Computational Geometry

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Module No. # 10

Arrangements and Levels

Lecture No. # 03

Arrangements (Contd.)

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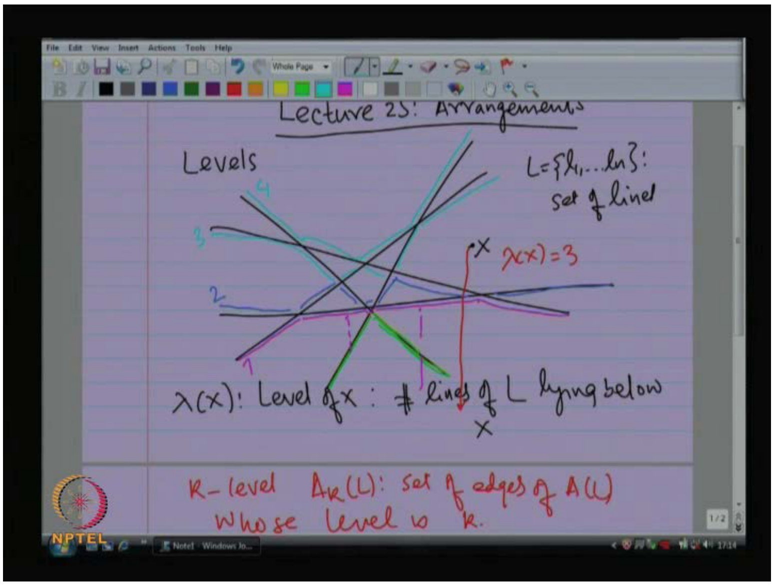

Let us continue. So, we will continue talking about arrangements today. And today I want to introduce a new concept called the conversion of levels, so again if you take a set of lines, so the level of the point. So, if I take any point X then I will define lambda of X, which is a level of X is a number of lines of L lying below X.

So, what it means is that if I take we will take a vertical ray downward, count the number of lines that lie below that point, I do not count the point itself. So, line if a point lies on the line, I do not cannot count that line. So, think of this is a vertical ray that I drew is an open ray. The starting point does not count. So, in this case for example, lambda of X is 3. I am right in this case.

Now, k level which I will denote as A k L is the set of edges of A L whose level is k. So, if I look at this picture. So, let us look at the what the levels look like in an arrangement. So, this line this edge if you look at this is level 0, because there is no line. If you all the points on this edge the level is 0, because there is no line drawn similarly, this edge also as level 0. Now, if I look at these edges, if I look at these edges they are level 1 because if you take it any point on any of these edges, if you draw vertical rate it will intersect exactly 1 line.

So, for example, if you take it here 1 line take a point here one line. Now, if I look at these edges this is level 2 and so on. So, now, I will draw level 3 and finally, this is level 4. So, what I did was if you look at the edges of the arrangements, I layer them these the partition them into levels and layers levels are polygonal change each level. You should look at each level it is a X y, X monotone polygonal chain, it starts from the left infinity and it continues rightward and finishes in the right end.

So, these are the levels and if you have n lines they will exactly n minus 1 levels because its 0 1 up to n minus 1 edges. Now, it’s also clear by the definition that each of this levels is a of course, it’s a polygonal chain. So, if you notice the way I do it and it was not accident that I follow if I want R o look at a level I follow an edge. So, I followed this edge and when I reach a vertex, I switch I was continuing on 1 line and I switch to another line. So, I then basically either I switch to this 1 and then this 1 and then this 1 and so, at every intersection point I reach I always switch. So, that is how it happens and its not easy its not hard to see that the levels are X monotone chains they will never go backward they will always continue going forward.

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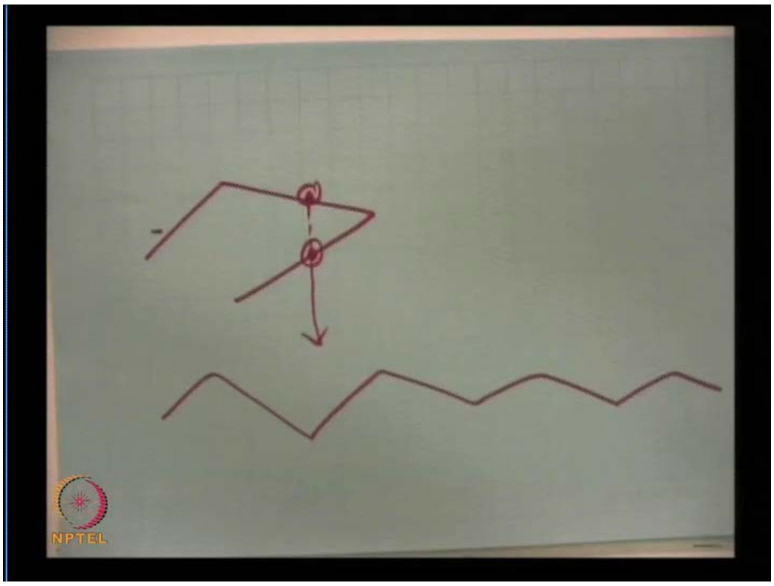

Because if you have a level which goes something like this, then you have a contradiction because what I am sort of saying is that these both of these points, has the same level and if you draw the ray starting from this point it intersect at least 1 more line. So, they cannot be the similar. So, the levels are always X monotone chains.

Now, what we know is the total number of edges in the an arrangement is n square am I and there are n levels. So, the average length of a level, if I took at random level or what is an average, the size the number of edges in a level will be linear because there are n levels and the total number of n square edges and each edge belongs to exactly 1 level. So, they only average size of the level is linear.

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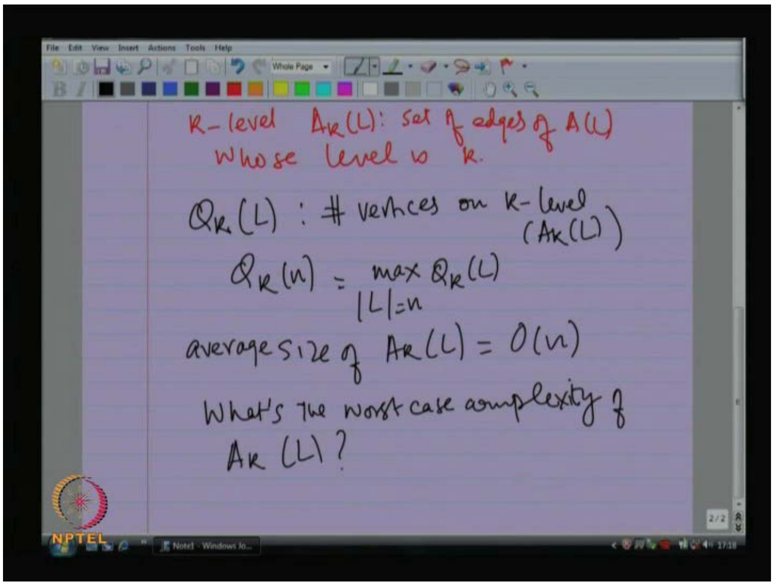

So, let me before say that let me see the sort of the following things. Let me denote (( )) phi k L is a number of vertices on k level. So, this is at the number of vertices on the level and that is defined about the complexity of the level's and what I said on average and as usual, define phi k n is a maximum of phi k L or over all lines or n lines this is the maximum complexity of A k level. So, what I what I said was that the average size of phi k L, average value or in an average. So, what I said the average size of A k L is the k level is linear, but the question is what is the worst case complexity and people believe that it is nearly linear, but it is not known and it is a big open problem.

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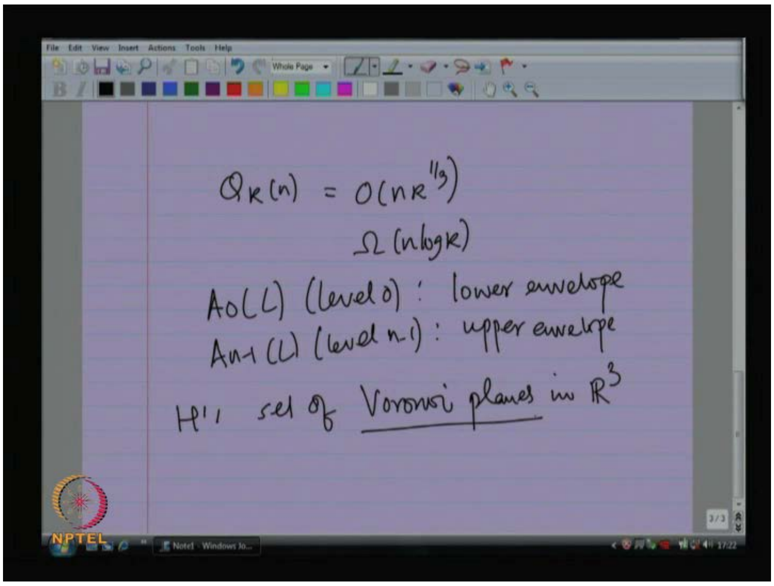

So, let me sort of say what is known is of following that phi k n is this is upper bound that is known and what is known, that it is n log k and this is also a about 10 years old result and 1 of the big open questions, in discrete geometry is to give a tight bound on the complexity of a level. Now, one thing I should say is that if you look at the level 0 which is a green line in this picture, green polygonal chain then 1 thing we notice is that A 0 L level 0 is nothing, but the lower envelope and if you look at the final level A n minus 1 that's a upper envelope. So, if you look at the final level the level 4 on this picture, then it is the upper envelope of these lines and if you look at the level 0 that is the lower envelope and remember that we talked about lower envelope and upper envelope in the context of Voronoi diagrams.

Now, the I have drawn levels for an arrangement of lines, but one can talk about in the same way one arrangement of one can define the notion of levels for an arrangement of plates using the same way. This the definition is just remains, the same instead of lines number of planes that lies. So, you have a set of planes. In 3 D the level of a point is a number of planes that lie below that, that point and if you remember that what I had said was Voronoi diagram corresponds to the low envelope nearest point to vernal diagram and I had asked the question what is so, if you look at the k th level. So, suppose if lets go back to the question of Voronoi diagram and if you remember.

What we had done was we started a set of points and mapped them to a set of planes and if you look at those planes, that was the tangent to the paraboloid and if you look at the k level in the arrangement of those planes. So, let us call them the following H is the lets is a set of let us call them as Voronoi planes. Voronoi planes also basically. What if you remember correctly, what we did was the following and I will draw the picture only in 2 D.

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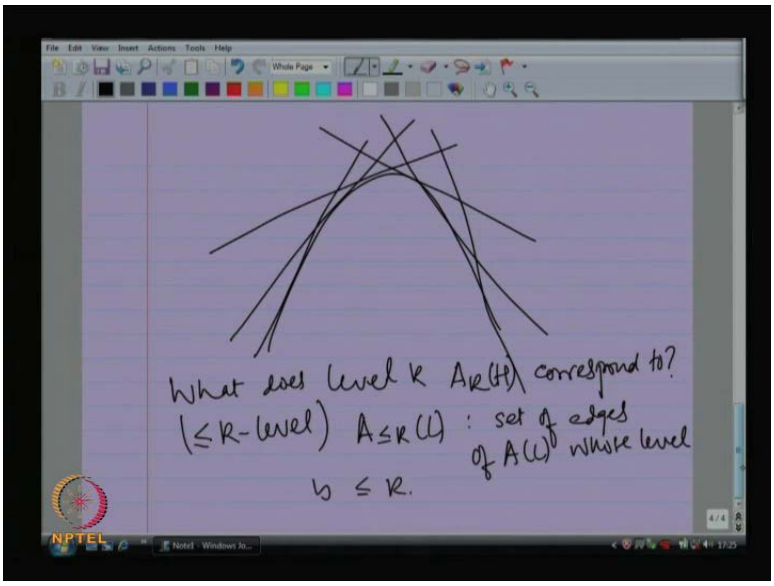

So, just think about the picture in 3 D that you had basically, instead of parabola you will have paraboloid and then you will have planes tangents to it and the lower angle corresponding to Voronoi diagram and now, if you look at the k level.

What will that corresponds to so, what does level k or A k H, it is A k th nearest neighbor. So, remember that what is Voronoi diagram? Voronoi diagram is a decomposition of a plane. So, the within each cell the nearest neighbor is the same that a Voronoi diagram k th nearest neighbor k th order that is called the k th order Voronoi diagram, it is a decomposition of the plane in which the k th nearest neighbor remains the same, so, if you take the level k and project it on the plane because in 3 D. When you take the k level in 2 D, it was a polygonal chain in 3 D it will be polygonal surface and take the polygonal surface project it on the plane.

You will get the k th order Voronoi diagram. So, that is 1 of the reasons, why the levels were studied because people were interested not only the nearest neighbor Voronoi diagram, but higher order Voronoi diagrams. So, we care about not only what happens to the nearest neighbor, but happens what happens to the second nearest neighbor and so on.

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No, it will still it will still bounded by lines k th bound because what happens is it, is a it still a plane, that you have planes set of planes when you intersection of planes is a line, when you project it you will get a line.

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No, cell will also still be a convex, but what will be the case is that a point may have multiple cells. So, each cell will be convex, but a point may have multiple cells. So, now, what I want to do is I do not know, what is the complexity or exact complexity of a level is and I said this notion that on average it is linear size. So, I need to I want to introduce the notion. What is called at most k level and you will see in a minute, why I care about this. So, so if I want to look at that at most 2 level not only look at the edges on level 2, but also look at the edges of level 0, level 1 and level 2.

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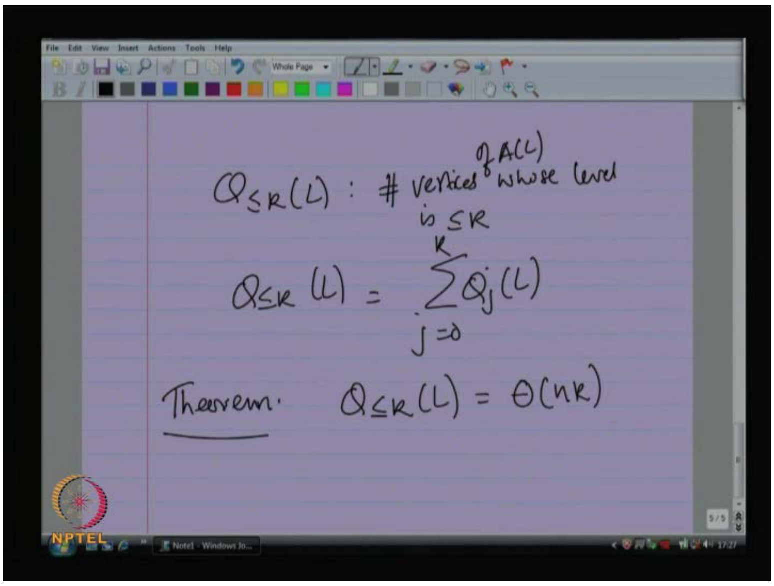

So, that is a at most k level and let me again, define phi k L is the number of vertices, whose level is less than k. So, the phi k L is nothing, but see look at the vertices whose level is at most k and the theorem is the following is the order on n k and actually, it is theta. So, this sort of is also verifies. So, supports a claim that the average size of the level is k because I cannot bound the complexity of a single level. Where I cannot bound the complex single level, but if I take a bend from any value up to k lets take that the first k level, then the complexity of the first k levels is at most an k and I am going to give you the proof of this theorem because not only its it is a interesting it is a own right, but the proofs that I will give you is a very general very beautiful technique, in the random sampling.

It is used in many different applications not only in geometry, but beyond geometry and especially, when you dealing with the problem with the clustering in out layers and similar, arguments are being used. So, so and the proof is cute and short, so I want to show you that how this 1 proves this theorem, but before I prove the theorem any questions about the notions of the at most k level or the notion of the level.

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The intersection points, the vertices intersection points because these are the vertices arrangement. So, these are the vertices.

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Complexity and number of vertices when I say complexity of something it means, how many vertices are there.

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You will see that why I care about the at most k level, because it is a 1 thing I said it is a same as A k th order Voronoi diagram you asked the question, you ask is What is the complexity of the k th order Voronoi diagram? We know that the complexity of the nearest neighbor Voronoi diagram is only linear, but how big the k th order Voronoi diagram will be and that is not linear, it is a it can be much larger and it can be you will, you will see that is a and you will see in some other why, why we care about at most k level.

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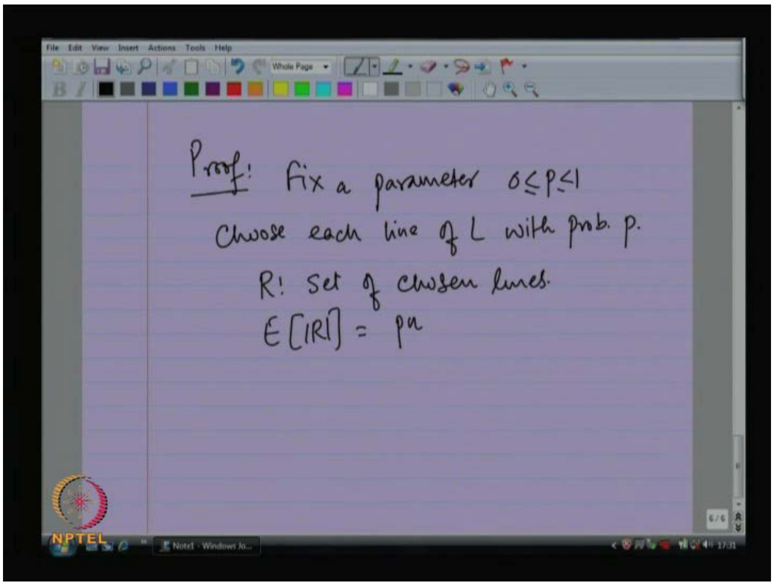

So, the proof is based on the random sampling argument and I will do the following experiment. I will fix a parameter p and which I will choose later and I will for every line, I will flip a coin a byes coin, which as a probability of being getting of head of probability of getting head is p and if I get a head I choose it. So, what basically what I am sort of I am saying is that I flip a coin for every line and I choose a line with probability p. So, fix a parameter, choose each line of L with probability p and let R be the set of chosen lines. So, what is the expected size of R, p times n am I right there is nothing deep there now. So, what happened was that you had a set of lines and I choose subset of these lines. Let us say suppose I this line was selected this was selected and lets say this was selected.

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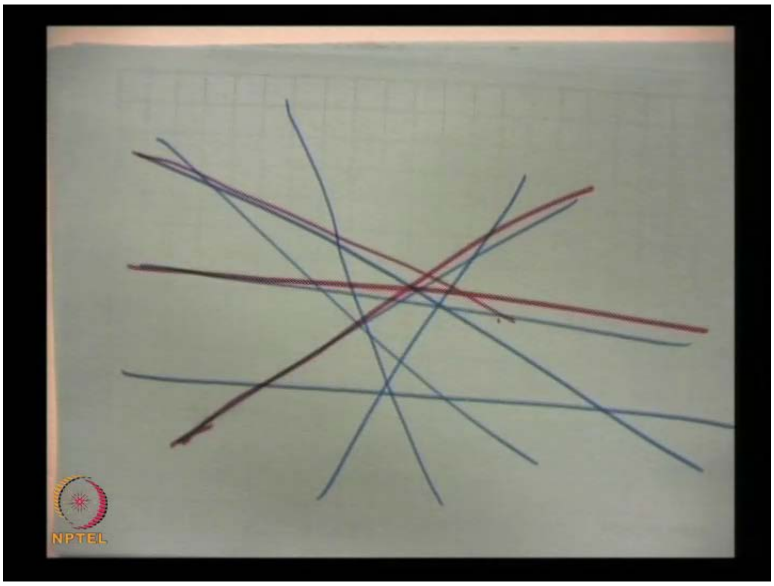

Let me try again I hopefully I will do better suppose, this 3 line were chosen. Now, what I am going to do is I am going to bound () the look at the. So, that is a level 0 of this set up R. So, look at the so, that is the level 0 of R.

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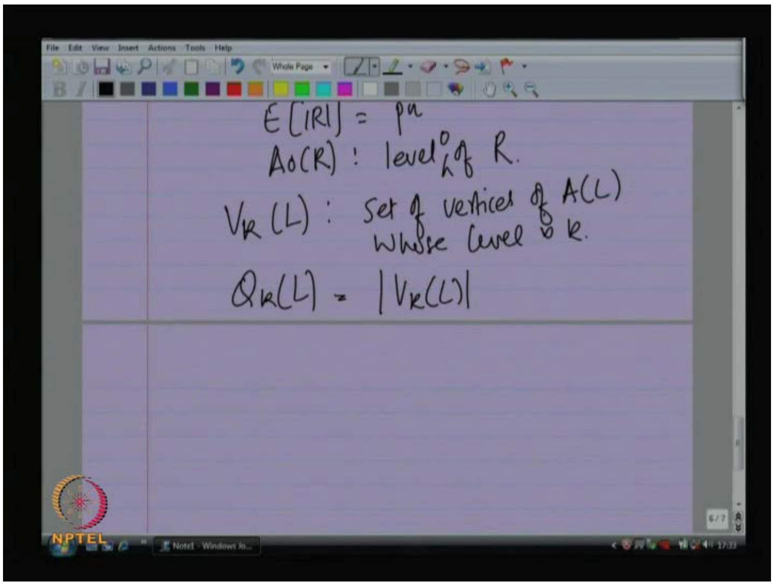

So, it means that this is the level in this case and let me do the let me introduce, the following notation. It will be just easier let V k L be the set of vertices of A L whose level is k. So, so let me and remember that the notation was an 5 k L is nothing, but the size of then.

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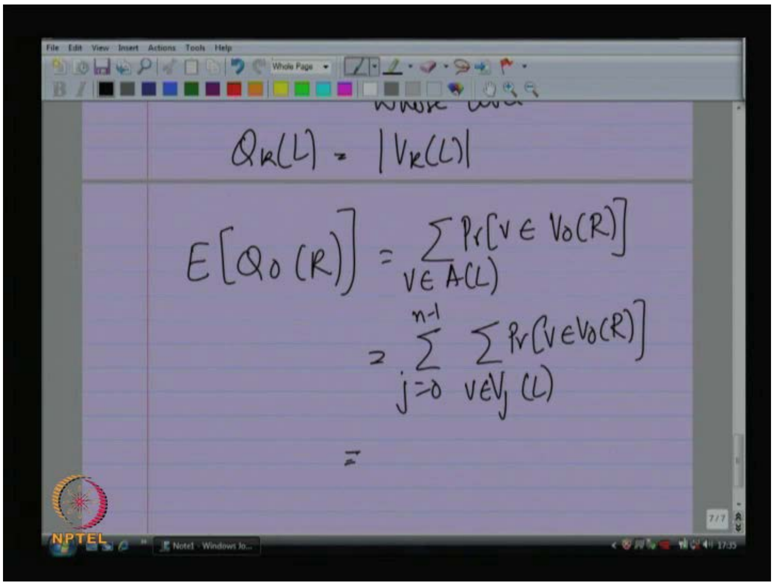

What I want to do is I want to look at the phi 0 R, the number of vertices on the low envelope on the random subset of the lines that I have chosen. Now, and I want to look at this expected size because it is a random subset. So, the phi R is the random subset and phi 0 is a lower envelope, which is random quantity and I am look at its expected value. So, it is expected value is nothing, but you look at all the vertices in the arrangement and this is basically, ask the question what is the probability v R belongs to v 0. Because that is a definition of expectation, that vertex on the random subset on the lower envelope, these vertices this is nothing, but the vertices of origin arrangement for each vertex, you ask the question what is the probability and because the vertex is chosen or not chosen will be there. So, it is a 1.

So, normally you say that p x in the value of x, but the value of the thus it is indicative variable. So, it is just a question of probability. Now, this what can I do is I am going to write it in following fashion. So, I took the set of vertices divided them into the levels because if we look at the all the vertices in the origin arrangement, they belong they have certain level as we talked about earlier and this is notation I used. So, far this is nothing deep has happened.

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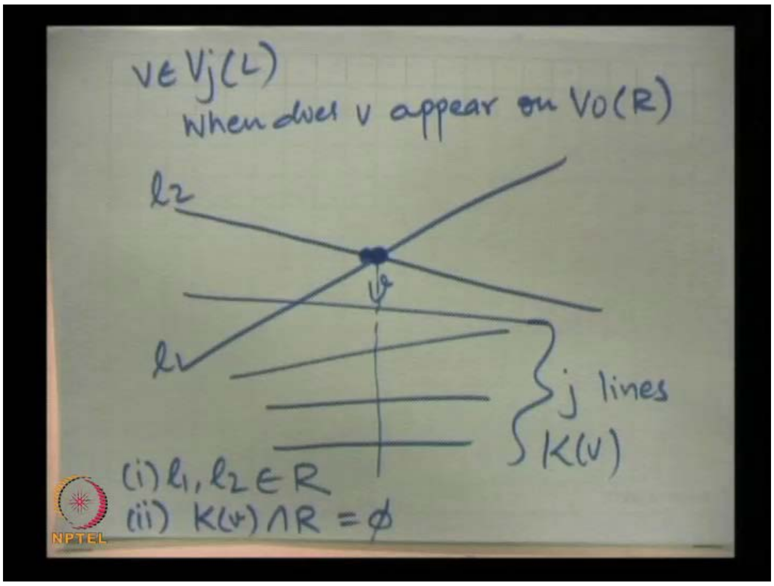

Now, let us ask the following question here you have a vertex of the origin arrangement, which lies on which is an intersection of point of 2 lines and let us say its level is j. So, so let us ask the following question. So, let me write the following question suppose, you have a vertex v and v belongs to V j L it is a it means, its level is j. So, when does v appear on. So, what is a when does v appear. This vertex appears of the random subset of lines yes. So, so what is the problem, when does some arbitrary vertex will appear here.

So, the 2 things have to happen. So, let us say. So, since its level is j then if we draw the lines below here there are j other lines that lie below it, 2 things have to happen for v 2 appear an envelope that these 2 lines should have been chosen and none of these lines should have been chosen because this any of these lines is chosen.

This vertex v will not appear in the lower envelope. So, let us write this. So, let us look at this call this line as l 1,l 2. So, one condition is that l 1, l 2 they are chosen in R and let us call this line as the killing set in the vertex because if any of these line is chosen. Then the vertex does not appear. So, these are the lines that kind of kill them appearing on this one and second set is K v is empty because none of these lines should have been chosen.

So, if you take an arbitrary vertex of the arrangement, blue vertex which is an intersection point of 2 lines, if its level is j then there are j lines that lie below this vertex in order for this vertex to appear on the low envelope, on the random subset of lines or these 2 lines should have been chosen and non of these lines would have been chosen. So, what is the probability of this happening not not p (( )) to the power j.

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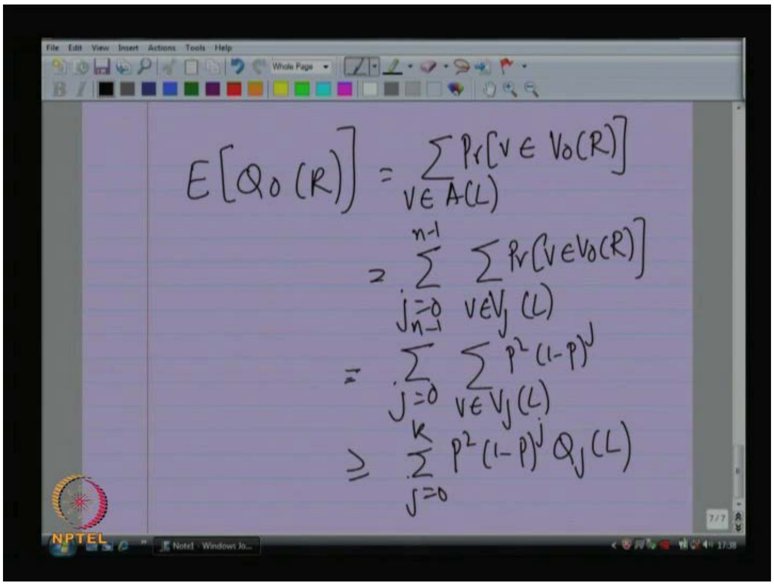

So, then basically we just write it this, this comes from the probability of this happening is p square this condition, this condition is at none of the line is chosen. The probability of the line not been chosen is n minus p and these are j lines. So, it is j. now, that is a main thing and now, I will do manipulation.

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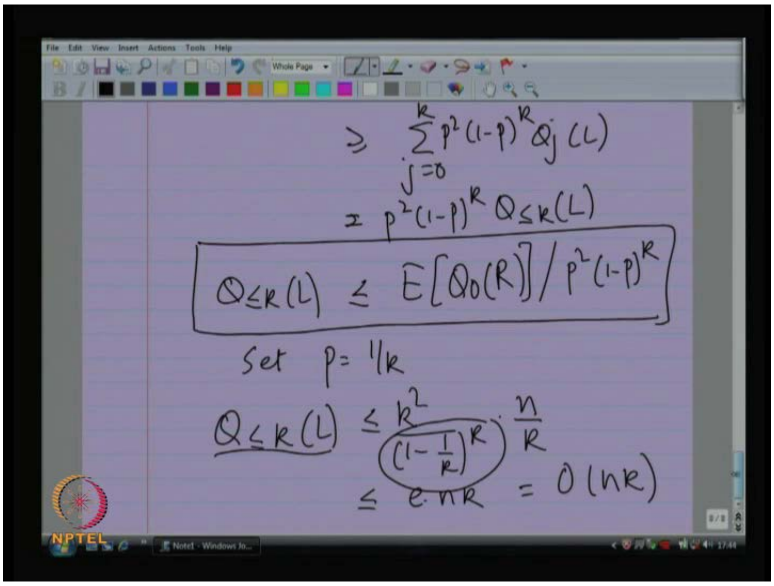

So, this is one can write it as j k. Now, this probability depends not on individual vertex it depends. So, this is p square 1 minus p to the power j number of such this basically, multiplies the number of vertex in the level, which is by definition is precisely 5 j R. The size of the j this one can write because k is at j is at most at k. So, I can write this way phi j L and I can take this first 2 terms outside. So, it becomes p square 1 minus p and this quantity j is equal to 0 to phi j L is nothing, but phi k L by definition. So, this was the definition of phi k L at most. So, it means is now put it other way phi k L is at most excepted value of phi 0 R over E square 1 minus p to the power R.

Now, what we do is we set p to be 1 over k then, what you get is p square becomes k 1 over p square. So, we get k square k. Now, what is the excepted value of the low envelope. Now, if you have set of L lines when you have set of whatever, lines you have low envelope is remember it is nothing, but the convex hull and you know the convex hull as the linear complexity. So, this is also a convex polygon, which is intersection of n lines and you know that its size is linear. So, its size is linear am I right. Now, what we know is that excepted size of R is p n, you said which is n over k. And size is linear.

So, we get n over k. So, what you get is what is can you give a bound on this 1 its roughly 1 over E. So, it becomes E. So, this is becomes E times n k. So, which is order of n k. So, notice what I did here. Now, I never use fact these were lines only thing, I used to bound this quantity was this 2 quantities this is 2 things were true that this that, I try to count this number of vertices and this number vertices were intersection point of 2 lines of 2 objects and the probability was that these 2 objects should not have been chosen.

So, these 2 objects should have been chosen these killing its killing, set that one has not chosen and one can define this an abstract frame work that you have a set of objects one can define for every object set of which is being derived from these object, geometric objects that what as called as defining set because this 2 lines defined this vertex one can talk about defining set and one can talk about the killing set and then, you can ask the question what are these probability of the vertex is been chosen (( )). So, this is very general frame work and other thing is the random sampling argument.

So, now, one can extend this argument also in 3 D and one thing you notice is until. The reason I wrote it this is the important part to think read because there is no randomization in this part, because this is a what we sort of, what is sort of what I prove to use the following this is what I prove to you this quantity. There is no there is no randomization in the part of L. The only thing was that is a that is a and this is what I am sort of saying that this is, that is why say it is a beautiful technique that randomization technique is being used to get an worst case bound on some geometric quantity, material quantity and that is what is sort of very powerful probabilistic argument. Now, how many of you seen probabilistic arguments to prove.

So, typically probabilistic arguments are used for intentional proof or to prove some probabilistic methods to show whether, the something is true or something is not true here. The probabilistic argument is being used to count certain things and it is used in many different sort of this technique is used in many different domains this.

So, what I want to say this is a important part equality. So, if you choose p to be k then I do not put this quantity of 5 0 R then, what one can write is the following.

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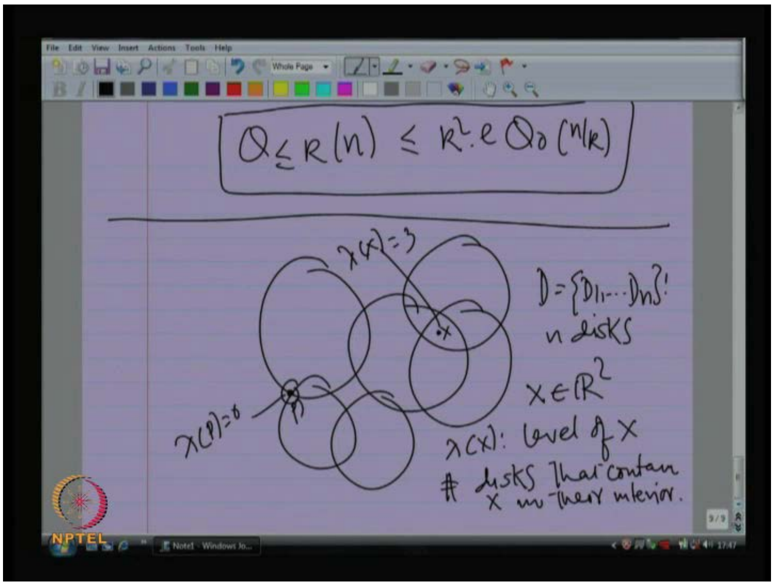

So, if I look in the worst case bound, so this is 5 k n p. I will choose 1 over k. So, its get k square and this is E I will just note this part, a square times E and expected I will replace it in the worst case bound because size of R is n over k. So, what you get is basically. So, if I think of an abstract frame work. If I have this notion of levels 0 or level 2 up to 1 k then basically, what you say is that. If some objects are being hidden by k objects, then this is the bound to that and I will, I will give you an application that why in more general static, but any questions about the proof.

It takes a while to digest this proof because it is a as you said it is a kind of contentive because I am using randomization to bound something that not randomized, this is deterministic in worst case bound. Let me give you one sort of example of this generality of this frame work and then I will give you 2 applications that, why we care about levels.

So, suppose you have set of discs. So, let us call D, n discs and I will define a level of a point and I will define level of point X this is a number of discs that contain X in their interior. So, for example, if I look at this point X its level is 1, 2, 3. If I look at this point or it lies in the boundary of 2 discs, but it does not lie in the interior of any disc. So, that is why it is a level 0. So, what is level 0, in this case say it again say yes. So, you look at the boundary of the union boundary of the union is level 0.

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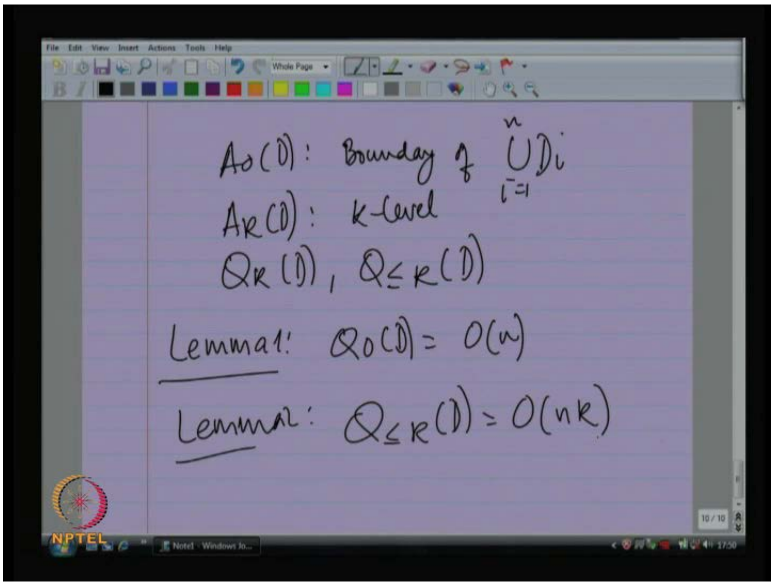

So, so again if I define the arrangement and I can talk about A 0 D is a boundary of the union. Now, you can talk about similarly, A k D and we can is A k level and again, I can talk about phi k D and also I can talk about phi at most k D at most k level. Now, first Lemma which I will tell you for now, you should believe me that if you have a set of discs. How many vertices can appear on the union on the boundary of the union there are n square vertices. There are because the total in the arrangement is n square vertices. How many of them can appear on the union does anyone have a guess pardon.

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10 n, 2 n any 1 else. So, how many of you believe it is a order of n. So, how many of you do not believe that it is not, how many of you believe it is more than it is a super linear, come on you have to have an opinion. This is democracy parden should be linear. Yes, it is linear and either I will give you home work problem, if I do not have time if I have time then I will give you the proof. So, phi 0 D is order of n then. Now, I can use this result to prove that the number of vertices that lie in the interior of at most k discs is only n k. And the proof is identical to what I gave here because again

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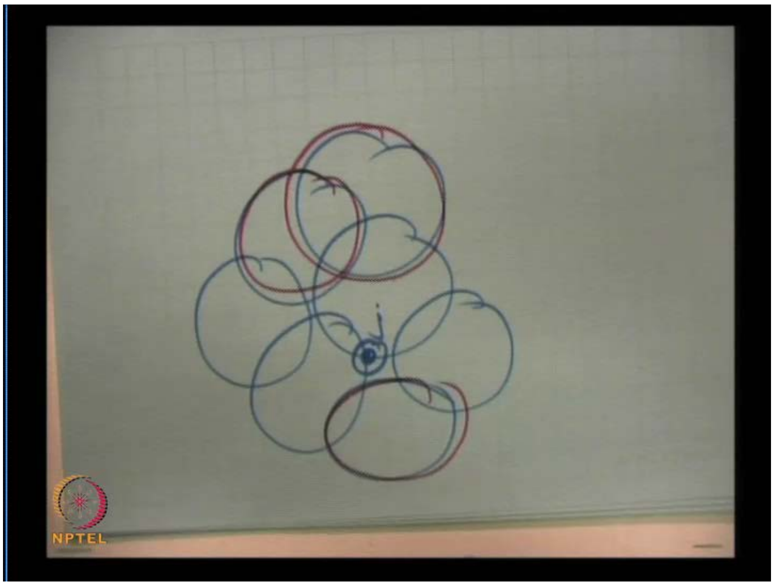

If you look at the look at instead of lines, if you look at discs, then if you and I choose a random set of discs let suppose, I choose these random discs and I ask you a question. What is the complex, what is if I choose a vertex of the arrangement of this blue vertex ask the question, what is the probability that it appears on the boundary of the union of the red discs and if its level is j it means, it lies interior of j discs.

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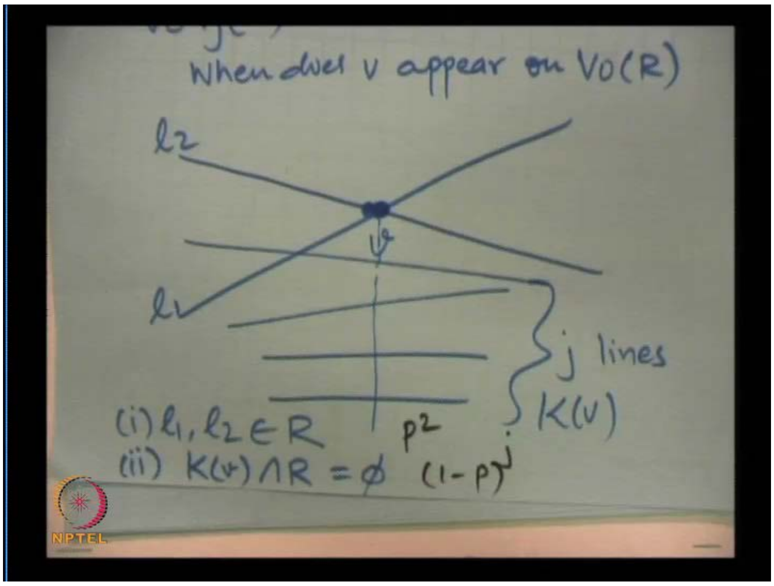

Then again, the same thing condition should happen is that well you remove from the lines. You remove it to say the 2 discs whose boundary defines this vertex, they should be chosen and none of the discs that contains this vertex in its interior their interior, they should be chosen and the same thing goes on. So, so basically in the same proof basically, just goes through and I will get to this. What I have written in the bracket square this line because actually, happen is the number of vertices whose level is at most k will be then the number of vertices on the union of the random subset times k square and now, I know by Lemma 1 by Lemma 1. I know that the boundary of the union has only linear number of vertices.

So, again what I can write here the way I wrote it here that phi 0, phi 0 R is n over k and the same thing will go through and that will do it. Yeah, there are other ways also, but this is a that cleaner way, nice way of doing it. Now, another that I give these 2 is stuff. Now, let me give you examples why we care about it.

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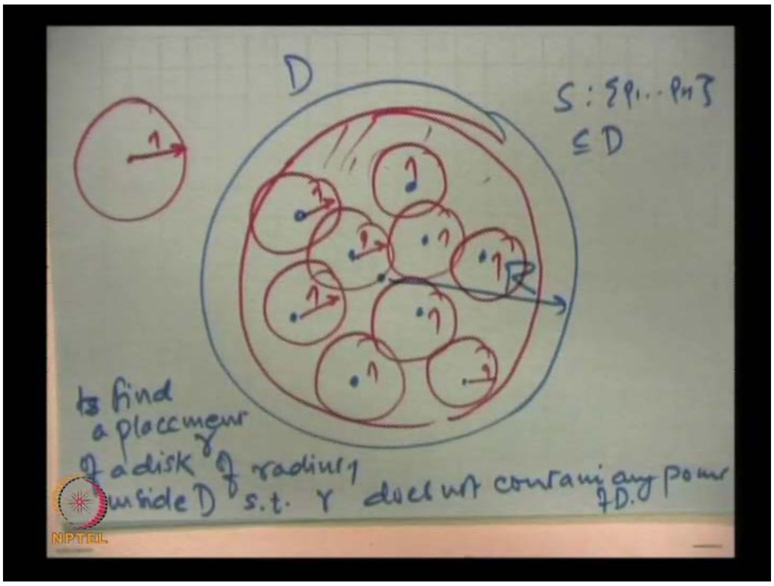

So, let us look at the following problem, which is suppose, there is a big disc which is some kind of region and you has a set of points inside this region. Let me start ask, start asking the simpler question. So, this supposes this is radius some big R, find a let us call this 1 as big D placement of a disc of radius 1 inside D such that, R does not contain and let us ask s is the p 1. So, I have the question, I want to ask is there are let us say disc of radius one can I place a disc of radius 1 inside this inside the bigger disc. So, that it does not contain any of the input points. So, think of the point as a sort of obstacles. So, that disc cannot contain any of these input points go ahead and then that is right.

So, what you do I said to reduce this 1 to R minus 1 and draw a disc of radius 1 around each point. And if there is any and then, what you do is you look at inside this bigger red disc and outside and come to the union of this disc radius 1 disc and see there is a point there.

If I find a point then it means, if you put a disc of radius 1 there it will lie inside the disc and will not contain any input point. Now, we know that the union of disc as a linear size right and I have not told you if I compute it, but when I give you the proof. How you, why it is a linear size? You will see the proof also how you compute it and then it will basically, do it. Now, if I change the problem something like this may be it is too stringent to require that, it does not contain any point that is it I allow it to contain k points that is why. So, it means that we want to compute at most sort of k at most level k.

Now, let me sort of flip the problem. So, this was the placement, what is called as the largest empty placement or you can ask the following question instead of I give the disc of radius 1 you can ask finally, largest disc that I can place inside this D that does not contain any point in which case, you need to do some kind of binary search on the radius and see how you find the larger disc or you can find ask the question. What is the larger disc I can put inside it. So, it does not contain the any of the input point.

So, think of these points as a constrain that you cannot contain any you ask the question and when you allow some output liars because when I say at most k points versus k points a kind of out liars. So, whenever you have to deal with the out liars, then this at most k level will shows up. So, here is another problem, which is a kind of a dual of this problem.

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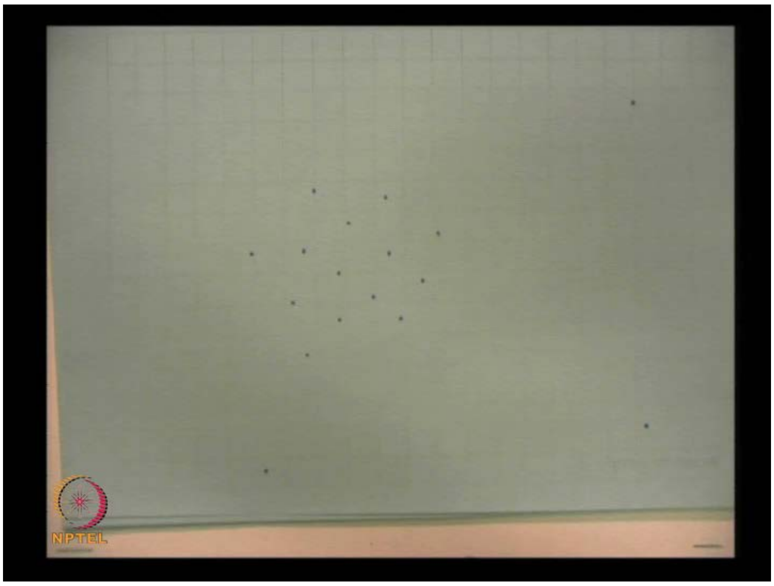

Now, I will give set of points. Is there are disc of radius one that contains all the points then what will you do its very similar, to what I asked earlier, but now I will ask a question is there disc that contains all the points.

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Intersection. So, that is the intersection. They do it now, what I ask the question again, I apply the out liar game some more time because there might be some out liars because most of the points are clustered here, but there might be some points because some measurement error something, they may happened which is like the. Now, we doing clustering you always want number of times, you have been out line. So, say I have asked the question is there a disc that contains all, but at most k point all, but k points then again I can define the level, but I have to define the level differently now or you can take about level n minus k, you can talk about. So, that is why the level at most k levels come here is a final problem that example I want to give you which is a classical problem, in classification and machine learning.

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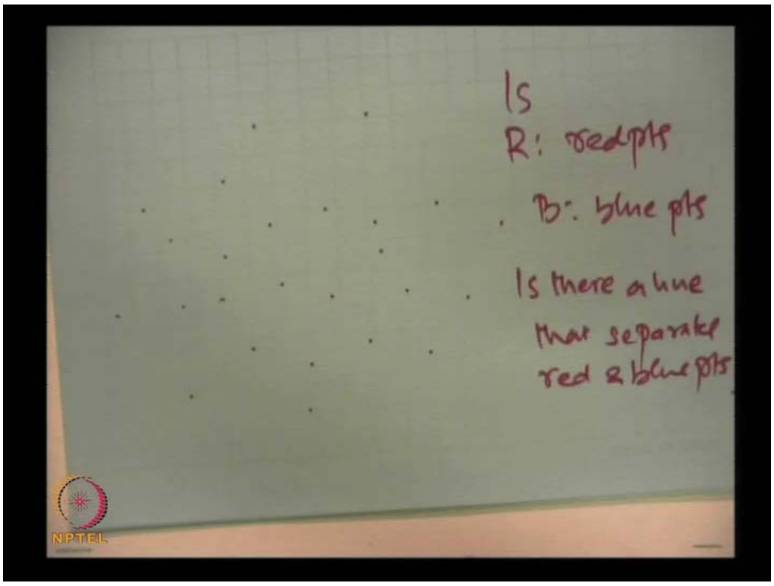

Is the following suppose you have some data, these are some positive examples and these are some negative examples. So, so for example, what happens is that, this is many it comes everywhere in the classification problem, but for example, like look want to look at them understand the disease. So, what you do is you have some symptoms, you think these are the symptoms that cause disease you want to check. So, you have certain defined some symptoms and then you look at the patients, who have the disease. So, the patients who do not have the disease. So, patients are the positive examples and the patients who do not have, that means, do not have the disease those are negative example and the question you want to ask is there a simple rule, that can separate them and the simplest rule that is what is used in machine learning classification called as linear classifier. So, what? that means, is there a line is suppose, this 1 a 2 dimensions is there a line that separates blue points from the red points.

So, the question I want to ask is there a line. So, R is a set of red points, B is the set of blue points is there a line that separates red and blue points.

Now, suppose let say the 2 possibilities if I align lets blue point, points lie above the red line above the line and red points are below the line.

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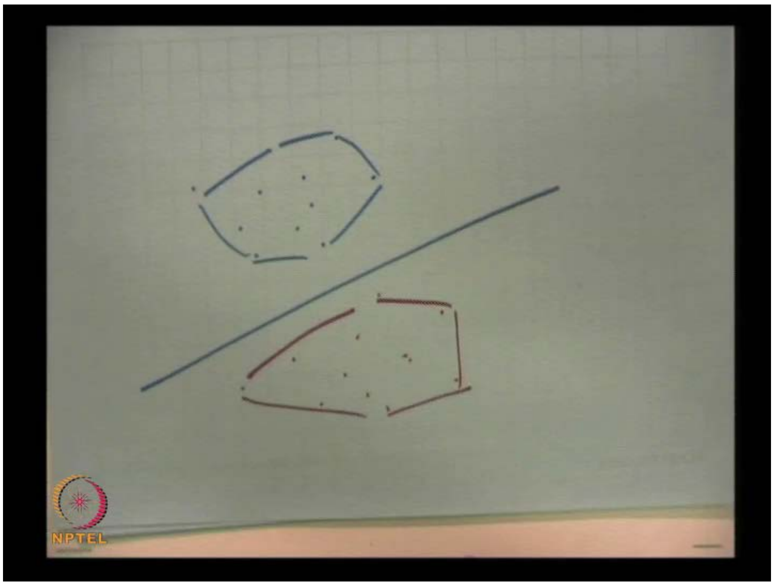

So, let us say suppose an our life was an very simple and you have a all the then, what instead of you want to check that what you, what you do is you compute the convex hull and you compute the convex of red points and you check with the polygon disjoints if. So, if the disjoint then there is a line that updates them. So, that will do it for you, but now again, there might be out liars.

So, it might be too stringent about what happens you cannot find a plane that is really separates them is a i allow k. So, where the side that contains the red points, you allow at most k the blue points you allow at most red k points in the side that contains red points, you allow at most k blue points then how do I do it. Think out duality if I dualism this red points and blue points.

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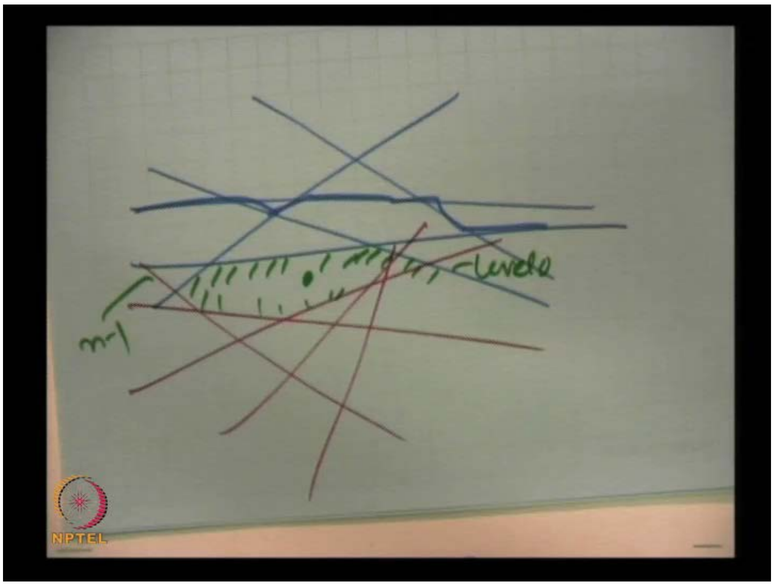

I am going to get a set of red lines and I am going to get a set of blue lines. If they are separable in this example when they are separable it means, there is a point that lies below the all the blue lines, and above all the red lines it means, it means as the below the level 0 of this blue lines and it as above level n minus 1 of the red lines now, but I allow some up to k out liars then instead of level 0 you want to look at level k.

You take the k level of the red lines n minus k level of the blue lines and ask the question there is a point, these are the 2 polygonal chains and the question you want to ask is that there a point is lies below this polygonal chain above this polygonal chain and that you can do. So, that is a sort of again we need the question. And the running time of this will be basically, depending on the complexity how fast you can compute those lines. So, that is I care about this are some of the reason, we care about the levels that happens any questions. Now, let me conclude by just give you I will just give you proving this theorem that the union of discs as linear complexity. I will give you 2 proofs, one you have seen it from which followed, what I have to talk earlier

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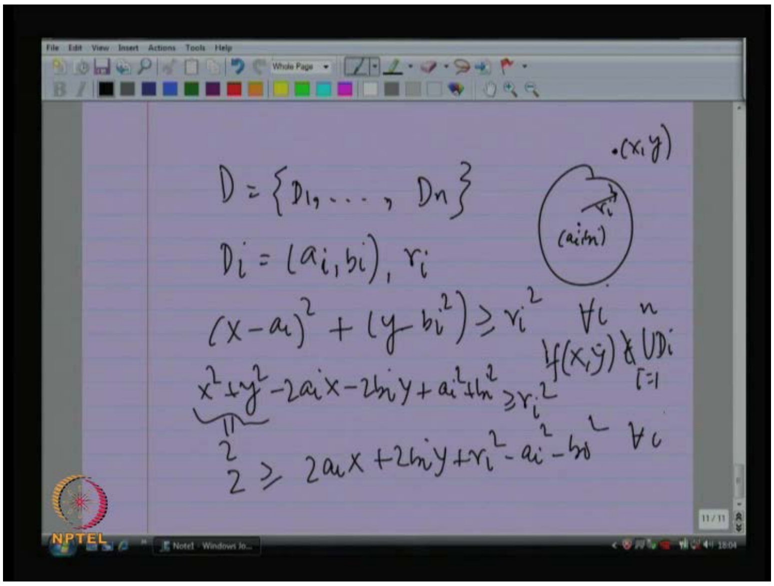

So, here is the sort of discs let say D i is a send a i and r i is its radius, so radius r i. So, the equation of the disc is x minus a i square. So, if the point as outside this disc it means, that the quantity this should be true for the point. For a point x y that lies outside the disc this inequality should be true and if it lies outside in the union, it should be true for all lines if a point lies outside the union this x. So, if you if you write this one as x square plus y square and you this one you map to see that our usual lifting transform then, what you get is sorry.

So, this is the equation of a plane and what it says is that, if a I took the circle and I took the disc mapped into a plane into disc mapped to half space in 3 D, this is the standard lifting transform that you have seen earlier, and if a point lies in the up stereo outside the union of the disc, it lies above this above half, half spaces and as you sort of seen in the intersection of an half a spaces as linear complexity as only linear number vertices.

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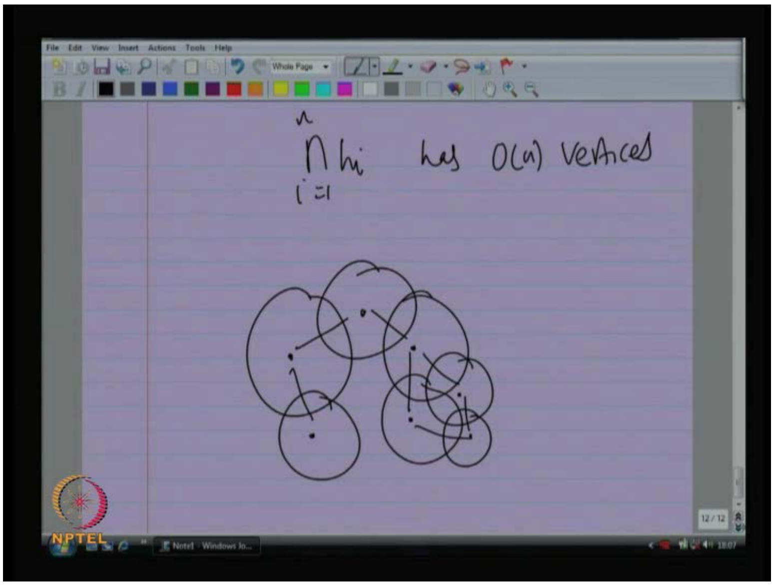

So, what we know is that let us call this as H i, then because this is a convex polytope in 3 D and its as a linear size that is basically (( )) and that implies that union of disc as linear complexity. Now, there is another proof which is lies and different proofs is the following. Suppose, you have a bunch of discs, you construct a graph as follows.

U in the vertices of the graphs are the centre points of the discs and you connect the 2 discs by an edge, you connect 2 centers by an edge. If the intersection of those 2 discs at least 1 of them appears on the bound on the union. So, for example, here you connect well let me do the following and this one also. So, this is the graph you get and the claim is at this graph is planer. I will not prove it is not very hard, it is higher school trigonometry, but using high school trigonometry, you can argue that this graph is planer and if a graph is planer it as only linear number of edges and that implies that union complexity has linear number of edges. So, that the reason that this second proof is nice is it does not use the fact that these are real circles.

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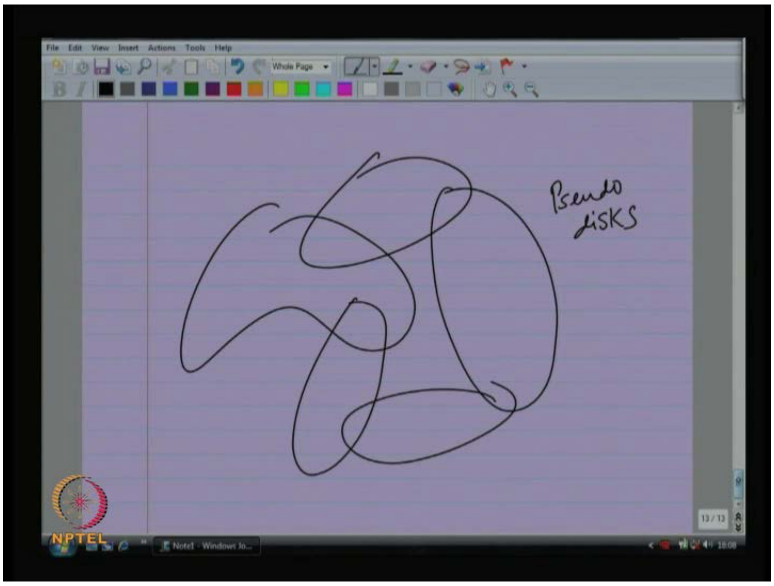

So, let me just conclude by saying the following suppose, you have some shape which is very strange, but it as the following property that if you take any pair of them, they intersect at most 2 points these are called pseudo discs and the complexity of the union of pseudo discs is also linear. The proof that I sort of the graph is planar, you can do at something similar here also sentence are not very well defined, but 1 has to be little careful and you can still say the graph is planar. And why this pseudo discs are important. Now, if you remember when I gave the motivation for the arrangements and one more thing I said was motion planning.

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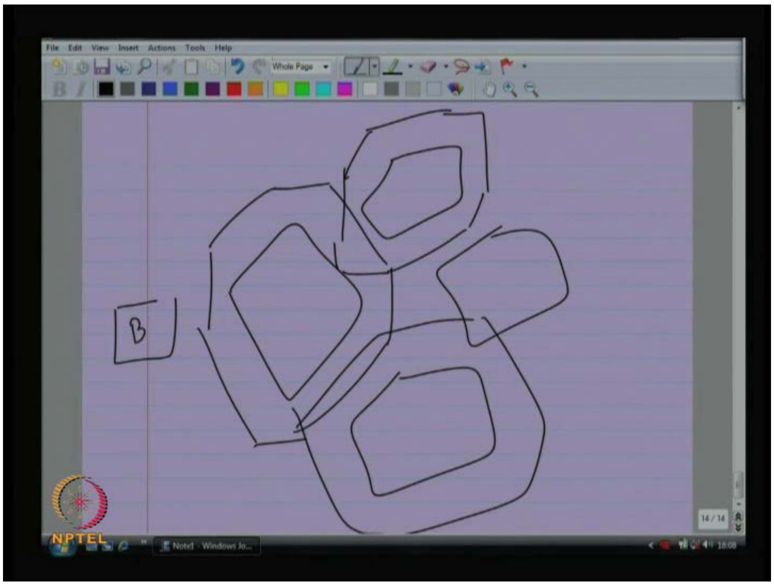

That you had the let say set of obstacles and then you had square B and when you when I generated this expanded obstacles, if you remember. Something like the obstacles you get, what you can prove is that. Now, obstacles they intersect because all the original polygons are disjoined, but after you expand them they intersect, but what you can argue is that, any pair of them will intersect and at most 2 points. So, these are pseudo discs. So, it means that you may looks sort of pretty nasty stuff after expanded, but the total number of vertices is still linear and you can compute them and that is what implies that you can still do a motion planning easily. So, let me stop here.