Time Series Modelling and Forecasting with Applications in R

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Week 11

Lecture 51: Nonlinear Time Series Models

Thank you. Hello all, welcome to this course on time series modeling and forecasting using R. Now, we are sitting in a fresh week and then again just to give you a very quick refresher as to what was covered the last week. So, that there should not be any disconnect between let us say last week and this week. So, of course, this week is entirely different topic, but again just to make sure that all of you are through with the last week right. So, last week was predominantly based on let us say modeling volatility or modeling changing variance in the underlying time series and there we sort of saw multiple different models let us say ARCH model or GARCH model their extensions like GJR GARCH, EGARCH, threshold GARCH, asymmetric power GARCH etcetera.

And down the line we talked about ARMA plus GARCH models right and in the very last session or rather the last session of last week which was the practical week we try to incorporate all the theory we study in the last week and then try to implement them on a practical sort of a framework right. And there if you remember we initially simulated from all the underlying volatility models like ARCH 1 or GARCH 1 1 GARCH 2 1 etcetera or ARMA 1 2 plus GARCH 1 1 right. And then down the line we analyzed one real data which was NYSC returns data. And one small homework I gave you in the last week is that if somebody is more keen in exploring about volatility models or modeling in general, then there was one other real data set which we did not have time to cover in the last session, which was Bitcoin data. So hopefully before you sort of start with watching this week's videos again a strong suggestion is that again you go back and try to complete that code just to understand that how do you sort of place all the visual graphs and charts and then how do you gather some information from let us say the actual model structure right

Then let us say the model implemented on the actual data then the forecast right so all these things you have to be completely sure of all right. Okay, so now proceeding ahead, so the topic for this week, this entire week would be revolving around this idea of non-linear time series processes. Okay, so far all the processes we have seen are pertaining to linearity or rather linear time series processes because linear means that You have sort of a slightly simplified sort of a structure. So, we started with ARMA or rather AR models, MA models, then ARMA, then ARIMA, SARIMA and lastly ARCH, GARCH.

So, all these models have a mean structure which is inbuilt in the model itself right and that mean structure could be either arma or arema etc and both arma and arema are kind of linear time series processes in a way as it sort of captures the linearity aspect so linearity means some sort of trend in the data or seasonality in the data right so these are some characteristics which are sort of corrected or rather modeled by arema kind of a model. OK, but as and how you have a slightly complex kind of a model structure or other real data structure, a linear time series model won't do the job. So for that matter, a non-linear time series process would be slightly more useful. So in the entire first session or other first or second session of this week, since the entire area of non-linear time series processes is slightly different, we will try to focus more on how exactly can you sort of differentiate between non-linear processes and linear processes and under what scenarios can you implement the non-linear time series processes rather than linear, etc. One more very important disclaimer is that if you are sitting in any time series lecture or rather a course, so I am not sure how many of you have done a time series course before this, but if you sat through and sort of completed the entire time series course in some of the earlier years of your education, then I am pretty sure that nobody touches upon the idea of non-linear time series processes.

And again, why exactly? Because understanding the nitty gritties or understanding the technicalities is not that easy when it comes to non-linear time series processes. So people do, of course, all the linear processes like ARMA, ARIMA, SARIMA right obviously they do include ARCH GARCH and also obviously there would be or there might be one chapter on spectral decomposition as we had earlier, fourier transformation all these ideas but covering non-linear time series processes is not a touched upon topic very much so again I am not saying that all the courses do that but then again So, the idea of non-linear time series processes is not easy to sort of understand or sort of explain. So, people tend to skip this from their usual time series course ok.

So, just to give you a flavor of what kind of different processes there might be and where exactly the linearity sort of fails down and where exactly does one require the idea of studying the non-linearity in the time series or rather some non-linear processes is kind of important which we will see in this week. So, first couple of slides would be a very brief introduction. So, the non-linear time series analysis extends the traditional time series methods to account for systems where the relationship between variables cannot be adequately captured by linear models. So, again just to specify just to push on this point is that wherever a set of linear models sort of fail, There exactly we make use of some non-linear time series techniques or non-linear time series models.

So, again non-linear time series analysis extends some traditional time series methods to account for systems where the relationship between variables cannot be adequately captured by some linear models. And second point is that non-linear methods are crucial for modeling some complex phenomena in areas such as finance, climate science, biology, etc. So, all these would be sort of work around areas or other practical areas where non-linear techniques or non-linear methods are cyano crucial ok. So, secondly we will try to understand that what exactly are the characteristics behind this entire non-linear set of models ok. So, obviously the very first important point is non-linearity of course.

So, relationships between variables are not additive and may depend on interactions or certain thresholds ok. So, whenever it comes to additive models, we will try to assume that additive models are sort of linear in structure or they can be sort of transformed to a linear kind of structure. But whenever the relationships are not additive between the variables and may depend on some interaction. So, interaction means two variables are interacting which means that one of the variable affects the other in some sense. So, in a way they cannot be added or their effects cannot be added as both the variables are proportional.

So, in that case they have to be multiplied. So, such multiplications are called as interactions. So, if the relationship is not purely additive and may depend on some interactions or certain thresholds then non-linearity comes into picture. Secondly, obviously complex dynamics. So, these may include some chaos, bifurcations or periodic behavior right.

So, if you want to analyze some complex structures underneath the time series of the practical dataset, then obviously putting forward a linear kind of structure would not make sense. So, one has to shift from a linear structure to a nonlinear structure. So, again, just a small disclaimer is that the entire area of nonlinear time series is completely different from analyzing linear time series, right. As in, how you have to—so exactly, the emphasis is more on how you sort of make the model structure slightly more complex or rather capable of incorporating some complex dynamics of the time series. Thirdly, state dependence.

So, what do you mean by that? So, current values might influence some of the future values differently based on the state of the system. So, this is called state dependence. So, based on which state the current time series is running in. So, again, imagine that you have one practical time series broken down into different states.

And then again, all the states are not exactly the same. So, you have some differences between states. And as in how the time series transitions from one state to the other, you might require a slightly different model structure. And it also means that the current values might have the capability of influencing any of the future values differently based on the different states of the system. And obviously, the last important point is non-stationarity.

So, the statistical properties such as mean or variance may change over time. So, of course, all these are mere characteristics, as in when one might have to look out for some non-linear time series processes rather than linear structures. Make sense? So, hopefully by this time, I have given you abundant motivation as to why one has to shift from a linear structure or rather a linear model to something like a non-linear time series process. Alright, so again, just a slide to start with: can you discuss some examples of non-linear processes? And the answer is yes.

So, all these are examples of some very important non-linear processes in the literature. So, the first one is threshold models. So, what do you mean by that? So, a very useful or widely applicable model in this context is the threshold autoregressive, or in short, TAR model or T-A-R model. So, for example, the threshold autoregressive models where different dynamics operate above or below certain threshold values or certain threshold levels.

So, think of a situation where you have some threshold, and then if the time series kicks above the threshold. So, let us say you have this progression: whenever the time series hits above the threshold, you have a different model structure, and whenever the time series hits below the threshold, you have a different model structure. So, can you sort of put forward a threshold value or a threshold level such that different dynamics are operating on either side of the threshold? And in this context, a threshold autoregressive model, or rather in short, a TAR model, seems beneficial. Secondly, volatility models, which we have already seen in the last week.

So, again, just to repeat. So, generalized autoregressive conditional heteroscedastic kind of models, or in short, GARCH models, for modeling changing variance over time. So, again, all these are examples of non-linear processes. Thirdly, nonlinear dynamical systems. So, let us say deterministic chaos, right?

And this is a very specific kind of example in the literature. Let us say Lorenz attractor. But again, the idea is that one need not go into details here. So, we will skip the details as to what you mean by Lorenz attractor, etc. But then you should understand that there should be some deterministic chaos in the model.

So, deterministic chaos means some complex dynamics in the model or some abrupt changes in the model, etc. So, these are some of the examples when it comes to extending a linear time series process or a linear structure to a non-linear time series process. The next example could be from neural networks. So, let us say all these are examples. So, recurrent neural networks.

So, I am kind of sure that almost all of you, or rather some of you, might have some confidence in analyzing neural networks. So, let us say RNNs, CNNs, or let us say LSTMs—long short-term memory. So, all these examples are examples of neural networks. And again, neural networks are not linear models. I mean, there has to be some non-linearity in the underlying model.

So, let us say LSTMs, RNNs, or some transformer-based models for capturing complex dependencies, etc. And then next is polynomial and rational kinds of models. So, let us say some non-linear regression models exist in literature, right, where one actually does some transformation. Let us say take some squares or some cubes of the variables which are included. So, again, a small disclaimer when it comes to the last point here is that in the regression literature, you have some set of models which are transformed to a linear model, right, by taking square roots, taking squares, taking cubes, taking logs, etc., right.

But all these are kinds of examples of polynomial models or rational models. So, even if you have a polynomial structure, right, by transforming, I can convert it to a linear structure. Make sense? So, this is—I will say that this last point is sort of a subset of the non-linear processes. So, the model itself is not non-linear because one can actually transform it to make it linear, right.

So, the correct way to put it is that all these are examples of polynomial regression models or, rather, non-linear regression models which come under the polynomial regression model heading. Make sense? So, all these are some of the important examples when it comes to, let us say, non-linear processes in general. So, again, a small disclaimer here is that this is not, or rather, these are not examples of non-linear time processes or, rather, time series processes. So, the word 'time series' should not be included here.

So, all these are examples of some non-linear processes in general. So, let us say neural networks, polynomial regression models, or volatility models, ARCH-GARCH kind of models. So, all these are called non-linear processes in general. So, these examples should not be confused with whether they are time series models or not. So, these are some general non-linear processes that exist in the literature.

So, now we will spend some time explaining or getting into details about what we mean by threshold models. So, the very first and most important kind of model is called threshold models. So, threshold models are a class of non-linear time series models. Now, again, threshold models pertain to time series models. So, threshold models are a class of non-linear time series models where the dynamics change based on whether the process crosses certain thresholds or not.

So, again, as discussed a short while back. So, let us say, again, imagine that you have some threshold here—some threshold value, right? And whenever the time series crosses the threshold, the dynamics change. So, it might happen that there might be some highly volatile nature, right? And when the time series again comes down the threshold or crosses the threshold one more time, there could be some calmness in the time series. Again, this is just one hypothetical example. So, the idea is that on either side of the threshold, different dynamics exist.

So, just to summarize again, the threshold models are a class of non-linear time series models where the dynamics change based on whether the process crosses certain thresholds or not. And secondly, they are particularly useful in capturing abrupt changes or regime shifts in the data, okay? So, whenever you have some regime shifts or whenever you have some abrupt changes in the dataset, right? So, they are particularly useful in capturing some abrupt changes in the data or regime shifts in the data. So, again, do not worry; we will spend a whole lot of time understanding what you mean by regimes, what you mean by regime changes or regime shifts in the model, etc., okay?

So, probably, this is exactly where we spend some time understanding what you mean by regimes first. But again, the heading is 'Key Concepts in Threshold Models.' So, we will again go deeper into understanding what you mean by threshold models, understanding the different ideas or different characteristics of threshold models, such as regimes, non-linearity, threshold variable, etc. So, the first ingredient is regime. So, what do you mean by regime?

So the time series operates under different regimes with distinct dynamics for each regime. Now, again just to simply put think of a regime being a state in a time series process where different dynamics come into picture. So, again think of a the complete time series or complete practical data which is broken down into these different states. And then these different states are called as regimes from a time series literature or time series context. And then these different regimes have different dynamics among themselves.

So two regimes need not be exactly same. So there has to be some dynamic change when you shift from a regime to another regime. So this is a small example, a small definition of what you mean by regimes. So the regime is determined by threshold variable, which may be the series itself or another related variable. So, this tells you as to how do you determine a regime.

So, regime is determined by a threshold variable. There has to be some threshold variable and then that threshold variable is nothing but that thin line between any two states. Again just to give you one visual example or one visual appreciation. So, let us say you sort of divide the time series into different states. So, these might be the states or rather these might be the regimes.

And then these vertical lines are nothing but the thresholds. So, think that these vertical lines are nothing but the thresholds everywhere. And the behavior of the time series would have different dynamics in each regime. So, each subset here is one regime. So, overall you have four or rather five different regimes.

And as in how a time series crosses the thresholds, let us say something like this, the dynamics might change. So, in this regime, it might happen that the time series is highly volatile. But again as it crosses a different threshold it might happen that again the time series is calm. Again it might happen that in the next regime the time series is highly volatile etc. So again just to summarize just to repeat one last time is that regimes are nothing but different states of the underlying time series where different dynamics or different dynamical systems are persistent.

Now, the next thing is threshold variable. So, this is that variable whose value determines the regime. So, again as discussed before, so all these vertical lines are nothing but the thresholds, let us say. And wherever you have a threshold and the moment the time series crosses that threshold, the dynamics change. So, threshold variable is nothing but that variable whose value determines the regime.

Often, it is a lagged value of the time series itself. So, often the threshold value is nothing but a lagged value of the time series itself. So, let us say if you have y t, then y t minus 1 would be a threshold variable or let us say y t minus 2 might be a threshold variable. So, as in how the time series sort of progresses or changes from y t minus 2 to y t minus 1 to y t, one can actually see different dynamics in the underlying time series. And next is non-linearity.

So these models exhibit piecewise linear behavior, making them interpretable while still capturing the non-linearity. So the threshold model is the idea is that even if you break down to different regimes, so inside each regime, it might happen that the model is linear. But overall, obviously, the model is non-linear. But then hence inside each regime if the model is assumed to be linear by making some transformations or by the view of it or by the virtue of it. So being linear inside each of the regimes make them interpretable while still capturing the overall non-linearity.

So, hopefully this slide is kind of understood now that what do you mean by regimes, what do you mean by threshold variables and how does the idea of non-linearity or other linearity sort of ties all these things down. More on regime. So, regimes and time series analysis refer to different phases or distinct phases or states of a system, each governed by different underlying dynamics. So, again I think the first point has been discussed number of times, ok. So, if you split a time series into different states or different phases, where each state or each phase would be called as a regime and then each regime is assumed to have some different underlying dynamics, ok.

So, in threshold models, these regimes are explicitly defined based on the crossing of thresholds by a variable, often the time series itself. So, again, I think that we need not spend much time on this slide because both points have been discussed a number of times: firstly, what do you mean by regimes, and then what do you mean by threshold values or threshold models. So, again, just to summarize one last time, whenever a time series crosses a threshold value or a threshold level, the time series enters into a different regime, which means that there is a dynamic shift or some shift in the dynamics of the underlying time series. Alright, so now we will try to understand what exactly are some of the regime characteristics, right. So, the first one is distinct dynamics.

What do you mean by that? So, each regime has its own set of parameters and equations that describe the behavior of the system. So, each regime is sort of distinct, and each regime has distinct dynamics, okay. What do you mean by that is that each regime has its own set of parameters and equations which describe the behavior of the overall system. For example, economic growth rates might follow one pattern during expansion and a different one during recession.

So, again, this is a very easy-to-understand sort of example where, let us say, if you are talking about economic growth rates, how many regimes are we talking about? We are talking about two regimes. So, the first regime would be expansion, the second regime would be recession, and obviously, expansion and recession are completely different. So, they have different dynamics altogether, right. So, for example, economic growth rates might follow one pattern during one of the regimes, which is expansion, and a different pattern in the other regime, which is recession.

Secondly, transition mechanisms. So, what do you mean by transition mechanisms? So, regimes change based on specific rules or conditions, such as crossing a threshold or a probabilistic switching mechanism. So, we have to understand how the time series switches between regimes. So, often—or rather almost every time—regime changes are based on specific rules, characteristics, or conditions, such as crossing some threshold value or more from a probabilistic switching mechanism.

So, we will try to understand what you mean by this, probably in the coming slides or maybe in the lectures this week. But then again, hopefully, crossing a threshold should be understood as of now. So, we spend quite some time understanding how the regime switches happen. So, the time series switches from one regime to another based on some, let us say, threshold values, or it could happen based on some probabilistic switching mechanism in place. Next is temporal persistence.

So, regimes often persist for a period before transitioning, leading to clustering of similar states. So, what do you mean by that? So, I will give you an example. So, let us say this is the time series. I will break this entire time series into two different regimes. So, this is the threshold value, let us say, and this is the behavior of the time series here.

So, let us say the time series behaves like this, and then the moment it crosses the threshold, you can see that the time series behaves in a highly volatile sort of nature, okay. But again, if you see or if you sort of look inside each of the regimes. So, let us say this is regime 1, R1, and this is my regime 2, which is R2. So, overall, you have one threshold value, which is this—this is T1—and then two regimes on either side of the threshold. So, R1 and R2.

Now, if you focus your attention on either of the regimes, let us say R1 for that matter. So, if you focus your attention on R1, the time series behavior has some persistence within each of the regimes. What do you mean by that? So, regimes often persist for a period before transitioning. So, before you see this transition, you see that the time series is sort of behaving in a persistent kind of manner.

So, persistence means that there is some long-memory property, right? Long memory means that there is some sort of behavior which repeats itself. Over a while, basically. And the same characteristic can be seen in the second regime as well. So, if you shift your attention to R2 here, since the time series is highly volatile in R2, the same sort of—or rather, the exact same kind of characteristic—can be seen throughout that entire regime.

Does that make sense so far? So, again, I can give you a counterexample. So, a counterexample could be, let us say, again, I will redraw the exact same sort of graph where you have one threshold value, two regimes, right? And it might happen that the first R1 is exactly the same as what we had earlier, but the moment the time series transitions to R2, it is highly volatile initially, but later on, the time series is not that volatile. And this might not be the case. This might not be the case.

And why? Because inside any of the regimes, the time series should be persistent. So, if it is following this highly volatile nature, it should be persistent over a while. Hopefully, this is clear. So, this is one of the regime characteristics.

And the second characteristic is non-linearity. So, the overall system may appear to be non-linear because of abrupt or smooth transitions between the regimes. Now, again, we will quickly try to understand what exactly the types of regime transitions are. So, the first one is called abrupt transitions or discrete switching. So, changes between regimes are instantaneous when a threshold condition is met.

So, whenever the time series hits a threshold, then it sort of switches the regimes instantaneously. So, this is called as abrupt transition. So, transitioning from one regime to another. And some of the model examples in this context are threshold autoregressive or TAR or self-exciting TAR or SE TAR. So these are some example models under the abrupt transition kind of a framework.

And some of the use cases could be economic recessions and recoveries, sudden market crashes or booms, etc., Now, the other kind of a transition could be smoother transitions or called a smooth transition. So, changes between regimes occur gradually over a range of values of the threshold variable. So, here the difference is that the changes are not abrupt as we saw in the earlier slide, right. So, you have a large difference between abrupt transitions and then smooth transitions.

So, what are the examples here? So, you have the star kind of model. So, smooth transition autoregressive models or in short star or logistic or exponential transition function. So, these are some examples. And use cases could be gradual policy shifts.

For example, changes in monetary policy effects or transitioning between weather patterns, etc. And the last kind of transition would be probabilistic transitions, where regime transitions are governed by some probability. or other probabilities, often modeled as some latent variables. For example, a Markov switching model. So, again, the Markov switching model's transition is neither abrupt nor smooth, but all these transitions are governed by some external probabilities, where you have some external random variables in play or so; all these are called latent variables.

And one very useful model in this context is called the Markov switching autoregressive, or in short, MSAR models. A couple of use cases could be stock market volatility clustering or hidden states in biological systems, etc. Alright, so again, just to understand where we are sitting at the end of the first session of this week: we tried to delve deeper into this entire idea about non-linear time series processes, understanding the characteristics of those, understanding what you mean by regimes, and how regime switches happen, right? And what are the three different kinds of regime transitions in place? So, abrupt transitions, smooth transitions, and probabilistic transitions.

Okay, thank you.