Computational Geometry

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Module No. # 08
Voronoi Diagram and Delaunay Triangulation
Lecture No. # 03
Delaunay Triangulation

Welcome back. So, we will continue with our discussion on Voronoi diagrams and the related dual called Delaunay triangulation. So, in the last lecture, those are otherwise discussed about the relationship between convex hulls and Voronoi diagrams and there is also a close connection with the Delaunay triangulation, namely, that if you look at the paraboloid transform, you take a set of points on the plane and you lift them to the paraboloid. Let us say, from the two-dimensional plane to the three-dimensional space on to the paraboloid.

So, the points are projected vertically on to the paraboloid. Now, when you construct the convex hull of the points on the paraboloid and when you project it back to the two-dimensional space, essentially what you get is a Delaunay triangulation. So, this fact was established in the last lecture. So, this is of course, one method of constructing Delaunay triangulation, that is reduce the problem of the 2d Delaunay triangulation or Voronoi diagrams to that of constructing a convex hull in the three-dimensions. However, for reasons that probably, somewhat beyond the hope of these lectures in this course, it actually is better or let us say, more stable to construct the Delaunay triangulation directly.

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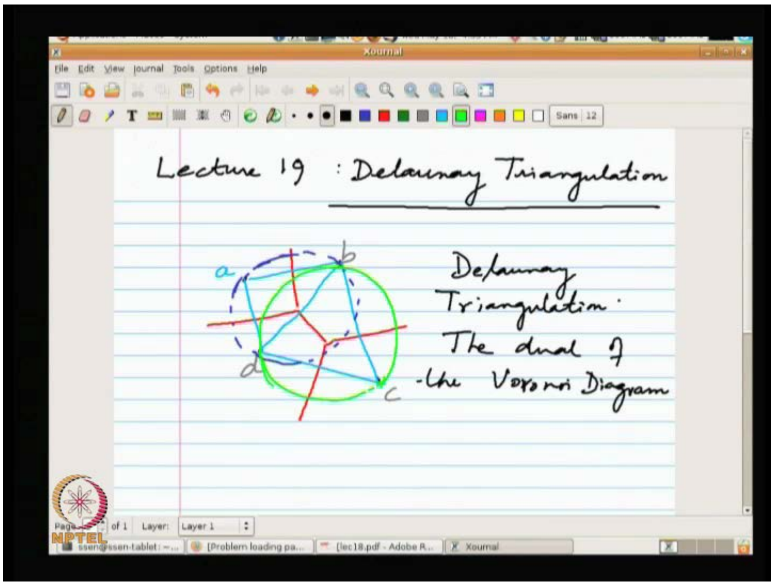

The basic intuition is the following that when you are constructing a Voronoi diagram or even the three-dimensional convex hull, you are dealing with higher degree curves. In the sense that, when you are trying to construct Voronoi diagrams, you are looking at perpendicular bisectors of the points, so that it is a higher degree. It is a line, but it still depends on the coordinates of the two points, whereas the Delaunay triangulation directly connects the points by edges right. So, for instance, if I have these points and my Delaunay triangulation, sorry my Voronoi diagram looks something like this, the Delaunay triangulation, actually it connects the points directly right.

So, here is a Delaunay triangle, here is the other Delaunay triangle. So, we are not actually, we are not defining this. Red lines are defined on the basis of, let us say these given points a b c d, but the red lines are the perpendicular bisector of the segments joining a b and so on so forth. So, the equation of this red line depends on these two coordinates, whereas the Delaunay triangulation edges are directly obtained by the line segment between a and b. So, numerically, when you actually implement these algorithms, it is easier to handle rounding errors etcetera when you construct a Delaunay triangulation directly rather than Voronoi diagrams.

So, we will today discuss, actually some direct methods of constructing Delaunay triangulation. So, Delaunay triangulation, of course is the dual, I mean one way of defining is that Delaunay triangulation is the dual of the Voronoi diagram. We will be restricting our discussion to two-dimensions, but even in higher dimensions, there are analog of this relationship between Voronoi diagrams and Delaunay triangulations. So, one very important or key property of Delaunay triangulation is that when you look at, let us say, this triangle constructed by the blue line segments, if you look at the disk or the circle, the circum circle of the triangles of the Delaunay triangulation namely. So, this is the circum circle of the triangle a, b, d.

Similarly, you can draw the circum circle of the triangle d b c and look something like, if I am drawing without compasses. So, you have to believe what I am drawing, but one thing you will know is, of course, this is only a small diagram of four points is, that this circum circle of a b d does not contain the point c. Likewise, the circum circle of d b c does not contain the point a. In other words, these two circles define disks which are called Delaunay disks. The Delaunay disks basically are defined as the interior of the circum circle of a Delaunay triangle. So, these Delaunay disks have this very nice property and which probably is not very intuitive at first site, is that they are empty.

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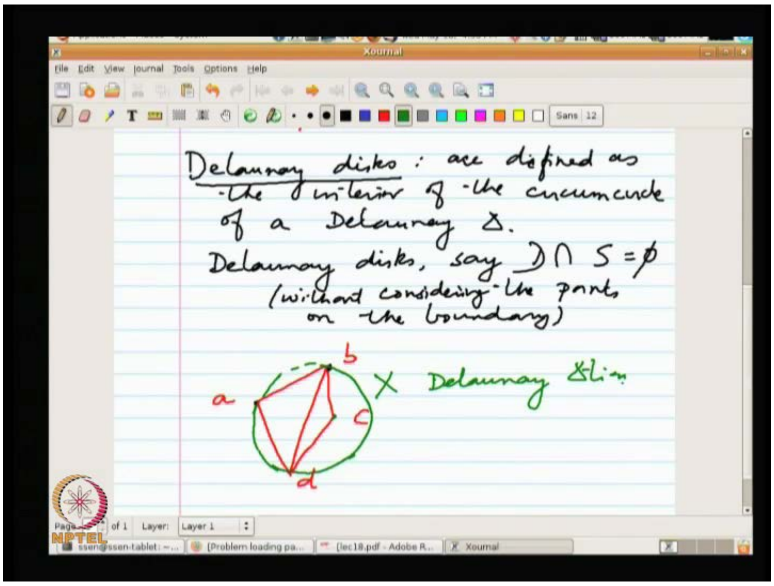

So, the Delaunay disks say d intersection with the set of points is empty, if you do not consider the points on the boundary, the points that defines the disk. So, this is a very nice property that the Delaunay triangulation gives and this is not true for an arbitrate triangulation. You know for instance, you can construct this example. So, take again four points. Consider this triangulation a b c d. Let me draw the circum circle of a b d and certainly, this circum circle or this disk, it cannot qualify or it is not eligible to a Delaunay disk because the point c is contained within the circum circle of a b d. So, this is not a Delaunay triangulation.

Therefore, it is not even obvious that Delaunay triangulations will exist, but on the other hand, you know if you look at the construction of a Voronoi diagram and if since Delaunay triangulation is the dual of the Voronoi diagram, this is an indirect proof that Delaunay triangulation exists. If one way to simply go about directly, in sense that prove that there exists a triangulation such that the circum circle of every triangle is empty. You know proving this property or proving such a triangulation exits could be extremely non-trivial task, but let us try to set of at least sketch, a kind of intuitive proof. Why you know such a triangulation exists?

Now, if you consider these four points a b c d, of course, this a b c, the circum circle of a b d is not a Delaunay disk because it is not empty, but instead of a b c d, if I did not draw this triangle and decided that I should draw this diagonal instead of b d. Let me draw the diagonal a c and therefore, even my circum circle has to be re-defined alright. So, let me try to draw the circum circle now. At least a free hand drawing should be convincing that none of these two circum circles contain any other point, namely that these two are empty.

The circum circle of a b c and the circum circle of a c d, both of them are empty. Therefore, by definition, they are qualified to be Delaunay disk and so, this triangulation does meet the condition of a Delaunay triangulation. Is this a coincidence? Not really. So, what is really happening is that if I take four points in convex position, so you can see that. So, I have these four points, obituary four points in convex position. You can prove regularly that either this drawing, this diagonal will be consistent with Delaunay triangulation or if this is not consistence with Delaunay triangulation that is the circum circle is not empty, then certainly the other one will be.

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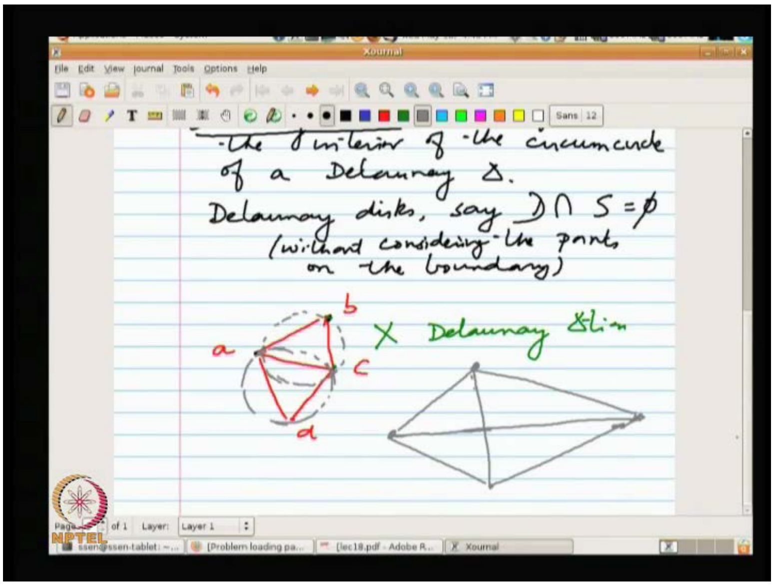

So, one of these two diagonals must satisfy the property of the empty disk. Why is it so? Well, if you think about the underline property is that if the four points to the four points lie on a circle. So, if I take a circle and put four points on it, so from our knowledge of high school geometry, let us call this angles alpha beta gamma delta.

We know that some of the opposite angles when all the vertices on the circle, they are supplementary right. So, alpha plus gamma equal to pi and delta plus beta is also pi. Of course, from our assumption that we made in the case of Voronoi diagrams of point sets and the same thing it holds for a Delaunay triangulation also. So, the one of the assumptions that we make when we design an algorithm for Voronoi diagram is that no four points are co-circular. That is the points are in general position.

So, the point sets does not have four co-circular points. So, therefore, this kind of configuration cannot exist. This kind of configuration, it also called the general position assumption. Now, the general position assumption is not necessary, but it is very useful actually to simplify many of the proofs and even simplifies some of this special cases. That arises when we have to deal with four points of circles. It does not really change the asymptotic complexity. It does not change the basic properties of Voronoi diagram, but it does have some bearing on some extra cases that what has to handle. So, it is convenient to make this assumption and let us not make life harder by doing away this convenience.

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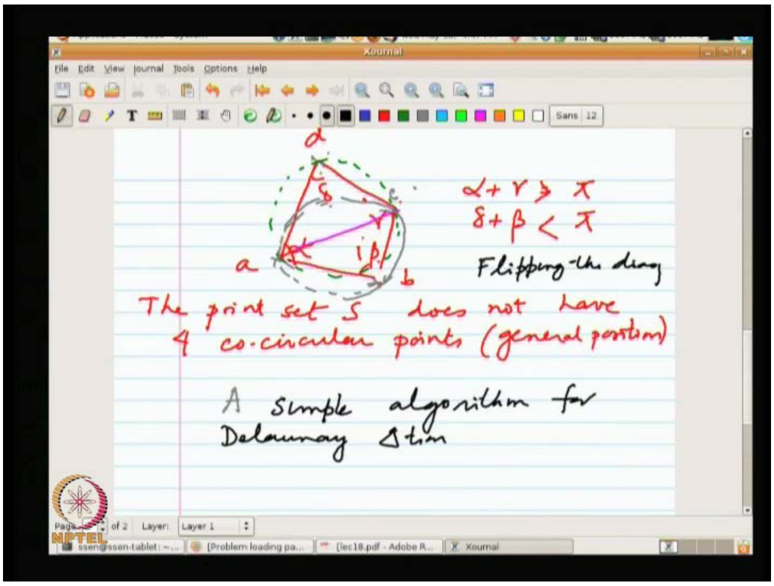

So, when this kind of configuration cannot arise essentially what we are saying is that either we have the satiation that all four points cannot be on the circle. So, either they will be inside it. It will be inside like this in which case you can see that when you consider the Delaunay circle. Sorry, that the circum circle if I drew this diagonal b d, then the circum circle of this diagonal would defiantly contains c has come inside now contain c. Therefore, this diagonal cannot define the Delaunay triangulation. Since, we pull the point inside again by our knowledge of high school plane geometry, we are actually now done the followings.

We have now alpha plus gamma is greater than, strictly greater than pi. Delta plus beta is strictly lesser then pi. So, this is the situation. So, if we draw the diagonal b d, then this situation. This basically corresponds to the case where this is not a Delaunay triangulation. That is we have drawn the diagonal d b and the angles lying on the opposite side of the diagonal, it exceeds pi. Since, this exceeds pi, the other pair of diagonal and diagonal angles delta plus beta is less than pi. Therefore, if we had drawn this one instead of this diagonal, we had drawn this diagonal, then by the fact that delta plus beta is less than pi.

Now, if you draw circum circle, this is no longer circum circle because it does not go to three points of triangle. So, I have redefined my circum circle and because delta plus beta is less than pi when you draw the circum circles by the property of the supplementary, some of the supplementary angles this point because delta plus beta is less then pi, b is going to lie outside of the circum circle. Likewise, when you draw the other circum circle same thing for the same thing propertyis getting little flatter, but I think you get the point.

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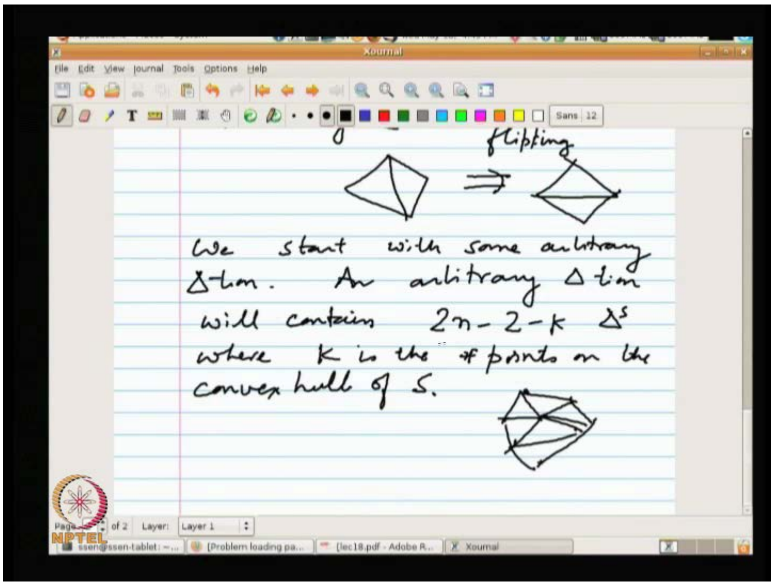

So, because the sum of delta plus beta is less than pi, the point d must lie outside the circum circle of a b c. Similarly, the point b is going to lie outside the point circum circle of d a c. So, this is a very basic property and that will be the basis for a very simple algorithm namely, let us say that we want to construct that Delaunay triangulation. So, a simple algorithm for Delaunay triangulation would do the following.

So, this drawing, the diagonal ac instead of drawing the diagonal db, is this can be thought about if as flipping the diagonals. So, this operation is called flipping the diagonal. So, we take two triangles that share an edge and if the two triangles do not meet these specification of a Delaunay disk that is the disk is not empty, then we flip the diagonal from this situation. So, this is basically flipping edge and this edge flipping is used as a basis for constructing Delaunay triangulation.

So, what do we do? We start with some arbitrary triangulation. What is that? An arbitrary triangulation using some of the methods that we are learnt in the course and will contain about 2n minus 2 minus k triangles where k is the number of points on the convex hull of s. So, in other words, we only triangulate the interior of the convex hull. If the given points are like this, this is a convex hull.

So, we are only going to construct the triangulation interior to the convex hull. So, whatever the triangulation you construct, some arbitrary triangulation these are the points in the boundary, the k point of the boundary. Then, the total numbers of triangles are defined by, can be, you can verify this is 2n minus 2 minus k. Anyways, it is still linear. The numbers of triangles are going to be linear in the number of points. So, we start with an arbitrary triangulation and then we do the following means we examine pair of triangles that shear an edge.

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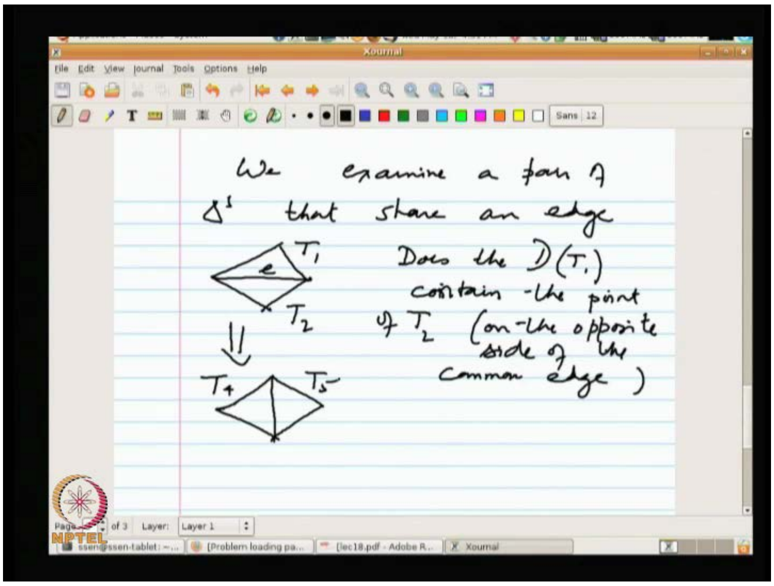

So, these two triangles share an edge T 1 and T 2. Now, what we check is, that does the Delaunay disk of T 1, let us call it the D of T 1, does D oppose T 1 which contain the point of T 2 lying on the other side of the common edge right on the opposite side of common edge. Call it e. If you will, so if this contains the points of the other side of the common edge, then clearly this is not a Delaunay disk. Therefore, we must flip the diagonal. So, then this must basically be changed to this situation.

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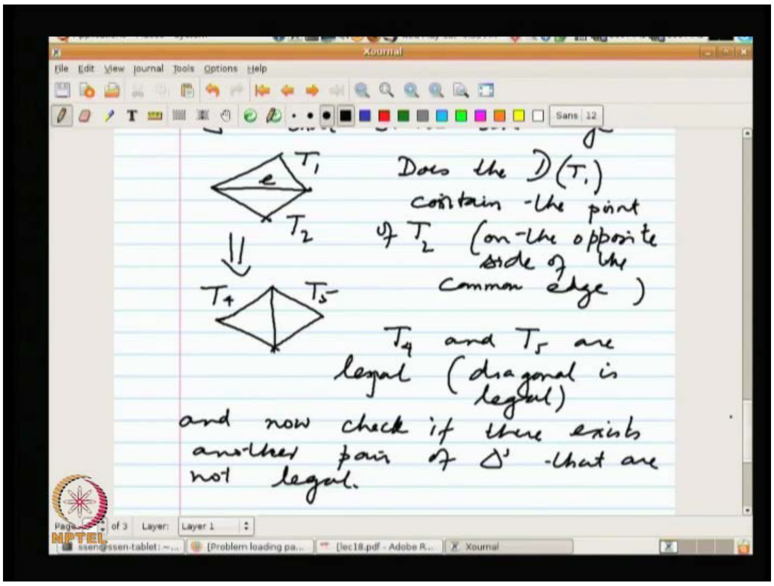

So, T 1 and T 2 will disappear and instead, there will be a place with let us say T 4 and T 5 and T 4 and T 5. As we have just observed will necessarily be satisfied the property of Delaunay disk. So, we have fixed the problem with just a pair of a triangle sharing an edge and we keep continuing to do on this. So, T 4 and T 5 are legal. So, T 4 and T 5 are legal in the sense that you know that there Delaunay disk do not contain the point. On the other side of the edge are legal or you can say that diagonal itself is legal. Now, we check if there exists another pair of triangles that are not legal. So, we keep this till we come to a situation where they are no more that all the diagonals are legal, until all diagonals are legal. So, we just keep continuing doing this thing.

Of course, it looks like a very unstructured process where we just picking up looking at some edge, looking at that two triangles that share that edge and just verify using some coordinate geometry whether or not the two disks are legal Delaunay disks. If they are, then we leave them as it is. If they are not, then we move on to another edge, look at the pair of triangles sharing that edge and so on so forth. So, first of all it is not even clear that this is going to converge. Why is it necessary that the algorithm, the edge once flipped does not get flipped back? If it does not get flipped back, then of course, it will converge. What is the proof or what is the argument that edge once flipped will not get flip back? So, for that we will make a few observations.

I would not give you a very rigorous proof, but something that can be refined proof. So, one thing to notice again, this is a very important property of Delaunay triangulation. Let us look back, go back to the, let us draw another figure. So, consider again some four points. Look at the two diagonals a b c d and suppose that b d is the legal edge. Suppose b d is the legal edge ok.

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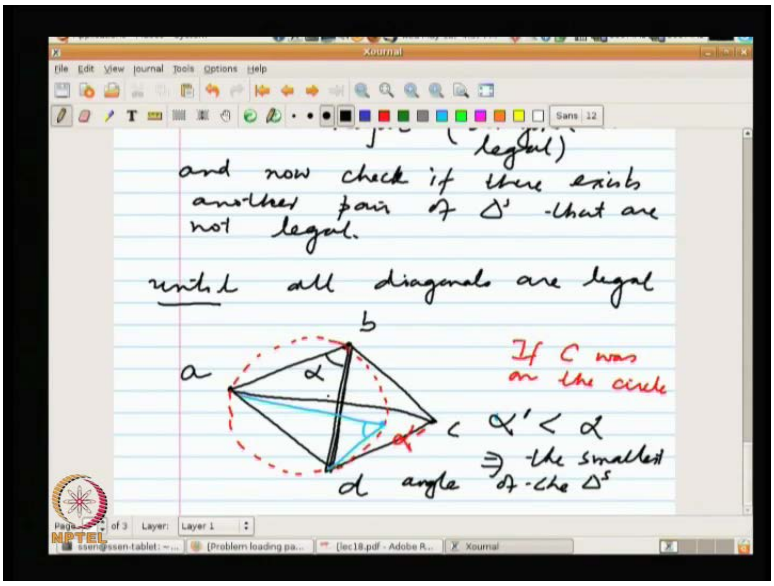

Now, consider this angle alpha. So, this is a legal edge. You draw the circum circle c should be outside of this circum circle alright. Look at this angle alpha prime alpha prime. How do you compare alpha and alpha prime? If c was actually on the circle, if c were on the circle alpha would have been equal to alpha prime right. So, if c was on the circle, then again from the knowledge of plane geometry that you know these are angles on the same are lying on the circles oppositely like this.

So, what I am saying is suppose c was somewhere here. So, then this angle would have been too equal to alpha because they subtend on the same arc, but c is actually outside. So, c is actually going to the smaller. So, the angle alpha prime. So, alpha prime is strictly smaller than alpha. Suppose, alpha was the smallest angle in triangle a b d, so we can do more case analysis. Suppose, in one case it is such that the triangle is such the alpha is a smallest angle in triangle a b d. What it implies is that if we had used the diagonal ac instead of b d, it uses that diagonal ac instead of b d, then alpha prime would have been one of the angles of the two triangles defined by the diagonal ac. Notice that ac is not a legal edge because we have said b d is the legal edge. So, if we try to draw the circum circle of a b a d c, it is going to contain b.

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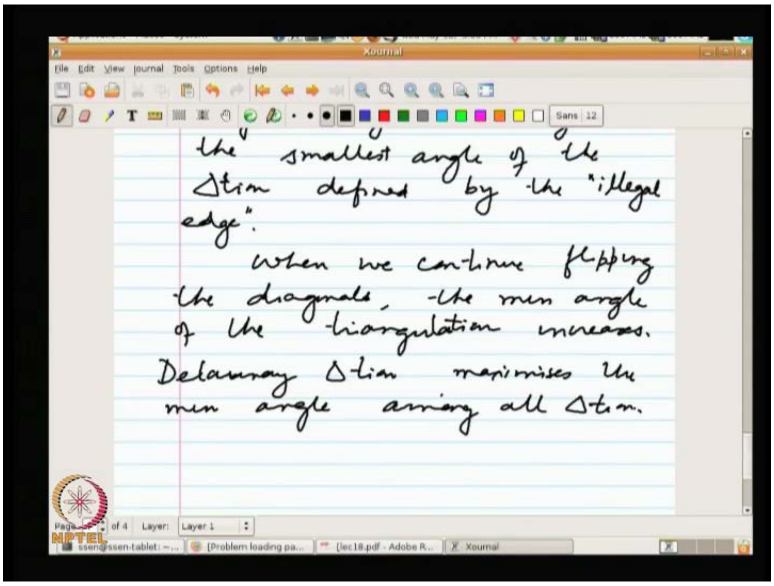

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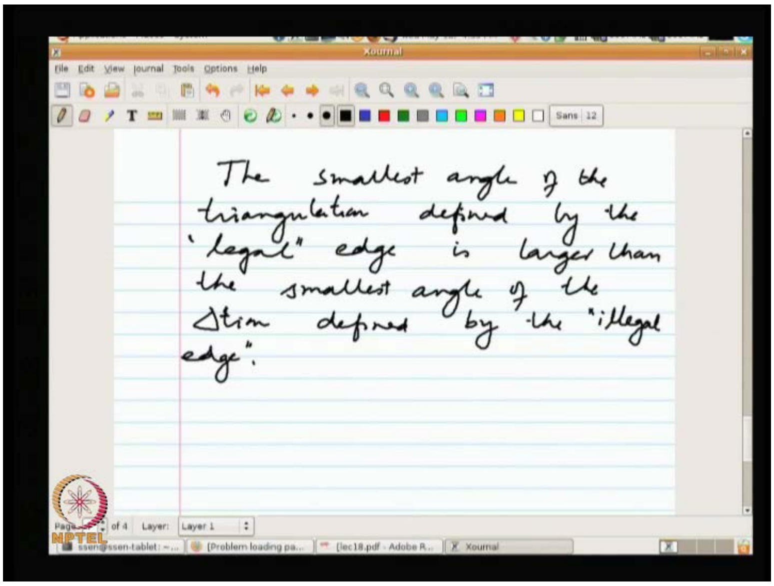

So, given that b d is the legal edge and alpha is larger than alpha prime, it implies that the smallest angle of the triangles. The two triangles defined by this legal edge. The smallest angle of the triangulation defined by the legal edge in sense of Delaunay triangulation is larger than the smallest angle of the triangulation defined by the illegal edge. This observation can actually formulize some kind of a proof which says the following that as we flip the diagonals, when we continue flipping the diagonals depending on whether it is legal or not, the minimum angle of the triangulation increases.

In fact, Delaunay triangulation maximizes the minimum angle of the triangles among all triangulations. So, this is another very important property of Delaunay triangulation that when you actually do the edge flipping, you are going to increase the minimum angle. So, every time you do edge flip, then minimum angle of the triangulation is increasing. Therefore, this is another proof or this is another property because of which this procedure converges. The edge flipping actually converges, but it does not tell you how long it takes. So, now, for that we will go back to the lifting transform. So, what the lifting transform does? So, let me try to draw a picture.

So, we have these two possible edges. So, here is a paraboloid and it has to be, well I mean it is actually a three-dimensional paraboloid, but you know I cannot draw that picture very well. So, when you are projecting on to it, the points just look at the four points. So, we saw in last lecture that the base circle defined by a triangle here or the disk is when you project it to the paraboloid. If it corresponds, if the circum circle corresponds to a Delaunay triangle, then all other points will be to one side of it.

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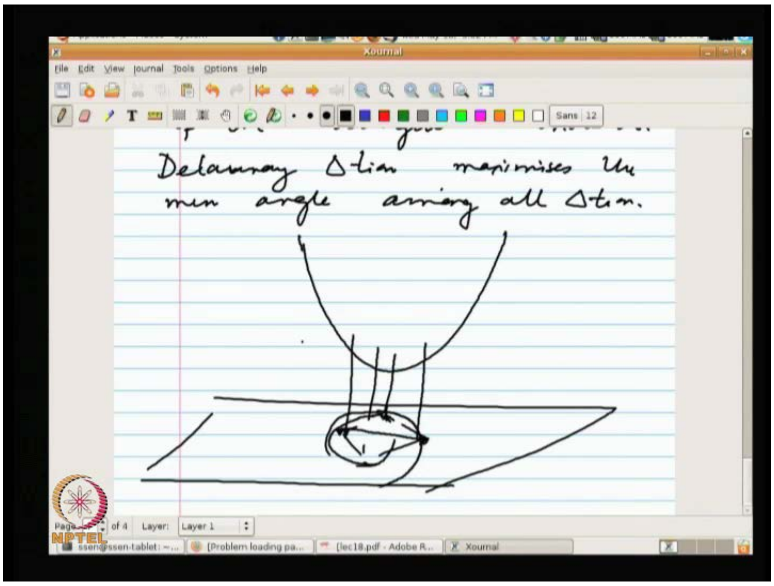

So, that is what actually why there this nice mapping between the three-dimensional convex hull, so that the three points actually turn out to be the face of a convex hull in the three-dimensional. Therefore, all the points have to be on one side of it and therefore, the Delaunay disk. If this point lies outside this circle which is the property of the Delaunay disk, then this point is going to be on the other side. Whereas, if it is other way round that is if it is like this, then this disk will contain this point and when you project this circle on to the paraboloid, this point is actually going to lie below the face to make it more clear. It is like this. You know in one case you know if this is so now I have to draw some kind of a three-dimensional picture. In one case, now the triangle is going to fold like this. So, a b c d.

So, this phase or the circum circle passing through this face, this point will lie below it. It kind of folds in this way, but if you drew the other diagonal, then it is going to fold in the other way. So, these four points I am going to lie on the parabola and will fold like this. It will be convex as above to concave. So, the legal diagonal actually folds it in the convex position, whereas the other diagonal folds it in the opposite direction. So, you can see that as you basically flip the diagonal, the diagonals are actually kind of coming down.

So, your triangles are getting closer and closer or the face of the convex, it is not a convex but basically the triangles are getting closer and closer to the surface of the paraboloid right. So, the diagonal, once it falls, comes below, it is never going to rise up again because this is an illegal diagonal. So, you cannot again raise a diagonal because the diagonals, all the points should come. Actually, it should be basically this point has come below the diagonal. So, if I raise the diagonal again, the point is going to fall below right.

So, a diagonal once it falls below, it is never going to rise again. Therefore, this is also an analysis of how many times a diagonal can be flipped. Since, a diagonal cannot be flipped more than once, the convergence takes order n square flips. Now, this is of course, module or data structure. So, you have to use some cleaver data structure, so that over all running time becomes about n square. So, we are only given a proof that the total number of h flips is n square edge flips. So, using an appropriate data structures you can obtain order n square time.

The good thing about this is that it is a very simple algorithm, but it is n square where as you know that a three-dimensional convex hull can be constructed in a log n times which means that a Voronoi diagram can be constructed in n square time n log n time.

Therefore, a good algorithm Delaunay triangulation should not be in more than log n time. Whereas, this one or the simple that is takes n square time. There is another observation which is also kind of important that if you are given a arbitrary triangulation, how quickly can you verify that it is a Delaunay triangulation with may or may not be, but your job is that I give you a triangulation where every face is a triangle connecting the points, given set of points.

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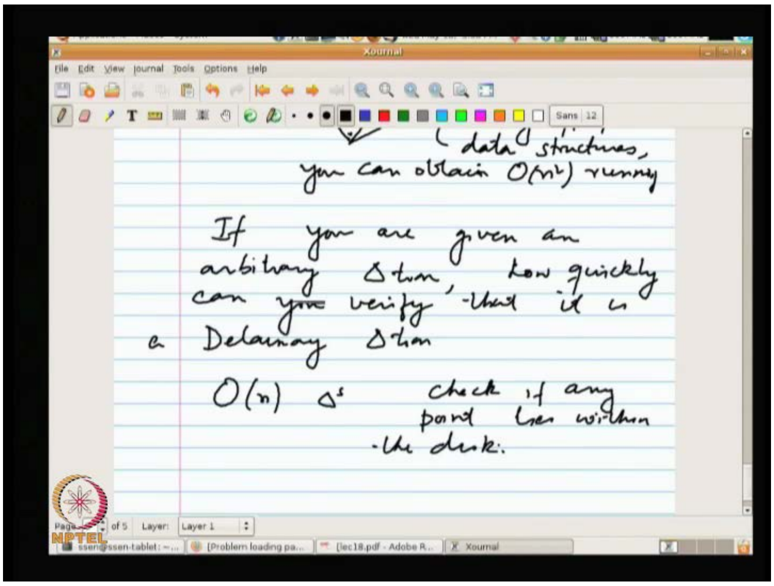

Now, your job is to find out if this triangulation is Delaunay triangulation. So, you can go back to the original definition. Basically, that you look at the circum circle of every triangle that is what is the disk and just find out whether if the disk contains any other point. Now, if you do that, you have about order n triangles. So, if you do this for every triangle, then order n triangles and then you have to check with this, with respect to every other point right. There are n points, so that means that check if any point lies within the disk.

So, again using some, let us say some appropriate analytical coordinate geometry formula. You can do the test for every point in constant time which means for that every triangle you will take about order n time to find out if it is really empty or not. They are n triangles which mean that again going to be an n square procedure right.

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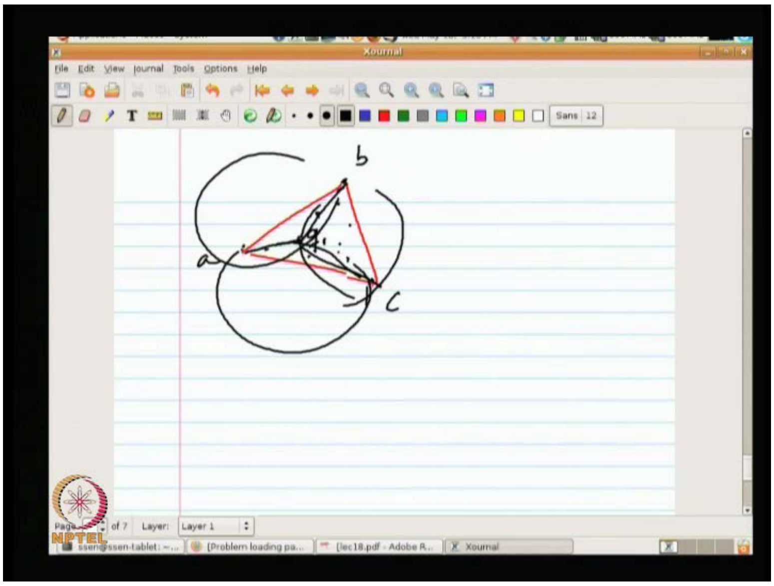

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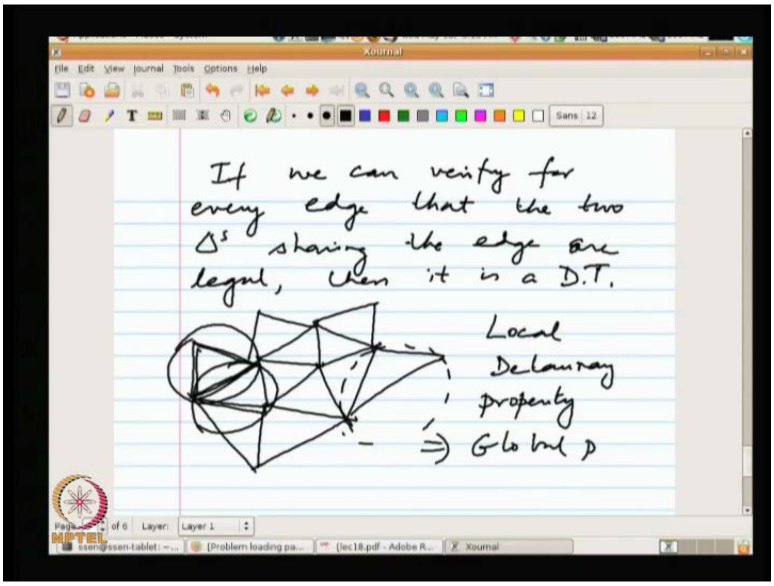

What I have defined is, basically a theta n square procedure, but there is a very nice property that says the following that if I can verify, if we can verify for every edge that the two triangles sharing the edge are legal, then it is a Delaunay triangulation which basically means that I am doing only local checks. I am looking at the triangulation. I am not looking; I am not trying to check with respect to every point. There is some triangulation. I am not going to take every circum circle and check with respect to every point, but I will only look at an edge, look at the two triangles sharing the edge, draw the circum circle.

If this circum circle does not contain this point and this holds for every such pair of triangles, then it is a Delaunay triangulation. In other words, what I am saying this is called a local Delaunay property. So, I do not need to check with respect to all other points whether they contain inside local Delaunay property. So, a local Delaunay property implies the global property. So, that is a very nice result. The proof is by contradiction. I am not going to do this.

Basically, what you can show is that suppose there is some Delaunay triangulation that, sorry there is some triangulation that satisfy the local Delaunay property. Suppose, there is some violation in the sense that although, it satisfies the local property. There is some points that lies within someone, one of the disk, then you can actually work backwards to show that it will violate some local property. I am not going to go into a restrict proof of this. What you know it can be worked on the basis of contradiction all right.

So, this is some of these nice properties about Delaunay triangulation. By the way, the minimum angle property is very useful for all kinds of modeling. If you can only have the points of the surface and you want to actually reconstruct the surface, it is usually very useful to do the triangulation of those points. So, as to minimize the minimum angle, sorry to maximize the minimum angle, so this is used all the time. Therefore, the Delaunay triangulation, the most popular kind of triangulation, when you are doing resurface construction, but then again let us go back to this problem of efficiency. So, this simple algorithm gives you n square kind of a running time. How about if I desire a better running time?

In other words, can I match the n log n running time using a direct proof rather than going to the lifting transform and building the three-dimensional convex hull? Can I directly construct a Delaunay triangulation in log n time? There is actually a very cleaver or a simple method based on a very general scheme called randomize incremental construction which also going to one of the lecture topics in your feature.

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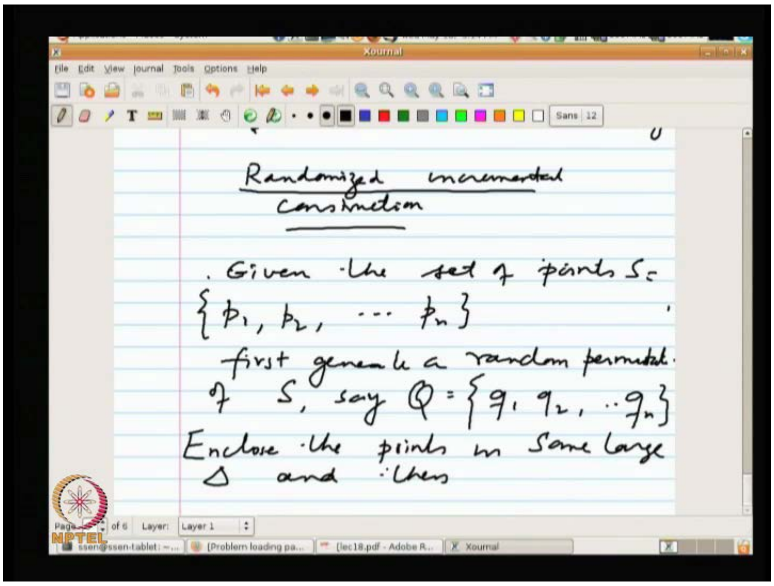

I will just give you a very brief description that randomized incremental construction. What you do is, you begin with an arbitrary triangulation. Well, actually do not even, so you just start with set of points, so given the set of points, S equal to p 1, p 2, p n. These are all two-dimensional points. Do the following first. Generate a random permutation of s say call it q and that now consists of a random permutation of the set of given points. Call them q 1, q 2, q n. Enclose the points in some large triangle and then pretend that you are adding these points in the sequence of q. That is q 1, q 2 upto q n and whenever you add a point, you should inductively maintain the Delaunay triangulation property right.

So, suppose I have so far added these three points. So, suppose these are the first three points to be added and well, these are some artificial points. The actual points are all inside and with one triangle of course, is nothing to be really worried about. These points do not exist. Actually, they are going to be added one after the other. So, the points add is q 1, let us say q 1 when I add a point, then the first thing I do is add some edges so as to make it a triangulation.

So, I add these edges right. So, now this triangulation consists of these artificial points a b c and the first point q 1 alright. Well, I mean in this case, there is nothing to worry about because if you draw the circum circles, they will not contain any of the points that have not been added. The other points actually do not exist. They are in some q to be added in feature, but if I draw the circum circle of this, certainly it is not going to contain a, n and vice versa. So, this can verified. So, right now there is nothing to worry about. So, as we go on, we keep on adding the next point and inductively maintained Delaunay triangulation of the points that we have added so far.

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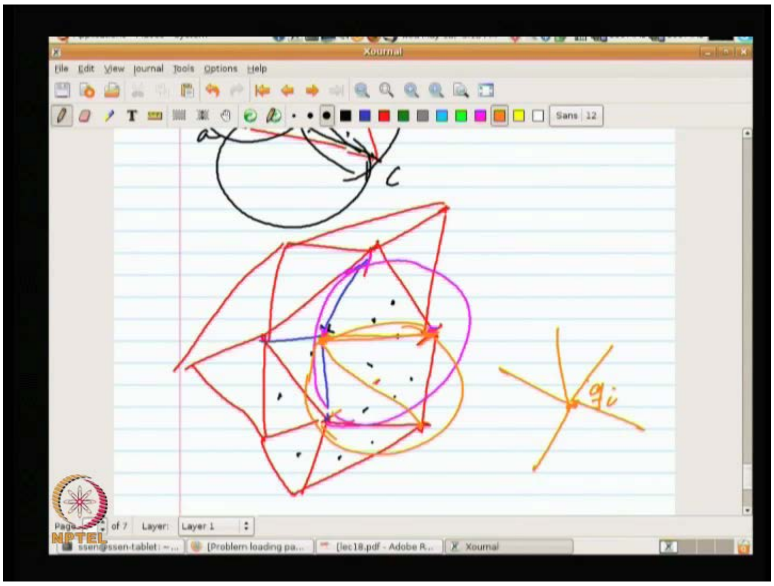

So, at some point, we may have the diagram like this. May be this, may be these are the points added already and this defines a Delaunay triangulation. These are the points that are not been added yet. The next point to be added may lying here. So, what we do? Let me draw few more edges. So, suppose this is the next point to be added, then we do what I exactly said before, we first draw these edges, but now there is a problem that you know if I draw this circum circle, it may contain some other point like it contains these points.

So, in that case, what I am going to do is, I will try to restore the Delaunay property by flipping this edges. So, this edge will be flipped. This is the new point to be added. So, if I flip this edge, I get this. Now, I have these two new triangles. Now, once we have new triangles, you may have, again if you look at this circum circle of this, it may again contain this and this may be violated. So, I may have, may be force radius and I add this. So, this process is going to go on till my triangulation becomes legal and notice that every time I flip a diagonal, one of the end points of diagonal is the new point because if you do not consider the new point, the previous triangles are all legal.

So, it must, it is valid. Something it must include the new point. So, in other words, all the triangles that we are going to add because of this and this is the new point. Is this the new point? Then, all the new triangles basically will have the epics in this. So, this is the new point. May be this was q y that was added. So, depending on the degree of this point, you will have to be doing basically edge flips. The number of edge flip will be equal to the degree of q i. Once we have actually completed legalizing this that is now up to from q and q i, this is a legal Voronoi triangle triangulation. Sorry, legal Delaunay triangulation and this process goes on the next we add q i plus 1 and repeat the process.

So, at every step when you add the next point, there is some work to be done depending on what is the final degree of the point. Clearly you know the degree of the point can be an n. So, in the worst case, we may be doing about order n work per i iteration which means that to add all the n points, we will be doing about n square work, but this is not the right way of analyzing this kind of algorithm. That is why I said this randomized incremental construction is a technique where we actually take advantage of act. That is not an arbitrary.

It is not a worst case sequence of point, but you know somehow the point that is to be added next there is some kind of an averaging arguments you can see it as follows. We will actually as I said in the later lectures, we will formulize this whole arguments that you look at the Voronoi diagrams that contains q. If Voronoi diagram, sorry Delaunay triangulation, a Delaunay triangulation is after all a planer graph and in a planer graph, the degree of a vertex is constant. The average degree of a vertex is constant. Therefore, when you add these points in random order, the expected degree of qi are constant order one. Therefore, the expected number of edge flip that you do in i iteration is constant. So, that is one of the things.

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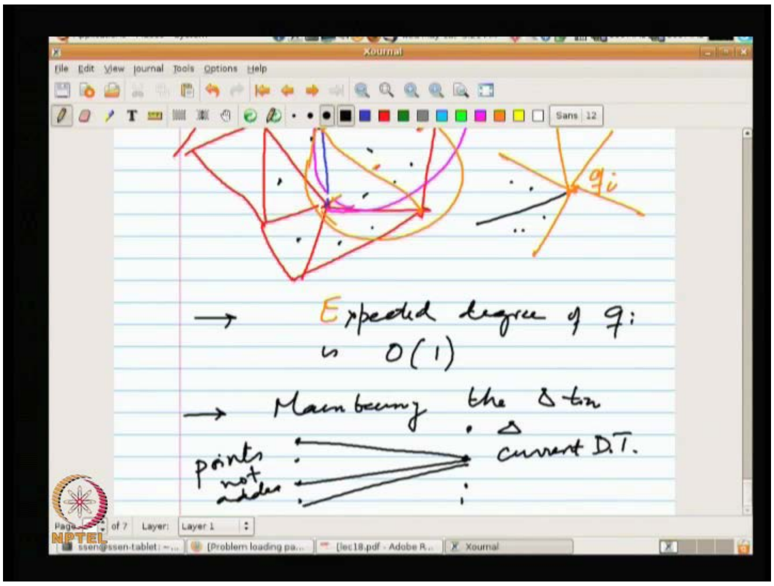

The other thing is that whenever we also need to worry about the points that lie within the triangle. When I picked up q i, I should know that q i lies in a certain triangle. So, I also have to maintain the triangulation. So, there is other job of maintaining the triangulation, so that when I have a new triangle. So, whenever there is a new triangles. So, whenever I add a new edge, you know the points have to be redistributed. So, I am always going to keep track of which point lies in which triangle. I will maintain a kind of a bipartite graph where these are the points not yet added and these are the current triangles of Delaunay triangulation.

A point I can draw edge between the points in a triangle if the point is contained in this triangle. So, of course, a triangle could contain many points and as you define the Delaunay triangulation, this graph is going to change. The amount of change is that will happen in the graph. Again you can bound it, of course, requires some proof that the amount of change that happens in the graph is expected to be about. Well, it is not quite constant or log n over the entire sequence of this insertion is going to be about n log n and that gives the order n log n running time of this incremental construction.

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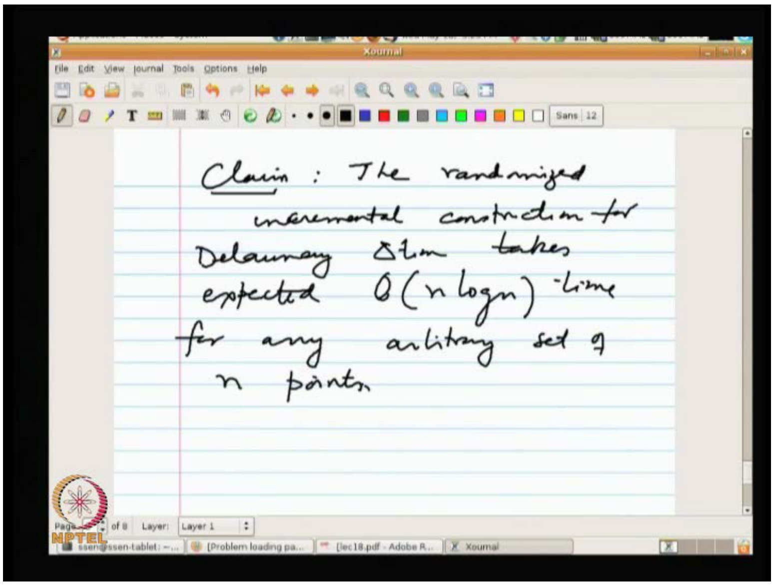

So, to summarize, so claim the randomized incremental construction for Delaunay triangulation takes expected order n log n time for any arbitrary set of points. That is very important. We are averaging over the set of points. We are only averaging over this sequence of insertion. Of course, a particular instruction sequence can be bad, but when we average over all possible instruction sequences and that is why we choose a random permutation. Then, the expected running time is only n log n which is as good as doing the lifting transform or doing any other algorithm because this also happens to be the optimal time for constructing Delaunay triangulation for any arbitrary set of n points.

So, what we basically looked at today was, we looked at some very basic properties of Delaunay triangulation. That was useful to actually define the prepositional Delaunay triangulation as well as also come up with some preliminary, but not very efficient algorithm for a Delaunay triangulation may the edge flip and algorithm. Then, we looked at another algorithm namely the randomized incremental construction which leads to an optimal n log algorithm where you maintain the Delaunay triangulation inductively by adding one point one after the other. The addition of the points is not done in some given sequence, but you actually permute the initial sequence. So, you had them in a random sequence and then you can prove that the expected running time of this procedure is n log n. So, that is all for today.