoin, and its price is affecting the price of the other, which is Ethereum. And in that case, both these financial assets or financial markets, in other terms, are highly dependent on each other.

So, we have to consider them jointly to better understand the dynamic structure of the global market. So now, if you assume that these two are the only currencies that one can trade—of course, this is a hypothetical situation. So, let's say you only have two currencies: Bitcoin and Ethereum. And then, Bitcoin and Ethereum prices—they have some dependency between them. So, why not put forward a model which comprises both Bitcoin and Ethereum?

So, why not put forward a bivariate model structure for both Bitcoin prices and Ethereum prices? And why exactly? To better understand the dynamic structure of the overall market. So, the overall market consists of these two, right? So, if you put forward a bivariate model rather than modeling them individually, you can actually understand the intricacies and dependencies between the two assets, okay?

So, knowing how markets are interrelated is of great importance in finance, of course, right? And one more point: if you have an investor—or a financial institution—holding multiple assets plays an important role in decision-making. So, these are some use cases or examples where one might go for an extension, like bivariate modeling or multivariate modeling, and so on. Now, a second example would come from economics. So, let's say one may be interested in the simultaneous behavior of interest rates, inflation, money supply, unemployment rates, etc.

So, the focus may be on the simultaneous study of time series of, let us say, GDP, the percentage of people below the poverty line, unemployment rates, female-headed households, crime rates, average income, minimum wages, etc. So, all these are economic examples, and then one might be interested in studying the interrelationships between any two of these. So, let us say, for example, the unemployment rate and then the average income. So, if you want to put forward a bivariate model comprising both the unemployment rate and average income. So, for that, we require some multivariate time series analysis tools.

So, hopefully, I have motivated all of you enough as to why you should transition from analyzing series individually, which is nothing but univariate, to something like bivariate, trivariate, or, in general, multivariate. Okay, so a couple of last examples: the first one comes from environmental sciences and agriculture. So, it is a joint study of time series observations of maximum and minimum temperatures, rainfall, or, in general, some atmospheric humidity, wind speed, direction, etc. And let us say the total production of wheat. So here you have two series.

The first one could be, let us say, maximum and minimum temperatures, rainfall, humidity, or wind speed and direction. And the second series could come from agriculture. So, let us say, the total production of wheat. Now, of course, one can expect that the total production of wheat in a particular month or year also depends heavily upon other environmental aspects. So, rainfall, temperature, humidity, wind speed, direction, etc.

So one may assume that the production of wheat, which happens, has to be highly correlated with some other environmental aspects. So putting forward a bivariate time series model comprising both one of these aspects above and, let us say, the production of wheat makes even more sense. And one small example from, let us say, health and environment-related studies is something like this. So, a joint study of air pollution levels or the number of asthma patients visiting hospitals and, let us say, the number of registered cars in a city, etc. So, one can actually find out a lot of interconnections between two variables to sort of put forward a combined model, which is a multivariate time series model.

Okay, so now we will quickly go over some practical examples. So here you see some European stock markets or European stock indices between 1991 and 1998. And then here you see a small index. So each and every color sort of points to a different index. For example, FTSE is the UK index.

Then DAX is the German index. Then SMI is the Swiss index. And then CAC is the French index. So, France, Germany, the UK, and then Switzerland. But then here the whole point is, even if you are plotting them on a single plot, can you somehow find out some interrelationships between any two series or, let us say, any three indices or, for that matter, all of these indices?

And hence, we can actually go ahead and try putting forward a multivariate time series structure to capture the dependencies between all these indices. One more example. So, let us say you have two series. The first one is Nottingham's average monthly temperatures, and then you have heating demand. So, the red curve gives you the heating demand, and the blue curve gives you the temperature data.

So, is there any relationship between the temperature data and the heating demand? Now, of course, ideally, there should be. So, why not study them together? So, whenever you have more heating demand or less heating demand, what is the effect on the temperature, or rather vice versa? So, when the temperature is higher, what about the corresponding heating demand? Or when the temperature is lower, what about the corresponding heating demand, and so on.

And probably one last example is, let us say, US economic time series. So, here you have three series. So, population, unemployment rate, and savings rate. So, this middle one, which is given in green, is called the personal savings rate. So, is there any relationship between the population, the savings rate, and the unemployment?

This again can be put forward in a trivariate kind of a modeling scenario and then we can understand them better, all right. So, again the idea is that these are some practical examples to tell you that under what situations probably analyzing them separately might not be feasible or might not be useful. So, as to one should go with some let us say bivariate modeling or trivariate modeling or multivariate modeling, all right. Okay, so these could be some potentially interesting questions that one can ask, right? I mean based on some of the earlier graphs and then in general some of the interdependencies between multiple series, then these are some potentially interesting questions that one can ask.

So, firstly let us say do some indices have a tendency to lead others? I mean, so what do you mean by that? So, let us say if you have two series. Then does one of the series sort of leads the other. So information which is passing in the first series is sort of ahead of the information which is there in the second series.

So does the first series affects the second series in a positive way or sort of in a directional manner. So such questions could be answered. So the second one could be are there feedbacks between the indices? Does the information in the first series has anything to do with the movement in the second series and vice versa? Third question could be something like how do impulses, impulses mean shocks etc. transfer from one index to the other.

So, again if you go back to the European indices example that we saw earlier. So, let us say you want to find out if the FTSE index has anything to do with the DAX index or is there any relationship between FTSE and DAX indices or rather if you see any shock in the UK market. So, does that or will that have any impact in the German market? So, these are some questions that one can try to answer. Last one could be how about common factors such as disturbances, yields, trends, risks, etc.

So, is there any common disturbance factor or any common shock which is there in both the UK market as well as the German market? And so on and so forth. So such questions could be easily answered through an empirical investigation using tools developed in a multivariate framework. Make sense? Okay.

So one very small disclaimer that before we sort of enter this week, so there will be lot of theory which would be coming in. And why exactly? Because one can imagine that if you want a univariate distribution, things are slightly easier. The moment you transition into bivariate, let us say bivariate normal, bivariate exponential, whichever distribution you take, there would be something which would be slightly advanced. So, I will try to give you some refreshers.

For example, the refresher that you see in front of you is a small refresher on matrix algebra. So, we will be using some matrix algebra here, right? And again, do not worry. So, do not be afraid. So, I will try to explain each and every point wherever, let us say, some high-fi matrix comes in or some advanced techniques come in, right?

Then I will try to explain them in a bit more detail, right? So, a small reflection on matrix algebra, and then we will see from the next session onwards how to put forward all these ideas and try to model using a multivariate framework. Now again, I will probably suggest that if you don't like matrix algebra or if you don't know much about matrix algebra, then pick up any basic textbook or even Google the basic ideas about, let's say, the idea of some basic matrices—I mean, what do you mean by inverse, what do you mean by determinant, let's say. Or, let us say, what do you mean by the trace of a matrix, or how do you find out the inverse, or what do you mean by a symmetric matrix, skew-symmetric matrix. So, very basic knowledge about, let us say, some matrices would be slightly beneficial to understand this particular week.

For example, eigenvalues and eigenvectors. So, where exactly can we make use of eigenvalues and eigenvectors of a matrix when we study the multivariate time series processes framework? But just a very quick refresher about eigenvalues and eigenvectors. So, let us say you have A, which is a matrix. So, A is a matrix which is n by n. So, A is a square matrix of order n by n.

$$f(\lambda) = |A - \lambda I_n|$$

And then we focus on this polynomial. So, F of lambda equals the determinant of A minus lambda into I n. Now, firstly, what is I n? So, I n is nothing but the identity matrix, right, of order n. So, the identity matrix of order n. Now, again, the identity matrix—if you are not very comfortable with it, go back and try to study it. So, the identity matrix is nothing but a matrix containing 1s on the entire diagonal and 0s on the off-diagonal, right?

And the order of I n is nothing but n cross n also. So, for example, this would be a 2 cross 2 identity matrix, right? Or this would be a 3 cross 3 identity matrix, etc. So, in general, I n is nothing but an n cross n identity matrix. And using that identity matrix and A, we are creating this polynomial.

So, f of lambda is the determinant of A minus lambda into I n. It is a polynomial of order n in lambda, of course, right? And lambda 1, lambda 2, lambda 3 up to lambda n are the n roots of this equation, is it? So, if you form this equation. So, the determinant of A minus lambda I n equals 0, then naturally lambda 1, lambda 2 up to lambda n would be nothing but the n roots of that equation. And hence, A minus lambda I n is a singular matrix because this determinant is 0, right?

$$(A - \lambda_i I_n) q_i = 0 \quad \text{or} \quad A q_i = \lambda_i q_i$$

So, A minus lambda I N is a singular matrix. And thus, there exists a vector, let us say q_i, such that this should hold true. Now again, I am not saying that you should go back and try looking at the proofs. I mean, one can actually prove all these things, but the proof is not the matter of fact here, or proofs are not really important here. I mean, of course, if you are more interested, then go back and see as to why there should be such a vector q_i and all these things.

I mean, this sort of depends on how much time you can spend outside the class and then try to refresh, let us say, some very basic matrix algebra. But again, coming back to just summarize this, A minus lambda I N should be a singular matrix. And thus, there should exist a vector q_i such that this equation should hold true. And going from this step to this step is not difficult. So, one can actually open the brackets, multiply by q_i, and then take one of the items on the right side.

$$AQ = (\lambda_1 q_1, \lambda_2 q_2, \ldots, \lambda_n q_n) = Q\Lambda$$

Where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Thus,

$$Q^{-1} A Q = \Lambda$$

$$|A| = \prod \lambda_i, \; trace(A) = \sum \lambda_i \,, \; A^m = Q \Lambda^m Q^{-1}$$

So, A into q_i equals lambda_i into q_i. So, this is a famous equation which comprises eigenvalues as well as eigenvectors. So, lambda_i, i going from 1 to n, are the n eigenvalues of the matrix. And then q_i, i going from 1 to n, are the eigenvectors of the matrix A. So, these q_i's are the eigenvectors, and then lambda_i's are nothing but the n eigenvalues of this matrix A. So, hopefully, the idea of eigenvalues and eigenvectors would be slightly easier to understand now that I have sort of given you a refresher on these ideas, hopefully.

So, going forward, again, let A be the same, let us say, n cross n square matrix. And let us say q1, q2, q3 up to qn be the n linearly independent eigenvectors corresponding to eigenvalues lambda 1 up to lambda n. Now, here, one small mistake is there: q1, q2, qn are n linearly independent eigenvectors. So, this should be eigenvectors here, right, because qi's are the eigenvectors, isn't it? So, qi's are the eigenvectors corresponding to eigenvalues. So, these lambdas are the eigenvalues.

So, corresponding to each eigenvalue lambda 1, lambda 2 up to lambda n, right, we are putting forward some linearly independent corresponding eigenvectors, which are q1, q2 up to qn, right. Now, let us denote this collection of all the eigenvectors as capital Q. So, capital Q denotes the collection of all these eigenvectors. Then what happens is that If you pre-multiply this Q with the matrix A, so AQ, then what would happen? So, for this, you have to go back a slide, which is right here.

And then how do you make use of this expression? Because you know that Aqi equals lambda iqi for every i, right? A into the eigenvector should be equal to the eigenvalue into the corresponding eigenvector. So, A into qi should equal lambda i into qi for all i, of course, this should be true for all i. Hence, cannot you write down something like this?

So, the matrix AQ would be nothing but equal to the eigenvalues lambda 1, lambda 2 up to lambda n into the corresponding eigenvectors. So, q1, q2, qn, right? So, AQ would be nothing but this collection. So, again, just to summarize, we started with this capital Q containing all the eigenvectors. And then A into that collection, AQ, would be nothing but the corresponding eigenvalues into the eigenvectors.

So, lambda 1 q1, lambda 2 q2 up to lambda n qn, and for this, we are giving a different notation. So, Q is still there because you still have the eigenvectors here, and then this is nothing but capital lambda. So, this notation is again a Greek notation which denotes lambda again. So, AQ should be equal to Q into this lambda. Where lambda is nothing but a diagonal matrix containing all the eigenvalues in the diagonal, right?

And then off-diagonals are zeros. Make sense? So, this lambda structure is like what? So, this lambda matrix is nothing but lambda 1, lambda 2 up to lambda n, and then zeros everywhere else, okay? So, this is a diagonal matrix containing the eigenvalues in the diagonal and all other values are zeros, right?

And then, using this equation or this expression that you see here, what would happen if you pre-multiply by Q inverse? So, Q inverse AQ would be nothing but Q inverse Q lambda, isn't it? If you pre-multiply by Q inverse on both sides, then Q inverse AQ equals Q inverse Q lambda. And, of course, Q inverse Q would be the identity. So, you are left with lambda.

So, this is a famous equation. So, Q inverse AQ equals lambda, where Q consists of all the eigenvectors, A is the underlying matrix, and lambda contains all the eigenvalues. And these are some of the properties that one can actually talk about. So, for example, the determinant of A is nothing but the product of all the eigenvalues. The trace of A is nothing but the summation of all the eigenvalues.

And if you have something like this, such as A to the power m, then this equation should hold true. That A to the power m is nothing but Q into lambda to the power m into Q inverse. So, all these are some elementary ideas in matrix algebra. So, if you pick a very basic book on matrix algebra or, again, let us say some elementary course notes on linear algebra or matrix algebra, then all these things would be there. So, so far, we have studied eigenvalues, eigenvectors, and then the relationship between, let us say, eigenvalues and eigenvectors, and then going a step ahead.

What do you mean by, let us say, this capital Q, or what would happen if you multiply capital Q by A, right? So, that is nothing but Q into lambda, where lambda is a diagonal matrix containing all the eigenvalues, and hence, eventually, we can focus on such an equation. So, Q inverse AQ is equal to lambda. And these are some properties that one can actually prove: the determinant equals the product of all the eigenvalues, the trace of the matrix equals the summation of all the eigenvalues, and so on. So, now what we will do is transition into the idea of a random vector, and then we will discuss some properties of a random vector.

So, firstly, you should understand what you mean by a random vector. So, let x be a p cross 1 random vector. So, a random vector means nothing but something like this. So, let us say you have x1, x2 up to x, let us say phi. So, x1, x2 up to xp.

$$E(X_i) = \mu_i \ \forall \ i = 1,2,\dots,p:$$

So, this would be a random vector. So, a random vector means a vector containing several random variables. So, each one of these items is nothing but a random variable. So, x1, x2 up to xp. So, let x be a p cross 1 random vector, and these are some properties of the random vector that one can actually follow or one actually has.

$$E(X_i - \mu_i)^2 \ = \sigma_{ii}:$$

So, expectation of xi. So, if you take the expected value of any of the elements of the random vector, which is nothing but equal to mu_i for all i. So, this is given by or this denotes the mean of the corresponding xi, right. Or further, if you have something like expectation of xi minus the corresponding mean whole square, right. So, expectation of xi minus mu_i whole square, this gives you the variance. So, sigma_ii.

$$E(X_i - \mu_i)(X_j - \mu_j) = \sigma_{ij}$$

Or if you have something like expectation of xi minus mu_i into xj minus mu_j, this is denoted by sigma_ij, which is nothing but the covariance between xi and xj. Now from here going forward, one should understand that the idea of a random vector would be very important because—and why exactly? Because you are trying to analyze multivariate time series processes. So which are more than one, isn't it? So, like we took the example earlier also.

So on one hand, you could have yt and xt. So how do you sort of combine these two individual time series processes in a vector, which is nothing but a random vector, right? Because each one of these—so xt and yt are random variables in themselves. So, if you combine them in a vector form, it is called a random vector. And then on top of that, I can talk about expectation of individuals.

So, E of y t or E of x t, or variance of y t, variance of x t, right? And one last thing we can talk about here is the covariance between x t and y t. So, all such things could be studied when we have some idea about a random vector, okay?

Thank you.