Riemann Hypothesis and its Applications
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Lecture – 13

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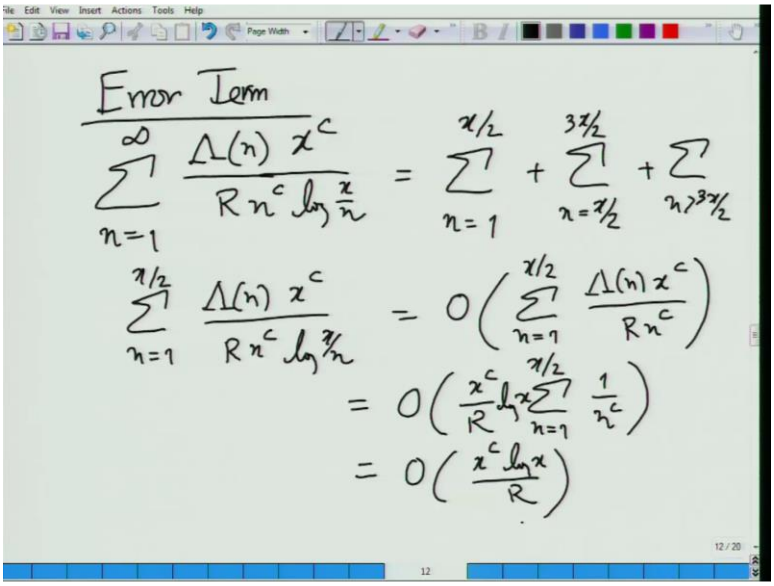

So, let us estimate the error term first in today, we had this is an error term. So, sum n going from one to infinity n by q at. So, here we have a very good handle on these values, this lambda n is bounded by log n always this is x to the c R, n to the c these are all very cleared. This is the slightly problematic on log x by n and the reason is that as n goes from one to infinity, when n comes close to x then this can become very, very small and therefore, the whole thing become very large.

We have to handle that aspect little carefully; otherwise bounding this is pretty straight forward. So, in order to achieve that let us plate this sum into three components n equal to 1 to x by 2, so everything reasonably below x, plus to 3 x by 2 and plus n greater than 3 x. So, the first and third are pretty straightforward to estimate just let see that.

So, here since n is always is reasonably smaller than x, the largest value that log sorry, this smallest value that log x by n will take is a constant, when n is exactly n x by 2 this would be log 2, which is the constant. So, denominator we would like to give make as small as possible. So, this is certainly order n going from 1 to x by 2 lambda n x to c divided by R into n c. And this we can this write as a x to the c by R, summation n going from 1 to x by 2, lambda n we can replace by log n at the most or make it even simpler lets take out log x.

Lambda n is always less than equal to log x take there are also 1 by n to the c. And this sum is what? Well c we remember is greater than 1. So, when c is greater than 1, this sum actually converges actually n going from 1 to infinity, 1 by n to the c convergence to some constant where is, so we just is.

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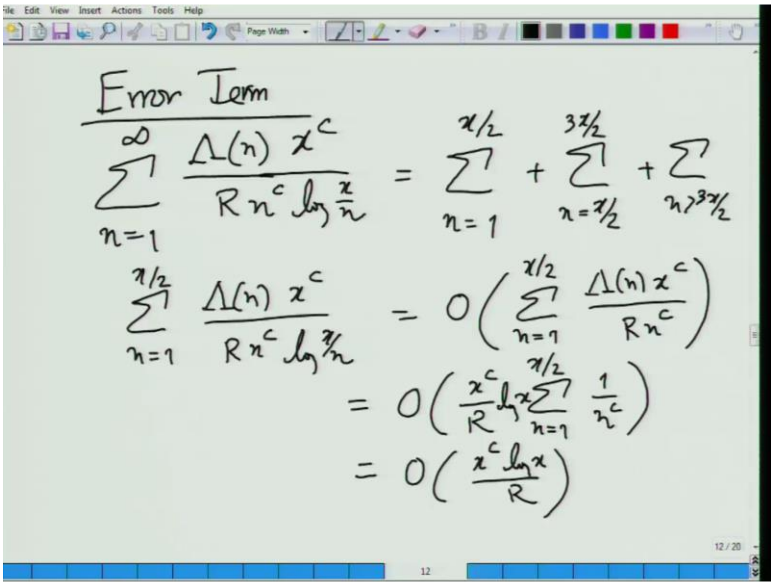

Similarly, if we look at this,, so here again this n is substantially bigger than x always. So, log x by n would be substantially as n increases log x by n will become a larger and larger negative number would larger and larger and here we are taking with a absolute values we did not worry about positive or negative.

So, again it will attain is minimum when x is a 3 by 2 x and again it could be a constant. So, therefore, we can again write this as correct and now here 1 is 2, 1 cannot substitute lambda n with log x lambda n q is keep increasing to infinity. This series the convergence will depends on exactly what is the value of c is that right, lambda n is at most log n to n good this converge, log n there is as for as smaller than any n to the epsilon this is going to converge anyway in does not matter which way you think as long as c is greater than 1.

So, this is going to be order x to the c by R for any c bigger than x y. So, that thing out only the middle term, which is this sum and this we have to handle slightly more carefully. Now, here the log x by n sort of is within a small range of values, but the problem is there it can become very close to zero, it may be very close to zero and then there is that is not a nice.

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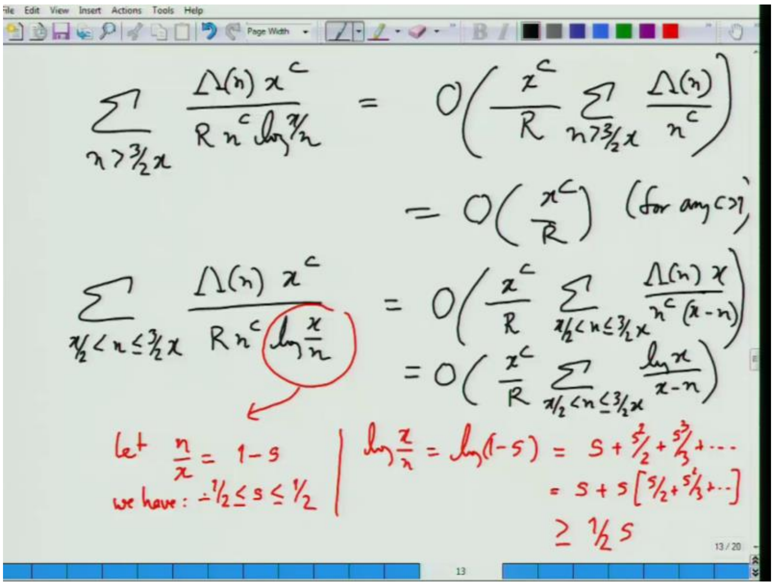

So, to handle that let us do the following replacement, it is again simple trick and this is will easily now can this in a more sensible form. If I just say x by n to be let us say what I want is goes between x by n where is between 3 by 2 to half right.

So, what I want here is 1 minus z or less z I am using else it is less 1 minus s. So, what we get is that minus half is less than s less than equal to plus half as x n goes between x by 2 to 3 by. So, x that is x by n.

X by n, where is from 2 by 3 or 2, it is make n by x, it does not matter actually here just by changing the sign you can replace log x by n to the by log n by x and so on. Now, also for this one this once you do this log x by n is same as log 1 minus s and log what is log 1 minus s that is a, the standard power series for this which is a s plus s square by 2 plus s cube by 3 plus and so on right.

And now, use a fact that s is in absolute value it most half. So, this is equal to s plus, I just say this sum of the rest of it is not going to be very large actually. Let just write it as s, s by 2 plus square by 3 plus 1 and since s absolute value at most half this will be at most first term really at most one quarter 1 by 3, 1 by 12, one quarter plus 1 by 12, if you take absolute sum, which is going to converge sum constant right, in fact is very small constant also I guess it will be less than maybe 1 actually not I guess I am sure it is going to be epsilon, because this is actually words in the geometric series. So, this is less than equal 2 s.

This what I want now I want I want to other way I want I want lower bound in this. So, this would be greater than equal to well if you just take s and plus this much. The rest cannot reduce it, so this would be, so take the first two terms fine this square it always positive. So, s plus s square by 2 that is positive and the rest can suppose it is all negative and try to subtract out as much as you can, you cannot you cannot subtract all more than half s from there, because just sum of all of these would be at most half s. So, I am leaving out some details for your workout. So, these are pretty straightforward. So, basically what you get is and s at most half. So, this would be s by this as s could be zero also. So, let us say at most half s it is half.

So, now, with this, we can write this as order summation on this things out x to the c by R summation x by 2 less than n less than equal to 3 by 2 x lambda n by n to the c and log x by n is replace by half of s which is half goes away. So, s, s is an s is x minus n divided by x, this is not quite what I want it, why they next coming then by anyway seems I cannot R of course that is right.

Now, n is at least half x. So, this is n to the c, I can replace by x by 2 to the c, c is a constant 2 to the c goes away. So, it is x to the c below and c is greater than 1, which cancels out with the x above. So, this is at most and this is log x, lambda n is since n is where is between x by 2 to 3 x by 2 this is log x by x minus n and as x minus as n, where is from x by 2 to the 3 by 2 x what happens to x minus n, yeah this is assume x is half it this is the this never zero.

So, what happens to a, what vary x by x minus n will go from minus x by 2, 2 plus x by 2 it right what, so we can fold up. So, and will take half integral value say, which is again it can take it to be make interval little larger and make it take an integral values, because you can always multiply out by constants and absolve it and then which means it is something like a sum of 1 plus 1 by 2 plus 1 by 3 up to sum 1 over x by 2, 1 over x.

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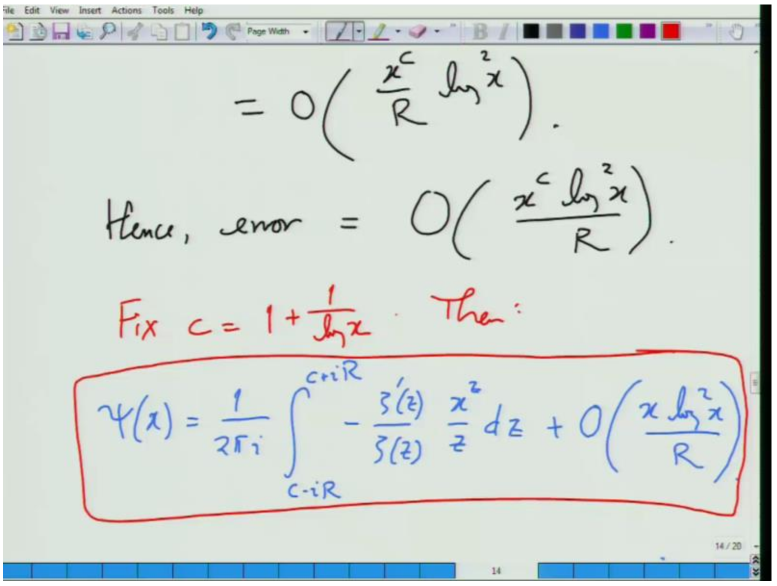

And that sum is if we just go back to this order x to c by R, log x here and this sum itself is log x, so that becomes log s by x. So, just adding all three what you get is of this error is a bit can be a bit too large if c is 2 or 3 then the error actually is much bigger than your main term. I call what was the main term? This is the main term right, whatever it is psi s what is the transient estimate psi, psi x is this plus the error term psi x would be how much at the most x log x cannot be more than x log x in the most silly counting you will x log x.

So, if see your c is a little more than say one point one then the error is already going through of course, assuming R is fixed number. So, that we do not want. So, let us fix once and for all the value c, which is going to be very close to 1, I cannot make it one unfortunately. I cannot make it 0.1, cannot make it 0.0001, because any of these will just be too much. So, I have to make it a function of x, it has to because, it has to bigger than one and yet not be a constant. So, this is what I am getting x, 1 plus 1 over log x.

And then what you get as look at this psi x is 1 over 2 pi i, now you have integral from c minus psi R to c plus i R minus zeta prime z by zeta z x to the z by z d z plus order what is x to the c for this value c as very simple, what is x to the c?

So, that will cancel out i, got it, yes you are right, you are right. So, is by does not matter I am saying it when if you take s to x 3, because I am going to choose x c to be 1 plus 1 over log x. So, what does this make x to the c called is e times x right, which is just I order x. So, x to the c is order x. So, basically you get x, logs square x by R and this is now, if very good starting point that I have manage to write psi x, which is kind of cousin of the prime counting function as I complex integral plus in error term, which is well which is till to bake, if R is small.

Of course nice thing is that R is in the denominator see I can always send R to infinity and make this vanish completely. And then the error term does not matter also and c the choice of c is also does not matter it can be any cost, but as we will see later on it is more beneficial to not send R to infinity instead set R to about square root of x, but that we will come to later, but for now let us this keep in mind that we will not be a actually sending R to infinity; however, if we do send R to infinity what happens then error vanishes and psi x s. So, this is certainly a corollary of this it is psi x equal 1 over 2 pi i is precisely the sending along this sign minus infinity.

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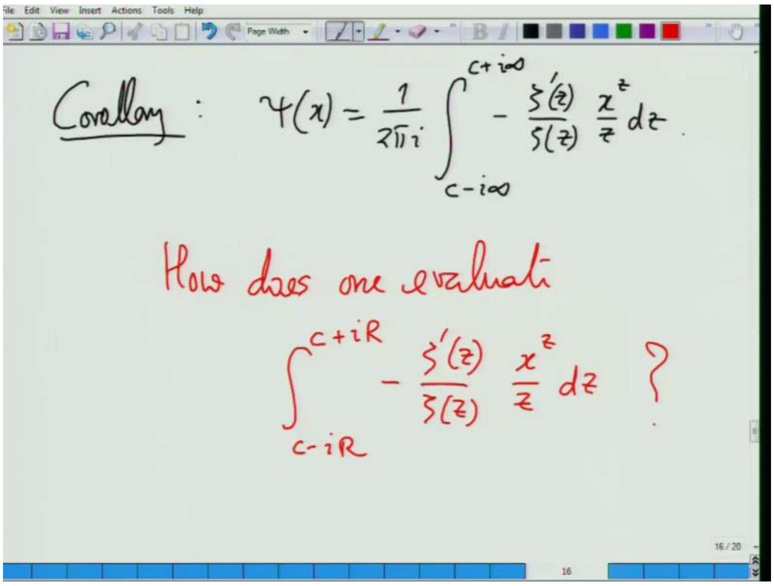

But I said this is only for fall we will not really we using it, because in any case as I said we do not expect in sort of precisely compute a value for psi x if n log psi, because that is allow we will compute precise from log n to psi is later on, but that is not going to very useful in terms of giving as a good estimate, to get good estimate it will actually use a finite value of a. So, that is the set up, we have psi x, we have a complex integral, which tells us what psi is, so the next vision is how to be evaluate this complex integral.

And here, I will revert back to c plus i R and c minus i R, because this evaluation is what is going to determine psi x for us all the nice thing we will be know how to evaluate this integral at least we know how to go about it evaluate this integral we already done this. We already have tools for this, what we need to do is make contour or made again domain whose boundary, one of the boundary is this c plus c minus i R to c plus i R close that make a contour integral around that boundary and count the poles inside that region, that is the value of the integral and then estimate the value of the integral on the remaining boundaries and somehow set it up, so that those integrals are very small. So, that at the towards they add up to some error term not substantially and then we will have our value of psi x.

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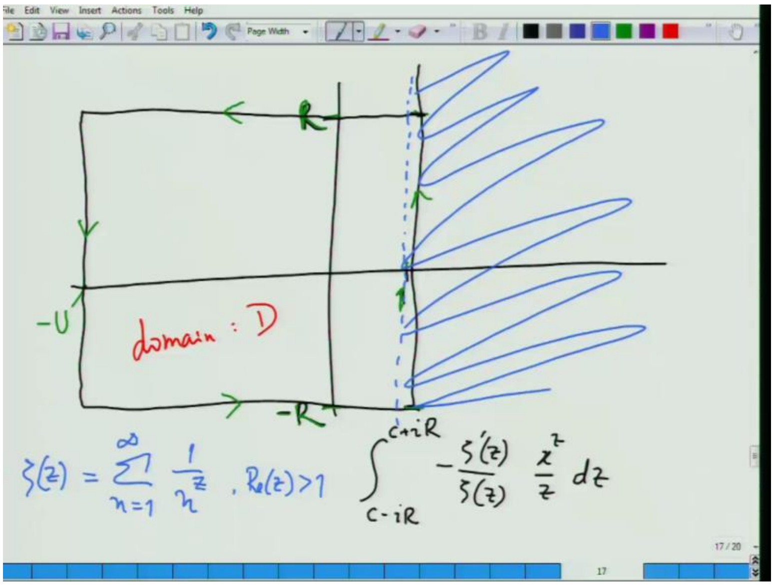

So, let a term that. So, here is the let me write the integral also c minus i R to c plus i R minus zeta prime, this is integral and we have one here and very close to 1, we have c we already chosen that minus R to plus R. And you have to integrate here to here this integrate. So, what is contour that, what is a domain that we should choose for this? We can extend that this site, we can extend this site we have seen it example of both actually, but if recall that example that example x to the z by z was the integrant. And depending on the value of x, we either went left or we went right and the reason was there if x is less than 1 then you need to go on this side positive side, because when you want to bound letter on those integral on the remaining boundaries, if x is greater than x is less than 1 then on those boundaries you can make z to be very large positive and large.

So, x to the z will come very small right and as you an expand those boundary there will vanish. On the other hand if x is more than 1, then this strategy will not work, because as you go further and further on the positive direction and actually z will keep glowing glow up. So, when x is positive you want to go on the negative site, because when and then large then when z takes large negative values again x to be z will become very very small and again you can try to bound it away a bound it to zero or small.

So, what this tells us is that we cannot hope to go on the right how this line to integrate this, we should try to go to the left of the site. So, the movement you conclude this there is the problem, what is the problem? So, let us defines this can domain, again positively this most simple domain that is possible exactly as we did in case of the delta function there is rectangle going all the way up to minus u here coming down and this is how you integrate along this counter clock wise direction.

So, that is the domain D, but I say said be immediately running to the clock, what is the problem? Look at the integrant, what is the integrant? Zeta prime over zeta, x to the z by z is the integrant defined on the domain, we want the integrant to be analytic on the domain, otherwise nothing works right that is a essential requirement.

Now, forget the being analytic is it defined on the domain. Words, yes, that is certainly true, but even it is works, x to the z is define in everywhere 1 by z is define everywhere except for the pole at z equal zero that is pretty well on this term, what about zeta z, zeta z is this infinite sum, where does it converge only for absolute value of z greater than 1, otherwise this infinity some does not even converge. So, which means this is the line z equals to 1 to the left of it including the law on the line zeta z is not even defined. So, nearly there is just this time is trip to the right of the z where this zeta z is define everywhere else is not define. So, there is really, serious problem here.

By the way do you know, do you believe that zeta z actually convergence for absolute value z greater than 1, the real z greater than 1 not absolute value greater than real z. So, at least we need to know, we need to ensure that converges on the right side of this line. So, are you convince that it does, you did it? Then it is perfect, then improve it any ways it pretty simple just take the absolute value of every term and then is a geometry comes out to be a bounded by a some geometric line integral and then the integral converges good.

So, zeta z is defined everywhere here, but there are problems in this site. Going further is find zeta z is define on the right hand side, but it is analytic on the right hand side on every point. How do you prove it be analytic? Ok, this is an assignment.

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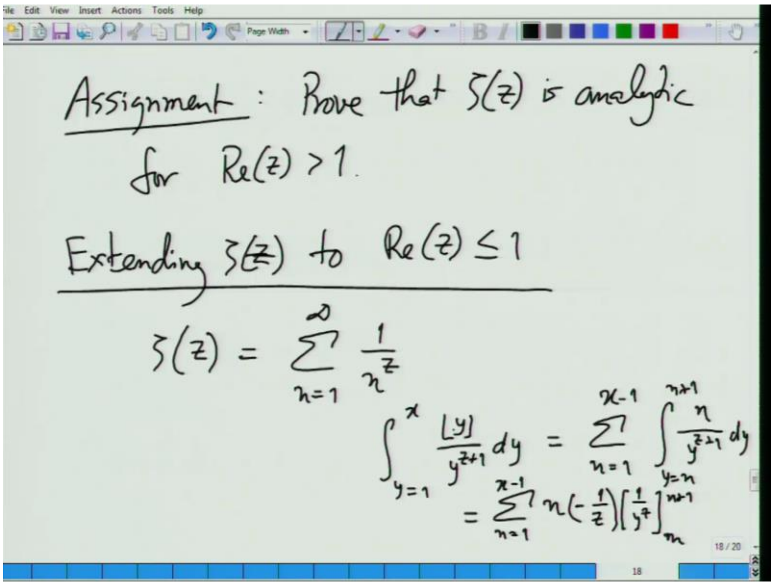

As the reasonably straightforward proof just use tools at you are learned, how do you prove something analytic and apply those things. So, now, we will believe that the zeta z is analytic on this half plane, which is on the real z greater than 1, but it does not even exist seemingly at least on the other half.

So, this approach appears doomed except that we know that analytic function can be extended beyond its domain. We have seen some examples on this and we know that it can be extended in a very nice way unique way actually. So, may be zeta has that property that maybe it can be extended, it is not define through this infinite series, but maybe there is a way to extend it from the other half.

Let us explore this, this is no half. So, let us write zeta in the slightly different form this I can write as let us see, let us make another attempt do it look at a partial sum this I want to write as a integral this is the key thing is to translated into integral, because that is easier to manipulate. And this trying to work out what is should be the form of the integral log in this integral. So, how far away is this sum from this integral, this integral I can write as let us series this equal to sin and where less area is all together this is equal to summation n going from 1 to x integral y going from n to n plus 1 and n going from 1 to x minus 1 actually and here we will have n divided by y to the z plus 1 d and this is equal to what is the integral value, this is a d y over y to z plus 1 is equal to what? d y over y to the z plus 1 integral of that is y 1 over z y to the z, here a negative sign somewhere.

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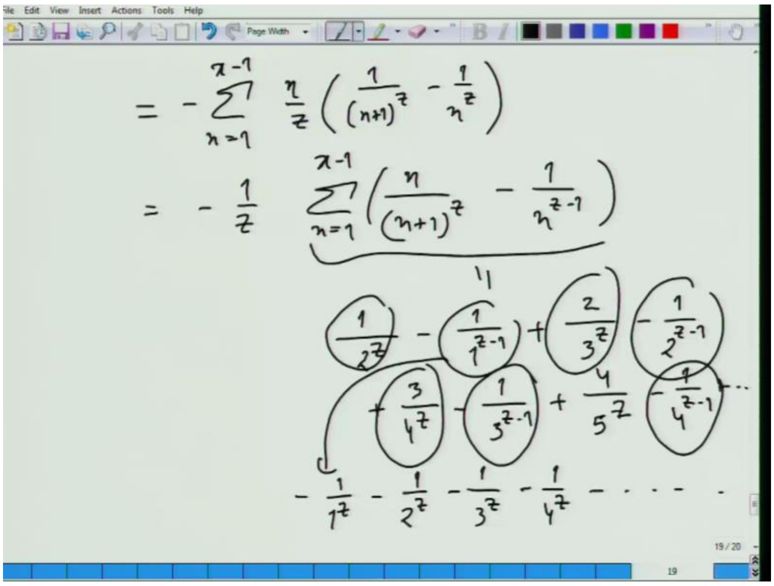

And this n to n plus 1 and this is equal to what? Now, what is this sum equal to, we want to expand this, this is equal to n equals 1 what happens 1 by 2 power z minus plus n equal to 2, 2 by 3 power z minus 1 by 2 power z minus 1 is it right. This does not seem we put now, this is not to very good.

Now, let us yeah may be it is not towards. Now, look at this and this, you get this gives you 1 by 1 power z minus of course, what does these two gave you, what about these two, what about these two? Which is exactly what you want, except there at the end? The last one this guy will survive.

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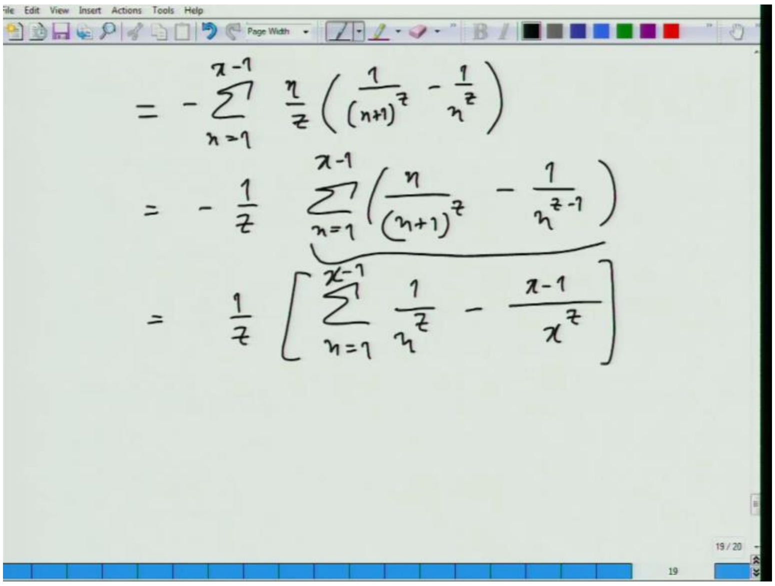

So, what I get is minus 1 by z and then here minus can go away, because at which make all the rest of the positive. Sum n going from 1 to x 1 by n to the z I think x minus 1. So, and then the last guy survives, which is n which n with is n equals x minus 1. So, that is x minus 1 divided by x power z. So, now let us get back to this good. So, now, we can write using this, what I can say is.

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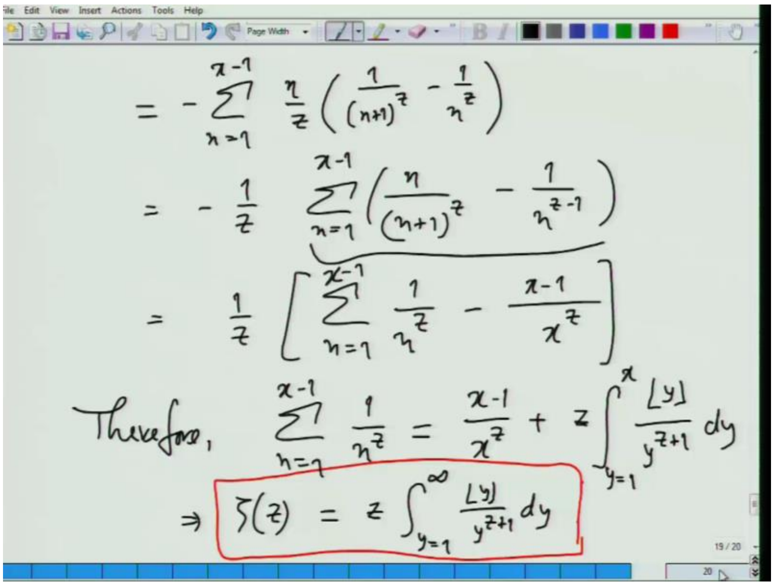

That is equal to x minus 1 over x to the z plus z times this integral y going from 1 to x. And now, sent x to infinity, I should not have chosen x here this is x I am using something else. This is a different x not to be confuse it x parameter x cosin. So, send x to infinity here, what happens the left hand side becomes the zeta function, what happens to the right hand side the first term.

See the absolute value z or real part of z is greater than 1, because of that this the first term goes to zero except as x goes to infinity. So, what I get is, zeta function equals this integral, thus nice to know although it does not solve our problem yet, but it is nice to know that there is a not so nice integral that I can write zeta function as, because not so nice, because there is a this flow of y sitting out as integrant, but any way we are not really integrate interested in getting a value for zeta z we only one to c if it can be extended beyond it is defining domain. And that we can use this integral follow, well how do you do that? Is this write this.

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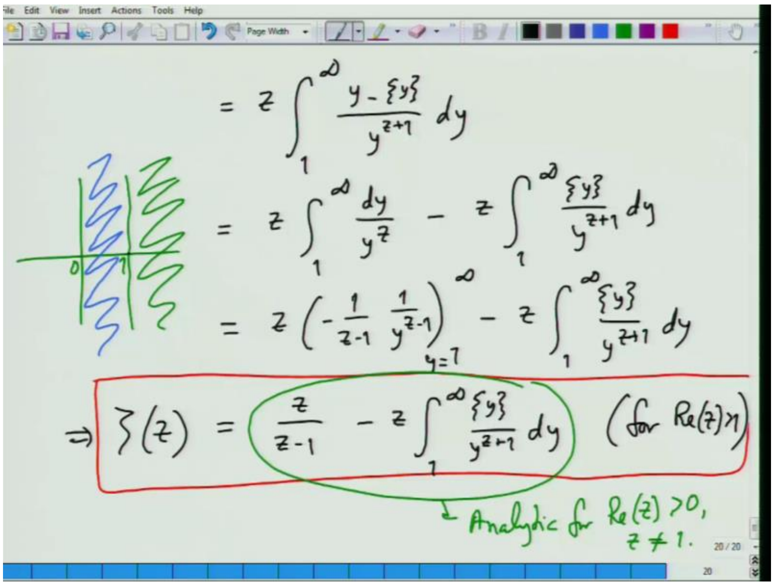

1 to infinity y minus bracket y, here bracket y is just the fraction part of y. Now, the first integral can be evaluated that is equal to minus 1 over z minus 1, 1 over y to the z minus 1 right, y goes to taking value 1 to infinity. Now, when y is 1 and infinity of course, goes away infinity at in pi equals infinity this is zero that is a z is a z minus 1 has a real part, which is bigger than 1, bigger than zero and y equals 1 this becomes of course all are this is real z greater than 1 and let me just write zeta z.

So, this is also important question, long let us look at the right hand side. Right hand side is an analytic function, why it is analytic? Well this is of course, analytic except for z equals 1 and the integrant here is bounded well of course, I have to be careful here. This part, this is analytic for real z greater than z, z not equal to 1, for z not equal to 1 there is this does not diverge. So, the first term is find everywhere else for real z greater than zero. Look at this integral as long as real z is greater than 1 sorry, real z greater than zero, y to the z plus 1 that real part of this is bigger than y.

Actually y to the 1 points something and the fractional y is bounded by 1. So, this integral converges absolutely as long as real z is greater than zero to sum finite value right for every value of z it converges sum finite value. And it is of course, the dependence on z is analytic, because it is the it only depend on z is y to the z plus 1, 1 over y to the z plus 1 is an analytic function is z. And we have seen that when you integrate analytic function, actually it is both analytic in z as well as in y or for every value of y is analytic and therefore, it integral is also analytic in z.

And it converges for every real z greater than zero. So, the right hand side therefore, is an analytic function, which for real z greater than zero and z not equal to 1. The left hand side is an analytic function for real z greater than 1 and these two coincide on the on real z greater than 1. Now, just invoke the uniqueness of analytic continuation this right hand side is the analytic continuation of zeta z for of course, this is still not fully done, this is 1 and you we had zeta z analytic here and we have the right hand side analytic also in the strife. Except of course this, a whole at 1. So, it has allowed as to extend the definition of zeta z slightly to the left not all the way, but at least slightly left. So, that gives us a hope that perhaps by doing something more we can extend it all the way to the left everywhere, which is what we do in the next class.