Computational Geometry

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Module No. # 08
Voronoi Diagram and Delaunay Triangulation
Lecture No. # 01
Voronoi Diagram: Properties

All right. So, last time I talked about Voronoi diagram and then introduced Delaunay triangulation; discussed various properties of both of them; today, I am going to talk about computing them. So, I will begin with Voronoi diagram and show an algorithm to compute them and then, I will say what it means in terms of Delaunay Triangulation. So, let us begin.

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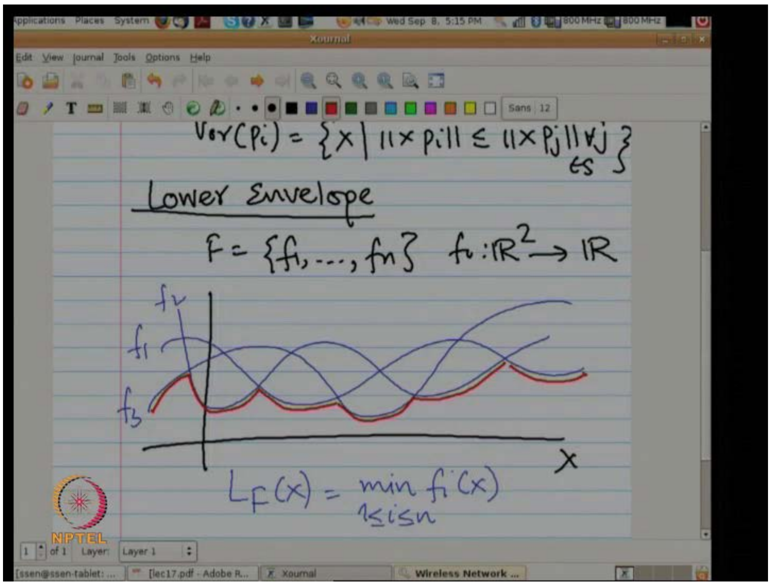

If you remember that you are given a set of points in 2D and the definition of; so, that was the definition of Voronoi cell. Before I describe the algorithm, I need to introduce a concept and what I will do is I will reduce computing the Voronoi diagram in 2D to computing the convex hull of a set of points in 3D and that is what I will do; and then I will say, well although you have not seen in the class, but you will see it later, how you compute the convex hull of a set of points in 3D. And, this reduction I will do works with any dimension and that will also describe or explain what I said about the complexity of the Voronoi diagrams in higher dimensions.

So, concept I need is what is called lower envelope and I will talk about lower envelopes more in next week. But, suppose you are given a set of functions; for our application, let us assume each function is a bivariate function, but it can be again any number of variables you can have.

It is hard for me to draw pictures in 3D. So, I will draw for univariate functions. So, let us say that this is the function f 1, this is f 2, so, this is f 3 or let me just keep only 3. Then, the lower envelope, which I will denote as L F of x, that is minimum. So, what I am doing is I am taking a point wise minimum of these functions. So, in particular, in this example what it means is that, I just trace the lower boundary; so, this red curve that you see, that is the graph of the lower envelope because I am saying the pointwise minimum. But, you notice that if you trace the lower envelope, then let us see if I can use this technology; never mind.

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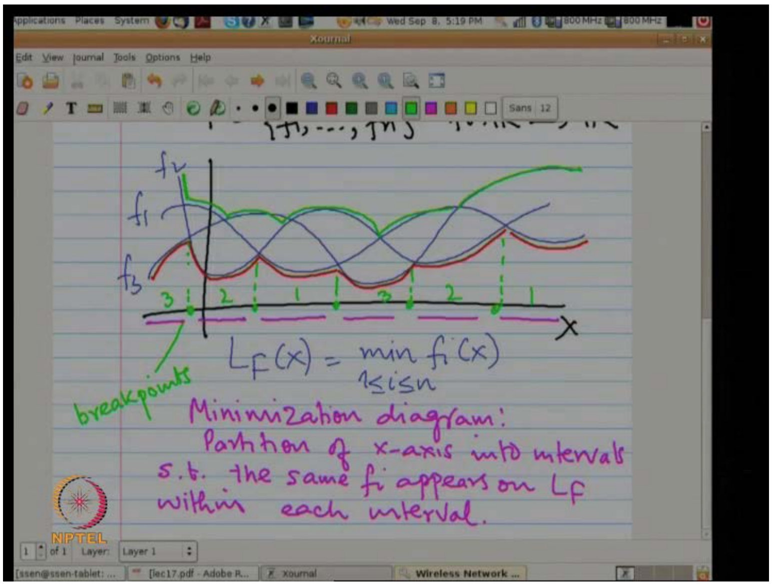

So, in the beginning, left side you have f 3 that is showing on the lower envelope. So, here you have 3; then function 2 shows up on the envelope; then the function 1 shows on the lower envelope; then again 3; then 2; then 1. So, these are called break points. So, if you trace the lower envelope, what is happening is one function is showing on the envelope, then there is a switch from one function to another function; and some other functions shows on the envelope and so on.

So, what happens is that if you look at the; since I have drawn the univariate function, what this lower envelope does is, it partitions the real axis or the x axis into intervals, so that the same function appears on the envelope within each interval. So, you have; this is one interval, this is another interval. So, you have said real axis partition into intervals; so, the same function appears on the envelope. So, this partition of the real axis into these intervals; that is called minimization diagram.

It will become a clear eliminate what has this to do with the Voronoi diagram. Thinking a little ahead, the way I defined the lower envelope, one can also define what is called the upper envelope of a set of functions. So, if I; let me choose my another favorite color; that one show, that is what I am thinking; let us see; let me do green. I think this is fine. So, that is the point was maximum. I will not write it down. So, if lower envelope point was minimum, you were tracing the lower envelope and the upper envelope is pointwise maximum of a set of functions.

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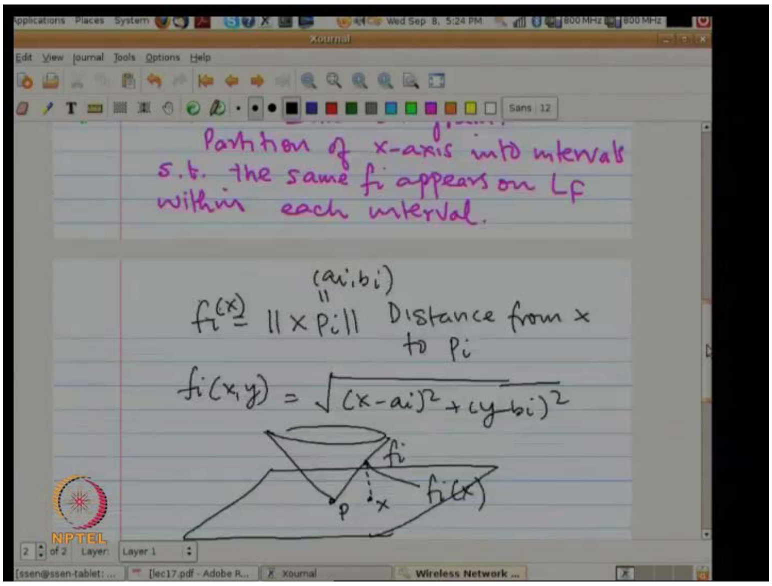

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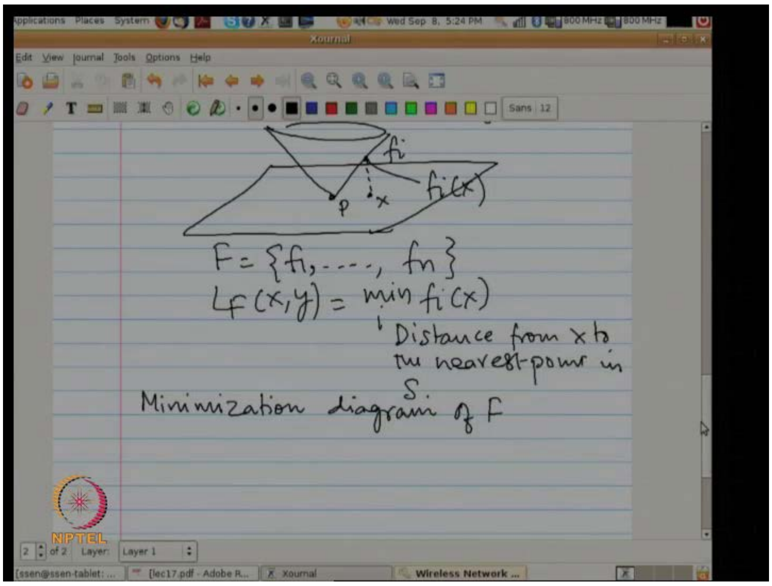

Now, why I need this? what I will do is, I will define a function f i; is a distance from x to P i. So, what is this function look like, if you look at the graph of this function?

I did not teach the inversion diagram yet; I am just asking that if you look at the function f i x, what is the shape of this function? So, it is a Euclidean distance you are talking about. So, if I draw this function what will it look like?

No; do not go to Voronoi cell yet. So, function f i, I have defined is a; if a given point x in the let us say, if this notion is confusing, think about f i x y and let us say P I; its coordinates are (a i, b i), then this is root x minus a i squared plus y minus () squared, right; that is the Euclidean distance, that is the distance function.

Go ahead. Pardon; no, it is not a sphere. Yeah, it is a cone, right; because sphere is a label set of this function. So, if I look at this function, that looks like. Well again; so, let me try to draw. So, if this is point P i, that is what the function look like this cone. So, this is function f i. So, if you took the point an x, this value is f i x.

Now, if I look at the lower envelope; now, you have these functions f i 1 for each point; now, you get the set of functions f n; you did this for each point. Now, I look at the lower envelope. So, this is minimum. So, what the lower envelope means? What is this function L, lower envelope of these functions mean? So, remember the lower envelope is a pointwise minimum, right. So, when I look at these particular functions, where f i is the distance from x to P i, then what is the lower envelope corresponds to?

It is the surface, but what does the point there it mean?

Yes. So, it is the distance from x to the nearest point, right. So, this is the distance from x to the nearest point it has and this is related to the Voronoi diagram; the way its defined is the minimum point. Now, if I look at the minimization diagram of F of these set of functions; now, here I have done in this figure. In this minimization diagram, I had shown was for 1 dimensional function, for univariate functions. If I want to do it for 2D functions, bivariate functions, then I just put it here and it looks.

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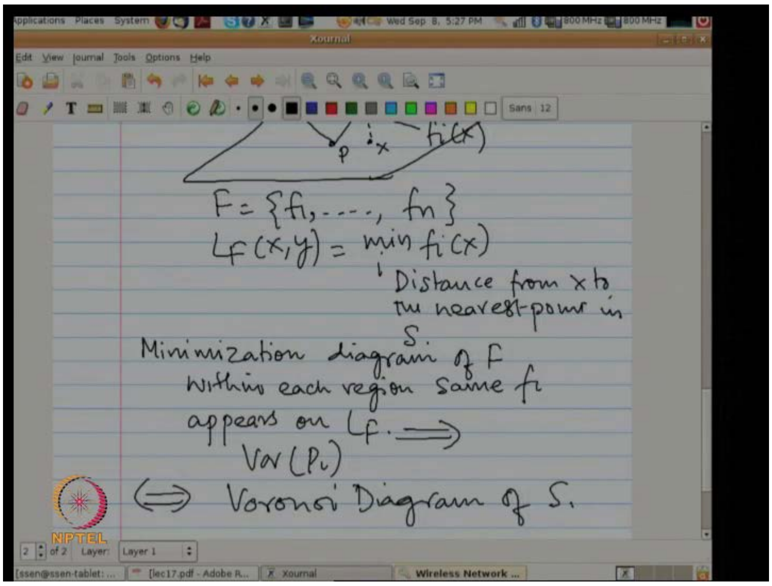

So, suppose say this is your xy plane and if you had the set of functions, you do the lower envelope; now, what you will get is, you get some partition of the plane into some regions. And here, let us say, f 1 is minimum and f 2 f 3 f 4 and so on in the different regions faces, different functions will appear on the envelope.

So, it will be instead of; when you talk about bivariate functions instead of a partition of line into intervals, what happens now is, you have a partition of the plane into regions, so that the same function appears on the envelope. All right.

Now, can you guess what does the minimization diagram of this function will correspond to?

So, if you look at the region; so, within each region, same function f i appears on the envelope. Am I right? That is the definition of the minimization diagram; that was the definition. So, what does it mean is that if you look at any region, if the same function appears on the envelope. And, in this particular case, a function appearing on the envelope; it means it is the nearest neighbour and that precise the definition of Voronoi cell. So, this basically means that this region corresponds to the Voronoi of P i.

So, if you take the minimization diagram and look at each region; each region in this minimization diagram corresponds to the Voronoi cell of a point. So, minimization diagram of F, which gives us nothing, but Voronoi diagram of s.

So, what I have shown you is, what the Voronoi diagram means is you define a set of functions, these are the distance functions; you look at the pointwise minimum and look at the minimization diagram which is nothing, but the functions when you project it on the plane, you get the diagram. So, one way of thinking about this is as follows:

You draw these cones; this is one point, let me do it on the piece of paper. Draw these cones around each point. While I have the drawn finite cones, but these cones extend to infinity and you look from below; you see some surface, that is called Voronoi surface. You project it down on the plane, you get the Voronoi diagram.

But, so what ? Yeah.

Yes. Because; that is very good question. If, it you will see in a minute; if you have a cone; if you take the 2 cones, these cones are identical. In general, that is not the case. If you take the 2 arbitrary cones, intersect them, it is not as; if you look at the bisect, the intersection curve, project it; you will get a parabola or some conic. But, in these special cones, these are 2 identical cones, they just shifted. Then, you get the parabola; you get of the intersection curve, then you project it, it will be a line which is nothing, but the perpendicular bisector of these 2 points (()).

But, you are asking a question; just keep it in mind, next class you will see the consequence of what you are asking, the question. So, it looks sort of miracle, but that is what really happens and you will see what is going on in a minute. Any questions about what I said so far? All right. So, it relates to this question, so, why these cones which are quadratic curves, quadratic surfaces, why does it lead to polygonal regions, convex regions? And here is a reason. So, I define; let me start a new page.

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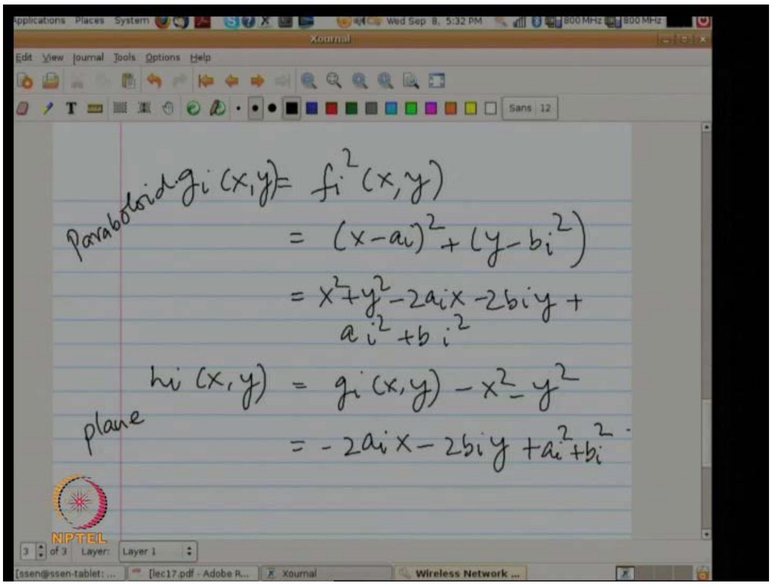

Let me define g i x, is a square of quadratic of a square of that function, which; in this case, I remember there was a square root, I removed the square root. So, this becomes; So, this is nothing, but the paraboloid, where I took the square; if you look at this equation, this function is a paraboloid.

Now, what I am going to do is I am going to subtract the quadratic term. So, I define y square, so, what you get; the quadratic term disappears. Thank you. So, I started with the cone, when I squared it, I got a paraboloid; when I removed the quadratic term, I got an equation of a plane. So, this is the linearic function, this is the plane and this was paraboloid.

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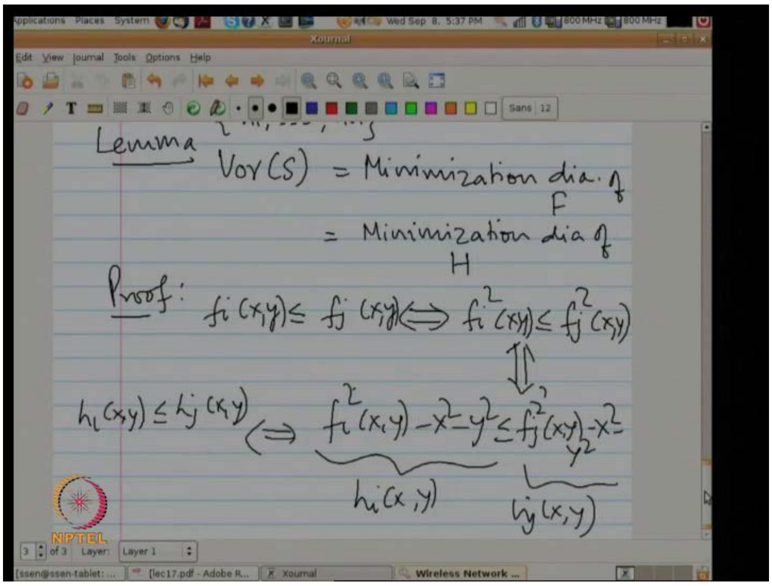

All right, so, what it all this good for and here is the main claim. So, let H be the set of n planes and here is the claim, Voronoi diagram of S, which we know is the lower envelope, is the minimization. So, what I am saying is that I started this functions F and in the two stages, I transformed them into function H. During this transformation, the functions have changed, but the minimization diagram has not changed. Can someone guess, can someone argue why this is the case? Why the minimization diagram has not changed during this transformation?

Not constant, x squared.

That's right. So, what I am doing is, as squaring does not affect anything and I am subtracting the common term; so, it does not affect. So, let me just go through it. So, here is the proof.

So, what I need to argue is that for any x, if f i is less than f j, that implies that h i should be less than h j; if I can argue that, then the minimization diagram will not change because of relative ordering of a function has not changed at any point. So, what I need to prove is this implies that the same (( )) equal to also will be true for h i x and h j x.

Well, I have this tendency of thinking of x as a vector instead of and a scalar. So, let me just continue with that. So, this implies; by the way, one thing I should say that f i and f j are non-negative functions; otherwise, there will be a little bit trouble. So, since they are non-negative, then this is the case because squaring both the sides does not change the inequality.

You see that what is the mess, I am creating, because by maybe I should be careful and put a y also there to avoid confusion. I feel like a kid playing with a magic slate that I used to do as a kid. So, it is f i squared x y minus x squared.

And this is nothing, but; because, this was the definition of this was nothing, but this is h i (x, y) and this is h j (x, y). All right. So, what it means is that the minimization diagram of these cones is nothing, but the minimization diagram of these planes; let me show you geometrically, what is happening. So, we started with these cones; let me draw the picture in 1D, because it is easier for me to draw.

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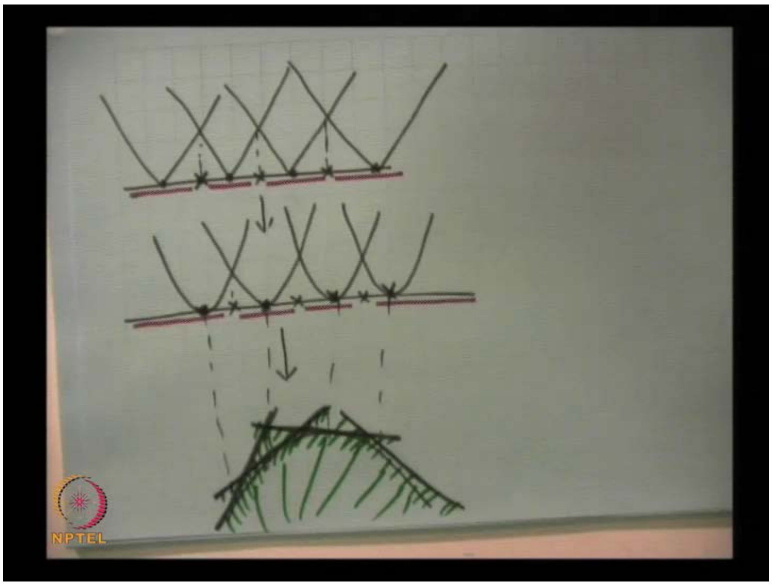

So, you had these cones and this was the minimization diagram; I squared it, I replaced them by a parabola; in that process, I did not change minimization diagram. And the third step what I did was, I subtracted minus x squared minus y squared term in this one minus x squared. So, what do you think about is that you think about this horizontal line x axis as being a flexible bar; what you doing is you are bending it downward. When you downward this horizontal line, it is becoming a paraboloid or in this one dimensional parabola, and this parabola is stretching to a straight line.

So, what happens is that, this becomes like this and (()) parabola, they become lines in this one tangent to this parabola; and same thing happens in high dimensions also. And the minimization diagram did not change in this process. But nice thing is that minimization diagram is not the minimization diagram of these lines, in this one. And if I think about; this is the set of points that lies below all the lines, so, it lies below each line. So, it is the set of points that lies below all the lines. If you think about half plane bounded by this line, line below this line then, this region is nothing, but the intersection of these half planes or in 3D, it will half spaces. So, let me go now to the next page.

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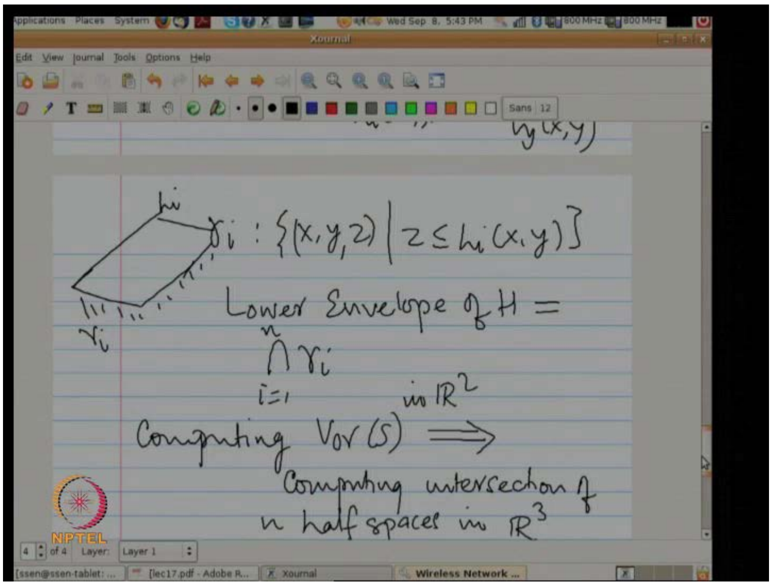

So, if I define gamma I; So, it is basically, the half space line below the plane h i. So, here was your, let us say h I, then the region lying below it, that is the gamma I. Then the minimization diagram on the lower envelope is the same as the intersection of gamma I, which is what I drew here; that the low envelope of these lines is nothing, but the intersection of this half space.

Now, you have heard about learnt about intersection of half planes or half spaces. So, if you take the intersection of half spaces it is a convex polytope and what you do is you take the intersection of these half spaces and you project it in the plane and you get the Voronoi diagram. And assume you have heard of duality. Am I right?

So, what it sort of says is that if you wanted to compute the Voronoi diagram of a set of points and what I reduced it to; so, this process I reduced it to computing a intersection of an half a spaces.

So, computing Voronoi of S is reduced to computing (()) . Now, how do I compute the intersection of half spaces? I do the user duality. Now, what I do is, just for some it will be easier to think about; I will reverse the z direction.

So, if I reverse the z direction, in this 2D as I reverse the y direction, then what happens? Earlier you were looking at the pointwise minimum, but if you look at the (( )) reverse y direction, you are looking at the pointwise maximum.

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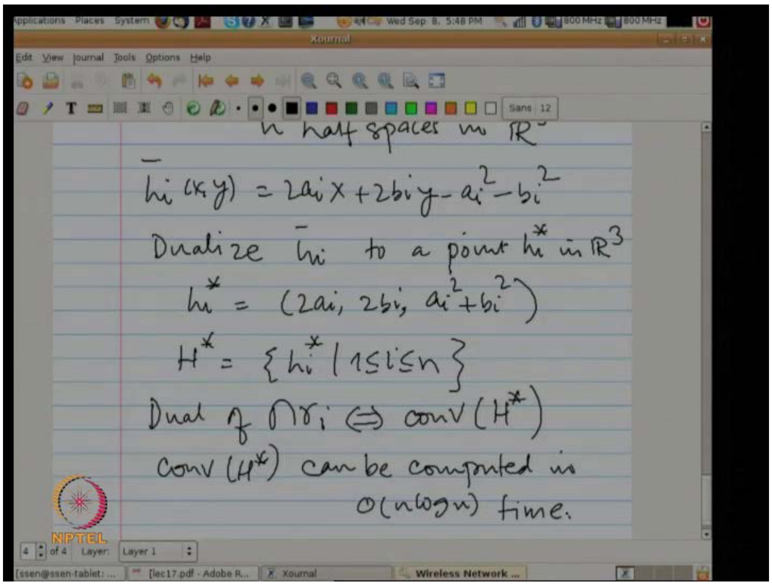

Am I right? So, if I defined, where was h i? h i was a equation was minus 2 i a x minus 2 b y plus a i squared plus y squared. So, let me define h i bar x y is 2 a i x plus 2 b i y i squared. So, this is; I just took the negation of that. And now, if I apply the duality, then what the duality does is the following; let me write.

So, the becomes 2 a i 2 b I, that is what it maps to. And let now, what is the intersection of half spaces correspond to when you look at the dual convex hull? Am I right? So, now, what happens is that you had the intersection of half spaces gamma i, I took these points, I dualized them, then dual of intersection of gamma i maps to convex hull of H star.

So, what I did was I started at a set of points in 2D, I wanted to compute the Voronoi diagram, I mapped them to a planes in 3D and I said Voronoi diagram is nothing, but take the intersection of these half spaces and project them; now I did that, use the duality, I mapped them to a set of points in 3D and I am saying it is a dual of the intersection of these half spaces is nothing, but the convex hull of these point sets.So, what I did was if I put all the pieces together, Voronoi diagram of a set of points in 2D maps to a convex hull of a set of points in 3D.

Now, you will see later in the class that you can compute the convex hull of a set of points in 3D in n log n time. So, convex hull can be computed in (n) log n time. So, what that implies is, for now that let us assume a black box, what this implies is that you can compute the Voronoi diagram of a set of points in 2D in n log n time. Now, these all; everything I said so far except the last sentence it works in a d dimensions also, in high dimensions.

So, the convex hull Voronoi diagram of a set of points in d dimensions maps to a convex hull of a set of points in d plus 1 dimensions. And there is classical theorem in convexity, that if you have set of n points and if you look at the convex hull of n points its complexity is n to the power d over 2 floor. So, what is called actually.

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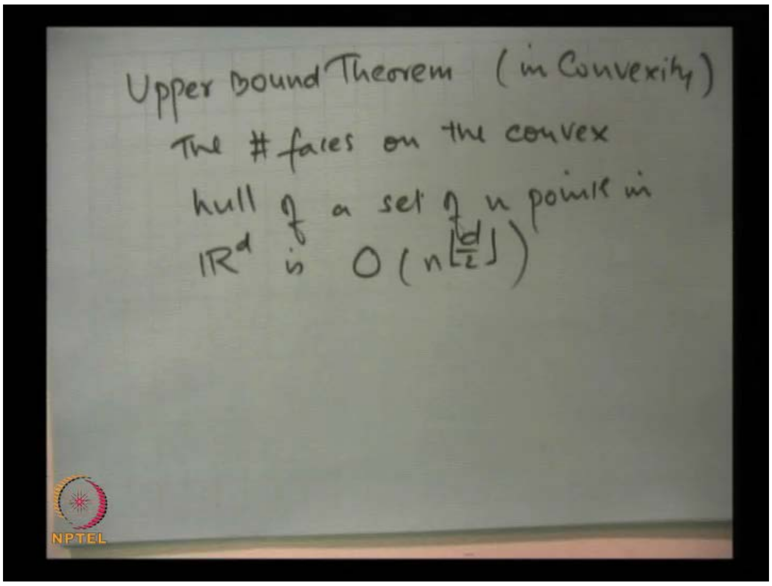

So, called upper bound theorem. The number of faces on the convex; now, in our case when you are looking the Voronoi diagram in d dimensions, it was a convex hull in d plus 1 dimensions. So, you take the d plus 1 over the floor which is same as d over 2 ceiling and that is what I had said last time.

Any questions?

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Well! You could try, but it would not work, because what is the case is that Voronoi diagram; by definition, by Voronoi diagram, as what I showed you was that. So, that is why I went through this instead of; the reason I went through this whole steps is not to give you as the mystery, why that is the case. Because what is the Voronoi diagram; Voronoi diagram is nothing, but this minimization diagram; inherently defined by the set of functions. When these set of functions in high dimensions and you taking the lower envelope and projecting them into d dimensions. So, Voronoi diagram involves the projection of d plus one dimensional object to one lower dimension; that is why, that is how it is coming.

And, here it is a convex hull in one high dimension, that is why it is in d plus 1 dimension. There is a generalization of Voronoi diagram, which I hope I will say it in one high dimension; sorry on Friday, what is called power diagram. Voronoi diagram is a very specific case of that and that is inherently is d plus 1 dimensional object and just happens to be a specific case.

For example, the another way of thinking it is the following, that the Voronoi diagram is the lower envelope of these planes, which were tangent to this parboloid. Now, in general if I draw some planes, they will not be tangent to paraboloid. One can talk about taking some arbitrary planes and take the lower envelope and it will reduce to some generalization of Voronoi diagram and that will be d plus 1 dimensional object and this happens to the special case.

Now, here I did the duality. Am I right? Because, for every day I do it; I use a duality here. I took this; I had the set of half planes and I dualise them to set of points. In the last class, I talked about the duality in the planar graph sense; I said Voronoi diagram and its dual wise Delaunay triangulation.

So, now what is the relationship between this convex hull? What will this convex hull look like? If you take this convex hull, so I am doing a duality here, right. Because, here is some saying that dual of this is intersection is the convex hull, but I had this convex hull in one high dimensions of this H star.

And, I take the convex hull. So, if I want to go back to Voronoi diagram, what I do is I have the set of points H star, I compute this convex hull, map it back the dual, I get the intersection of these half spaces and then I project it down; I get the Voronoi diagram. But, if I take the convex hull and I project the convex hull back down to 2 dimensions because H star is a set of points in 3D, so, convex hull it is a 3D convex hull. If I project it down to the plane, this convex hull what will I get?

I will get a Delaunay triangulation of a point set and to see that you have to be little careful because;

But I have a question here.

Yeah,

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So, in general, they are not; but this is the magic in this case; it just works out like a magic in this case. But, in general you are right; it does not all work like this. So, this is a general theory. And I am not just showing the whole theory; I am just showing you some kind of a special case of that and that is why it look like a magic and I will not have time to really go through the whole stuff and to talk about it when this really works.

So, if you remember that h i was 2 a i 2 b i a i plus b i squared; what I do is I scale it by factor of 2.

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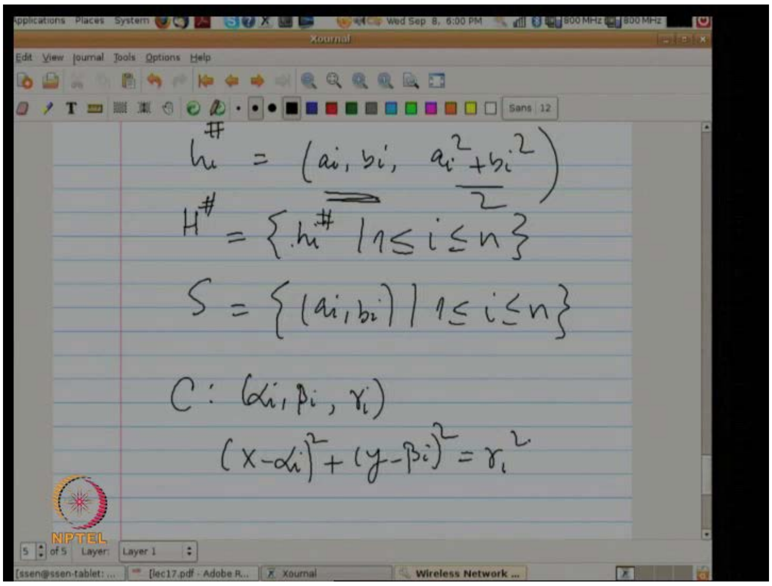

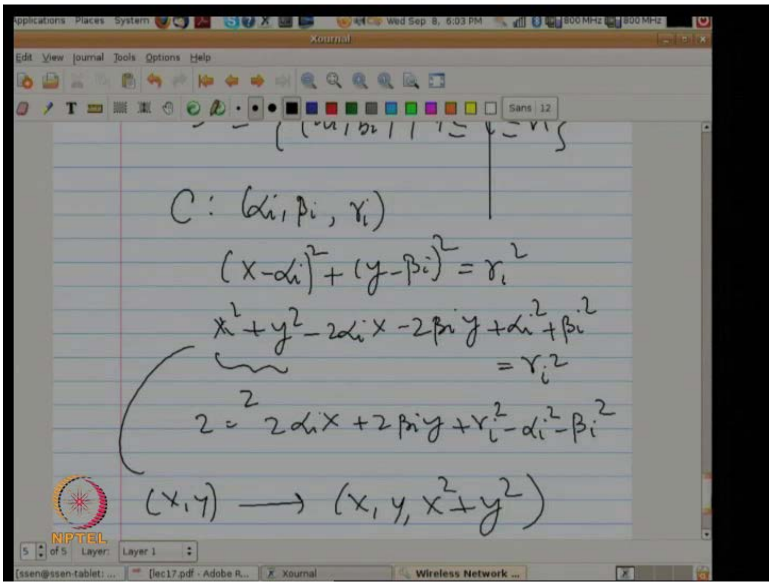

So, let us define the set of points, let us call it h i sharp; I divide each co-ordinate by factor of 2; so, I just shrink everything by factor of 2. So, then what happens? It becomes a i b i and notice that a i b i was similar with the co-ordinates of the i x point. So, what happened? So, if I look at them. So, if I define H; and remember that what was that S; S was to remind you as a i b i.

So, if you look at the set S and H sharp, the x and y co-ordinates are the same; S was the set of points in 2D and H sharp is the set of points in 3D, but the x y co-ordinates are the same. But, they have only z co-ordinate. How do you get H sharp from S?

So, this is the lifting transform that you have seen in the class, I believe.

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Yes.

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Precisely.

So, what you are doing is the following. Again I will draw only in 1 dimension, so, I will write the equation in general, in 3D; it is 2 and if you had the point P i, just lift it to the parabola. So, if you want to compute the Delaunaial triangulation, here is the one way of doing it; take the set of points, lift them on the paraboloid; z is equal to x squared plus y squared over 2, take its convex hull, project it down; you get the Delaunay triangulation.

So, there is a deep relationship between Voronoi diagrams convex hulls, Delaunay triangulation convex hulls; they all related concepts, but just we have to think in one high dimension.

Any questions?

Sir, how do we know Covex hull is a Delaunay triangulation?

Good question. I will answer your question in a minute. Any other questions from what I said so far, before I answer her question? So, why is this convex hull is a Delaunay triangulation? I will not give you the complete proof, but I will give you partial proof and then you can fill in the details. Remember what I said for Delaunay triangulation; elaunay triangulation had the following property. If this was a triangle p q r; if it was a triangle; if you take the circumcircle, it does not contain any points, all the points lie outside this circumcircle. Am I right? So, that was the property.

And let us now talk about what this correspond to in 3D? So, what I did was I took the set of points in 2D, I mapped them on this paraboloid. So, ignore the. So, if you have a circle, let us say circle C, whose center is, let us say alpha i beta I; this is center and this r i is the radius. So, we have circle in 2D and this is r i.

So, the equation of the circle is; assume this; you might have seen in this one. And the points lies at the circle that correspond to; if a point x y lies at the circle, you replace equality by lesser than equal to or if it lies outside, it become greater than equal to.

So, now, if I open it and if I replace this one with equation z, then what you get is, you can write it. So, what happens is that I took the equation of circle and mapped it into a plane in 3D. So, circle becomes a plane in 3D and this is the general techniques in algebra, that is used a lot and this is called linearization technique. And it comes in many different areas, that number of times it is hard to deal with the higher degree terms, variables and if you convert it, if you have a function which is not non-linear function polynomial, you can map it to a linear function in high dimensions. And that is what is going on here. So, you are dealing with the cones or circles and you are mapping it to planes in one high dimension.

So, that is what I did; I took the circle and map it to a; it becomes a plane. And, if you look at, what I did was the transformation where x and y did not change, but x squared plus y squared became z. So, this transformation, lifting transform; this is called point x y is being mapped to x y x squared plus y squared, which is like this the factor of 2 issue here, but this is very similar to what was going on earlier. Am I right? a i b i a squared plus b i squared 2. So, I am again taking the set of points and mapping it to paraboloid. And a circle now maps to a plane. Then, what really happens is if you look at the convex hull and of these points that all sitting on this;

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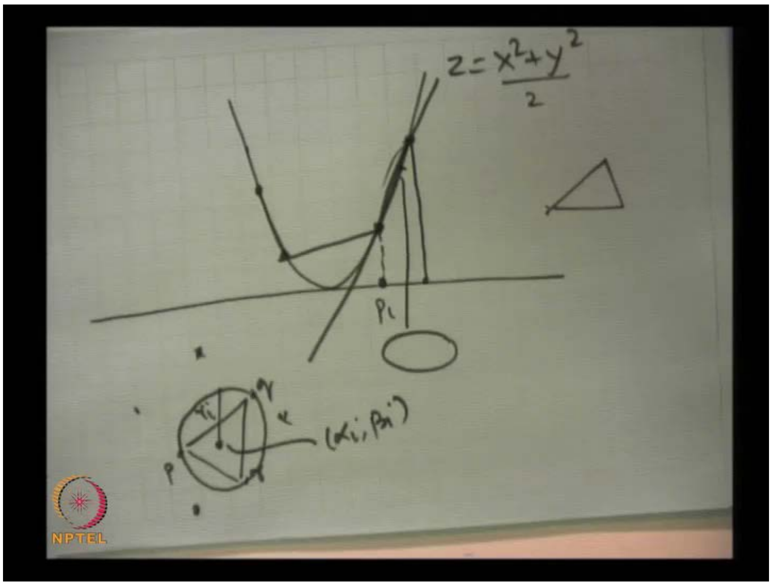

If you take the convex hull, which in 2D it is a polygonal chain; but what happens is in 3D, you will get something like triangles and it will be. I am not good in drawing these shapes, but if you think about the set of points on the paraboloid, you will get some triangles and you will get some convex polytope.

Now, if you take the plane containing one of the triangles; what this happens is, if you this plane, if you project it, take this intersection with this paraboloid plane, intersect by, when you map it down, you get a circle back. And the point lying above the plane which means point lie outside the circle in the plane.

So, if you remember that the condition for the Delaunay triangulation was that all the points should lie outside the circle and what it sort of means here is that all the points lie above the plane and that is the case because of convex hull. If you take any triangle of the convex hull and if extend to the plane that all the points lie above it, none of the points lie below it; since it is a plane supporting the convex hull.

So, what it means is that when you project it back, the planes are mapping corresponding to the circles all the points lie outside the circle and which it means it is a Delaunay triangulation

Now, I see that I have about; How much time do I have? Few minutes or I am out of time? Couple of minutes.

So, how do I construct a convex hull of a point set and let me describe in 2D and then you can imagine how it happens in 3D?

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So, think about the special case when all the points are in the convex position. Am I right? Because if the points lie on the paraboloid, on the parabola, all the points all the points will be vertices of the convex hull. So, then what you do is; suppose you have computed the convex hull of some points. So, think about in 2D, you have computed some point. Now, I am adding a new point, let us say; then, what you do is you compute this tangent in 2D and you are going to remove this portion and replace it at this portion. That is what happens.

Now, the way you think about; the way one think about; physical way of thinking about is you have a convex hull which is a convex polytope in 3D; new point think about the light source. Whatever it can see, when you put the light, some part of the polytope is being seen; some part of the polytope is not seen. The portion that is being seen, you remove it, you get a hole which is the (( )), take each (( )) is an edge, take this edge, connect to the point; you get a new set of points.

In 2D, that is what is happening; you have these edges that you can see, you remove them; in this case the (( )) only consists of 2 points, you connect them by side by side. But in 3D, the (( )) will look like some convex polygon and here is your point and you are going to this one.

Now, what is this correspond to; I said this convex hull is nothing, but if you project in the plane is nothing, but the Delaunay triangulation. Then, one can ask do I really have to go through all these mapping or I can think directly in terms of Delaunay triangulation? What is going; when you insert a point, what really goes on. And in the next class, I will tell you what it means directly in Delaunay triangulation. So, I will stop here.