Riemann Hypothesis and its Application
Prof. Manindra Agrawal
Department of Computer Science and Engineering
Indian Institute of Technology, Kanpur

Lecture – 22

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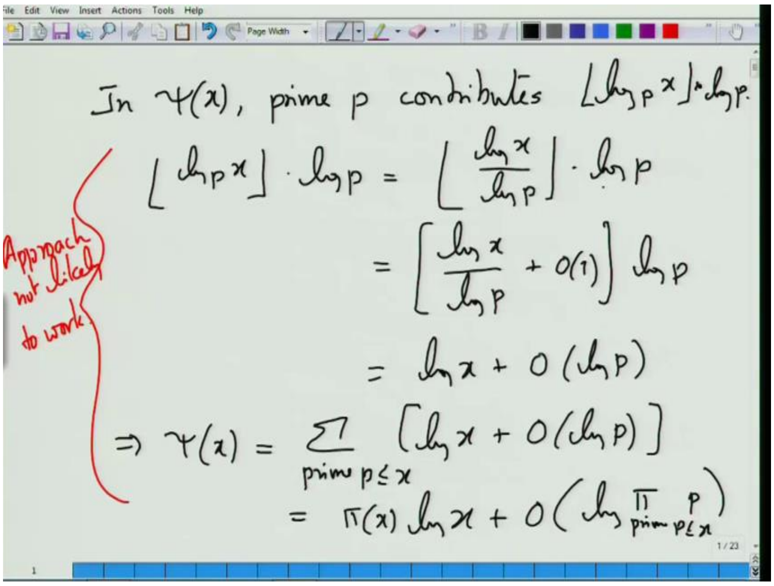

Let us work out your idea. So, let us say in psi x prime p contributes log t x floor times log p, now log p x floor times log p equals, of course log x by log p floor times log p. Now, this is log x over log p plus order one, there can be it can differ from log x or log p by a first one right in the worst case, it will differ by something close to 1. Then, psi x equals sum over prime p less than equal to x log x, this is equal to pi x log x plus order log of product of prime p less than equal to x

Now, what do we do with this quantity there how do we estimate that product of all primes less than equal to x x log x. Then, that is not very nice see the error term, we want to be square root x log square x order psi x by log x.

This means order x by log x plus something plus which is way you want x by log x if we look if we wanted constant, it is very easy to show that pi x. For example, psi x is between 1 by 5 x and 5 x, so you did not have to do any of this Riemann hypothesis and complex analysis. Forget all this, just do simple counting one can show with a little bit of cleverness this is what we well before Riemann that pi x is greater than equal to x by log x and less than equal to pi x by log x.

Clearly, it is a good point, let that be an assignment that is show that pi x is x by 5 log x is between x by 5 log x and 5 x square log x and just do simple counting nothing else. Of course, here see we have been very pessimistic in approximating the floor here, we are saying that x log x by log p floor is always log x by log p plus order one can bond this with one sure. That does not help there are would be times for different p, this will be very close to 0, the error and there would be times when it is closer to one, but then it is going to be very hard to say for how many p's.

It is closer to 1, how many p's it is closer to 0, so unless we do an estimate of that in some clever way of doing that do not think we can conclude much about this. So, this is likely, where in this approximating this you mean, but this already look pretty high, in fact if I forget order log p here, if this was order 1, then what do you get pi x log x plus order pi x. That is already too much because what we are looking for is a much tighter relationship, so if psi x is x plus order square root x log square x.

Then, I want pi x to be also the error term to be also close to square root x, in fact we are now this has to fail because see this floor unless it is exactly not necessarily. It has to fail this is not right, but it is much more likely to fail because even if the log x has a slight differ slightly from a multiple of log p. Then, the error would be at least one over log p if error is at least 1 over log p, then multiplication by log p will give you error 1 for 1 prime and then when you sum up you get order pi x and then that is too much.

So, we were to do something else and I tried something last time which clearly was wrong because you cannot differentiate the error and I promised that I will find out about it today and do it. I had no time in the morning to find out, so I am at the same position I was yesterday evening, but we can still try that is see you sure last time the error. We had approximated certain error and we said psi x if you plug this in then differentiate, then we get this, but we cannot differentiate the error.

So, let us go a step back and before arriving at that error we had some expression for psi x, which we then the error part we do it an approximation. So, instead of doing an approximation for the error, let us go back to the original expression of psi x differentiate it there because there was an exact equality and then do approximation maybe that will help. So, if let me re jig your memory what was exactly psi x, that is that is one thing, but psi x.

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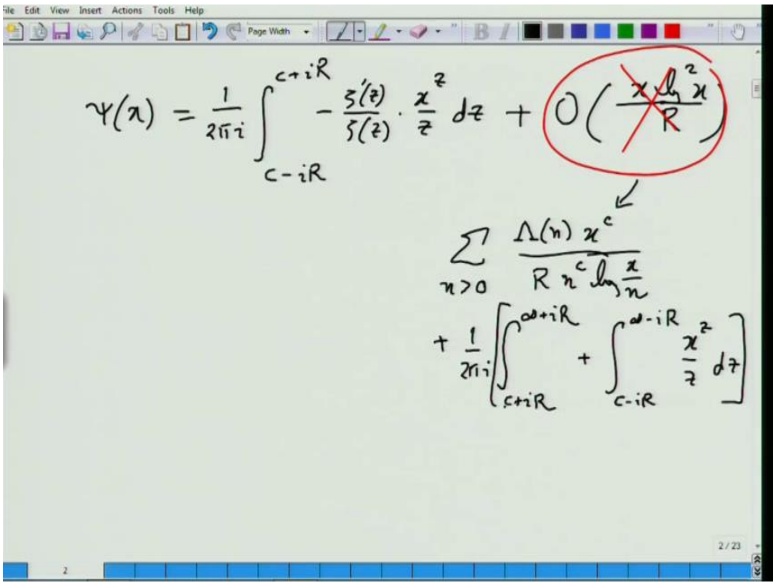

In the beginning we derived if expression for psi x which was something like is c minus i R to c plus i R and of course 1 over 2 pi i here, then we had minus zeta prime z over zeta z x to the z by z plus, what was that. Now, I am not saying that get an approximation of that of something like I think that was like x by R log numerator. So, I do not want this because this I can’t differentiate, I want that expression which we approximated to get this log x x log square x, there was some infinite sum if I remember correctly summation over yeah.

Why order of the whole sum what approximation go the order no go back why order because delta is also approximate. So, let us go back, what was the delta if there was an approximation there. We said like from c minus i infinity to c plus i infinity that integral of x to the z by z is precisely the delta, but when you truncate the integral at trumps.

We just look at it from c minus i R to c plus i R then you get an approximation of delta and that truncation error is what we were trying to we did estimate it correctly, in fact we did see how we did calculate this definition of delta that was by looking at these two rectangles right.

One c minus i R to c plus i R and going to positive side and once going to the negative side and then we said that if we evaluate this integral to the three sides or three edges. It tends towards 0, but now one side of rectangle we can send towards 0 and this these arms which go from c minus i R to u minus i R that u can go to infinity, but R my R stays there. So, integral along these two braches one the vertical one and two the horizontal ones vertical ones is what I want the horizontal one is the error and the horizontal one stays the error stays. So, what is that integral value actually we can just write it as may be just as a integral you see let us see maybe it is a good idea to revisit the whole thing and just write it as it is.

So, first of all just answer me the question is this exact apart from the problem in the approximation in delta is this part that you get is this exact. There is no approximation involved here except for the delta function which is like c plus i R to these are the two integrals which are for delta approximation. Actually, this integral is c plus i R to infinity plus i R, so it is going on the right hand side depending on x whether it is less than 1 or greater than 1. Actually, there will be an integral going to be left hand side also, but is not that already here that is what I am confused about in this sum is this not already incorporated, we approximated this integral to get this order.

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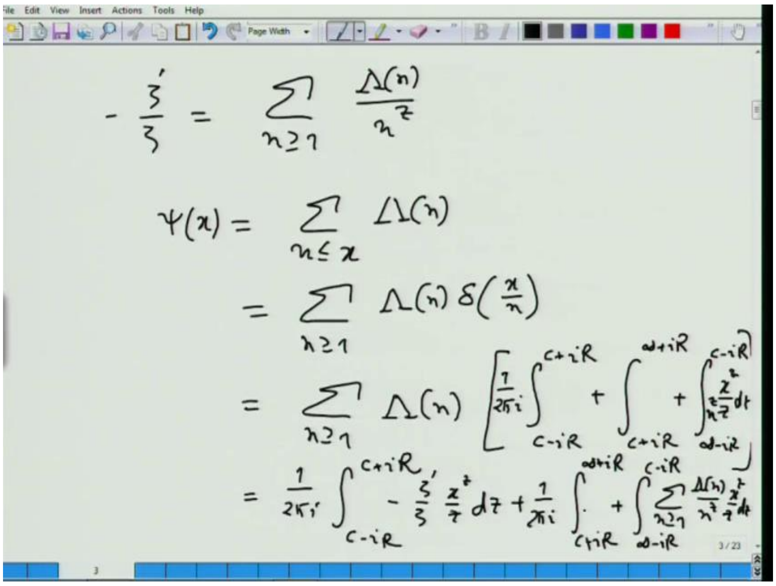

Let us just start from the basics zeta prime over zeta is summation m greater than equal to 1 lambda n by n to the z. Moreover, we also know that psi x equals summation n less than equal to x lambda n which you said is greater than equal to 1 lambda n delta x by n. Then, delta x by n is 1 over 2 pi i integral c minus i R to c plus i R plus infinity plus i R to c plus i R to infinity plus i R plus infinity plus i R to c minus i R infinity minus i R x to the z by z, so that is delta.

So, we get therefore, one over two pi, this is x by z divided by n n by n to the z and this should be minus this. So, that is a good part and yes these integrals are what giving me that approximation, so this is exact, so this is the error part which is these two integrals now after that we focus on this part and we derived an expression for this as well.

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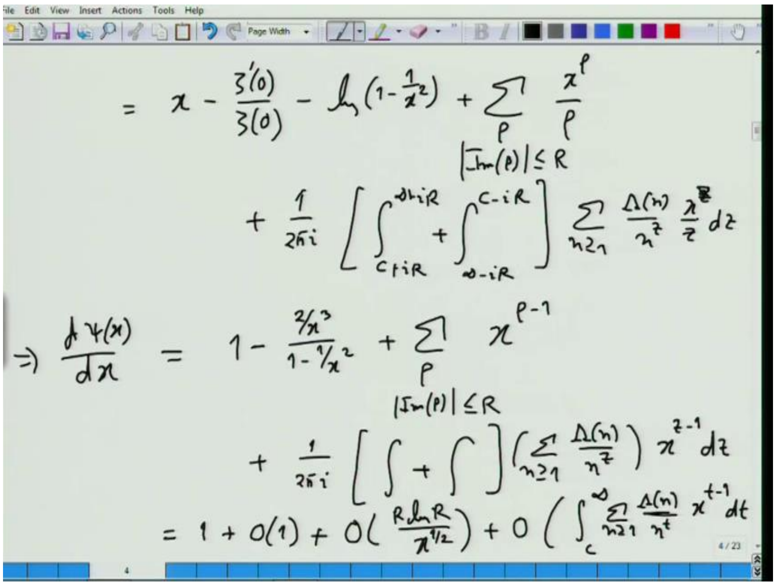

Well, what was that it was in terms of the 0s, the residuals in that big rectangle and that came out to be I think x minus zeta prime 0 or times zeta 0 minus or plus I do not know whatever it is. Then, there was another minus of log of 1 minus x square, these are all trivial 0 s and then there was a plus over rho x to the rho by rho rho's are the trivial non trivial 0 s and this is it that is the expression for integral.

I am not sure about the signs here and maybe not sure about this one also, this probably but these are not important anyway these are very tiny small numbers plus this error fine now this is exact no approximations anywhere. Now, let us differentiate psi the psi x or psi t whatever it is then what is d psi x by d x 1, this vanishes what happens to this, this is like 2 x upon 1 minus x square. What happens to this is the rho minus 1 and what happens to this x to the z by z, now what is the approximation what is the error?

The error is this whole thing it is 1 plus the error, so what is the error 2 x by 1 minus x square, this is 1 over x square, so then will differential of this would be 2 by x cubed by 1, which is same as saying that you have multiplied the whole thing by x cube. Then, you get 2 over x cube minus x not 2 over x cube minus x, it is a is bounded I mean for any sensible value for x this is order 1, so this is gone.

What about this x to the rho minus 1, now when you look at the error you just look at the absolute value which is we just look at the x the real part of rho minus 1. Now, assuming the Riemann hypothesis to be true this would become x to the minus half and sum over all rho, such that imaginary part of rho is less than equal to R. This we have already estimated, so that is just that we do some calculations for this we must have because an error here or we did one over rho take it from me it is like R log R. We will prove it is the numbers of 0 s actually there is a very nice expression, we can very precisely define or you know write down the formula for the number of 0 s of zeta functions at height up to the height bar.

So, it takes care of this, now what remains is this how do we estimate this order of x cube, so that is like whatever that see it goes c to infinity and c to infinity is imaginary part is really playing role c to infinity is on both sides. When we look at that anyway the abstract value you get t so basically what you get is order c to infinity sum over and greater than equal to 1 lambda n over n to the z n to the z is also of course, bounded right n to the z is absolute value.

We get n to the t and x to the t minus 1 d t t is the one that is going from c to infinity t is the variable parameter that is being integrated. This is going there is integral, but only the real part is varying the imaginary part is always fixed and the moment you take the absolute value the imaginary part anyways goes away.

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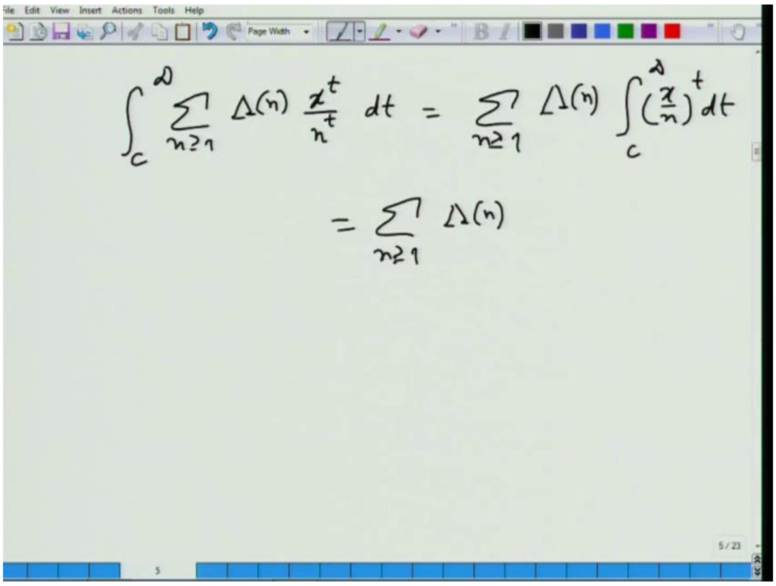

Then, integral c to infinity and lambda n, you have x to the t divided by n to the t d t all this is simply integrated as 1 lambda n can be taken out and then you integrate c to infinity and this would be of course I have strictly speaking. I should have split this sum into two parts n less than equal to x and n greater than x, because depending on n less than equal to x and n greater than x.

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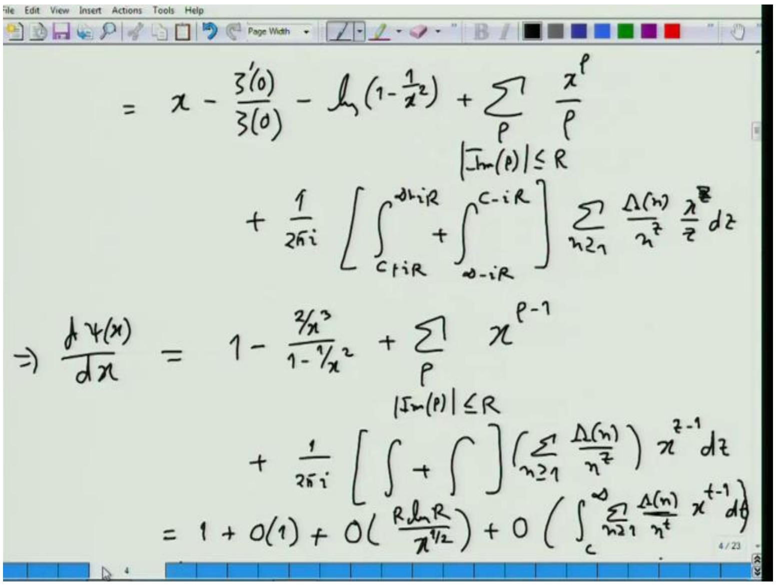

The definition of delta function will itself change in the sense that this integral I have just said it is going from c to infinity the other thing occurs that is n is less than x. Then, this would go from c to minus infinity if n is bigger than x, then it goes from c to infinity. Remember that the function for n less than x delta n over is 1 because you are going from the negative side. Then, there is a pole that you are pulling it in inside the rectangle if n is bigger than x, then you go the right hand side, where there is no pole and delta value is 0.

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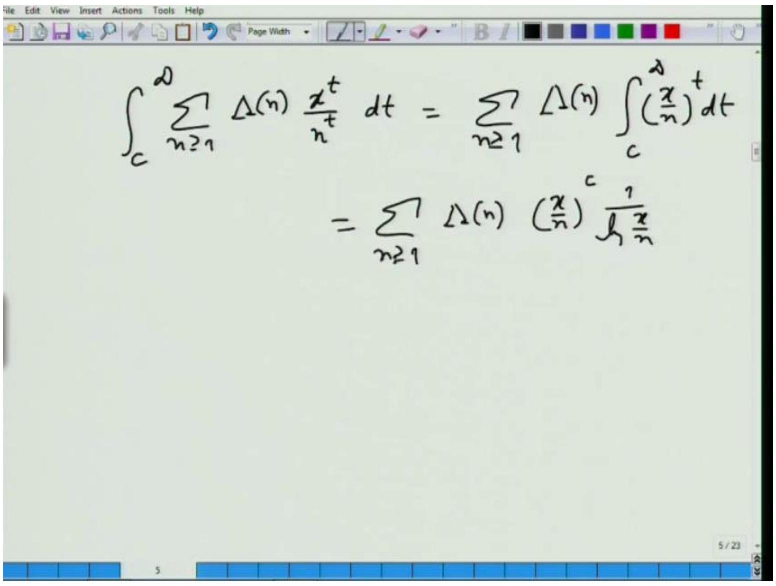

So, this sum actually this integral should come inside the sum and with these two splits, but effective, but the effect of that is not going to be, look at this integral. So, here think of n always being bigger than x because only then this integral will converge, otherwise if this goes in infinity, then n is less than x then it diverges, but that is fine because whenever n is less than x this integral. The limit changes to minus infinity which has a same effect so integrating this gives you what the same thing x by n to the x actually 1 over log x by n. So, this is more or less the same thing more or less why I am calling exactly the same expression that we call earlier except.

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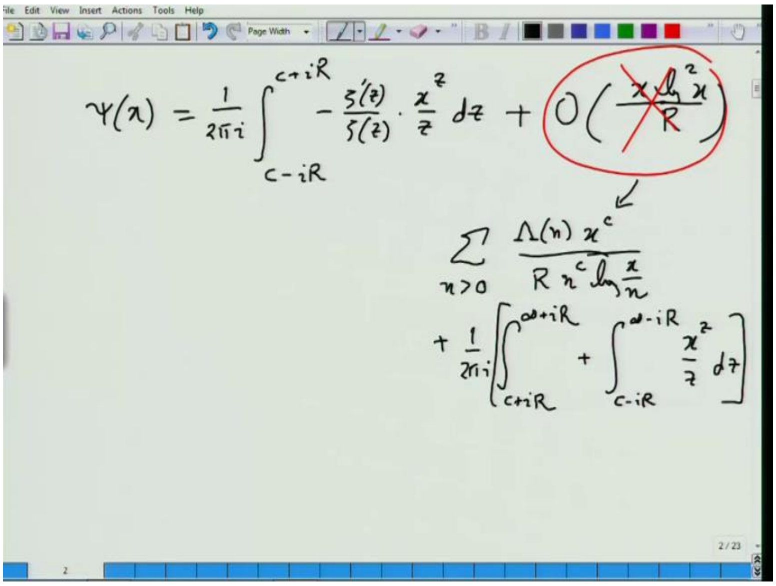

There is something there has to be something missing here, there is an R that is missing that is what happened to the poor R of course, this is x to the t minus 1 there is an x here. So, instead of an R there is x that is the only difference that has happened, so we can use exactly the same analysis to derive that this is order log square x, x log square x divided by capital R, but now is divided by x.

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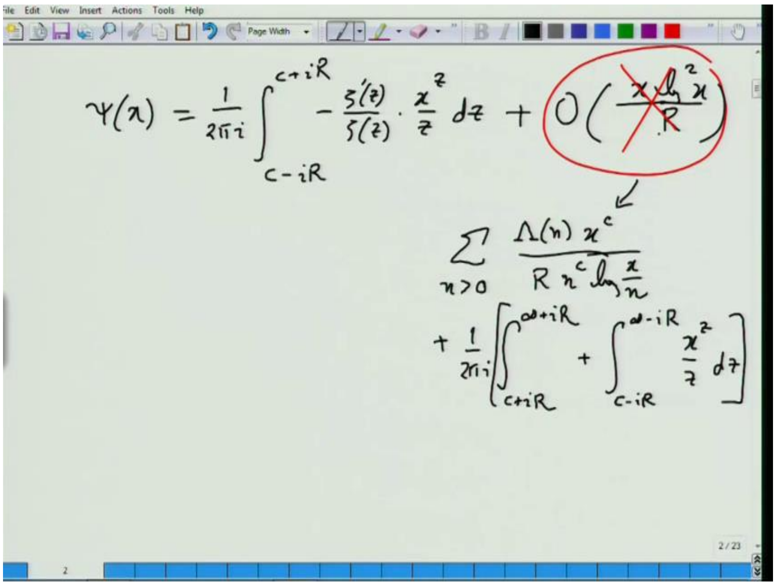

The earlier error which was this is order x log square x by R and this is this expression the only change, now is there is that instead of this R we have an x that is it.

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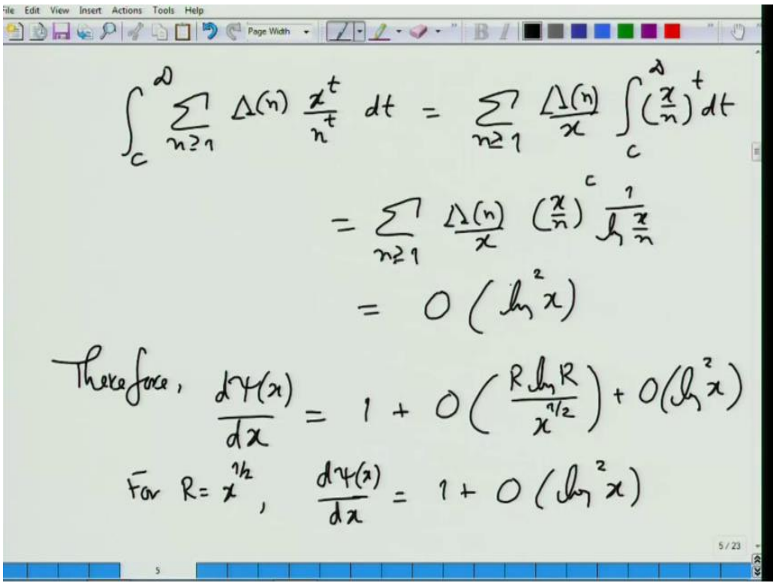

So, this is it becomes a x log square x by x which is order log square x and hence is one plus log square x. Now, also I remember that we are finally, plugging in R to be square root x when you plug in R to be square root x we simply get the error to be just order log square x, you agree with me is there any question, you are to ask now if there is any doubt.

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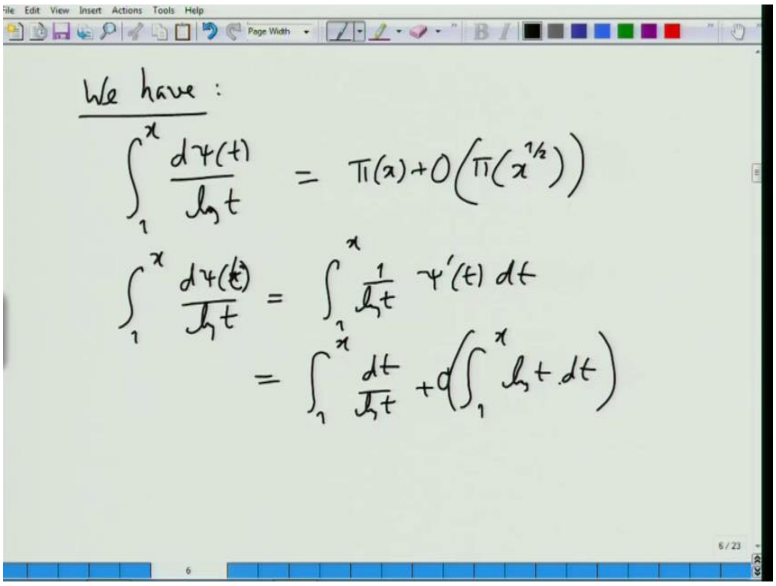

Now, let us come back to the correct analysis and what we had was this side t by log t integral going from one to x pie x let me just stick order by now plug that in 1 plus order, we know one plus order log square t. Now, we back to the good situation this is the real this error is or I should not say trivial something is wrong here and you see one plus. order log square x. What does it mean, it means no sense this is much smaller than this error is completely bizarre, we miss many things R is of course we miss this is also not good.

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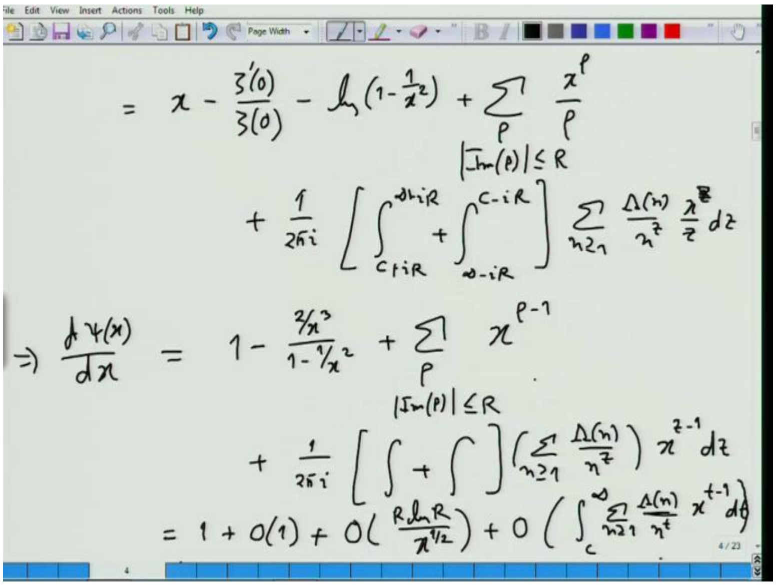

We did this estimation and we said this is equal to order R log R by square root x no it is completely unacceptable. This does seem to what it means this will be a 1 over square root x, when you take the absolute value and then sum over all. Let us just count all the 0s of zeta function they have to log rand, then we missed the R again when we differentiate this with respect to x, you lose that cancel out that z in the denominator that is how you lose that R.

That is worse, but even this is if you can get rid of that then this is one here I should not be writing one I should simply be writing something like order 1 over x cube because x is parameter, you take the absolute value. So, absolute value x to the minus 1 is square root x for any rho, so that comes out square root x is common you just sum over all rho’s the number of rho’s is as I said is less than equal to R log.

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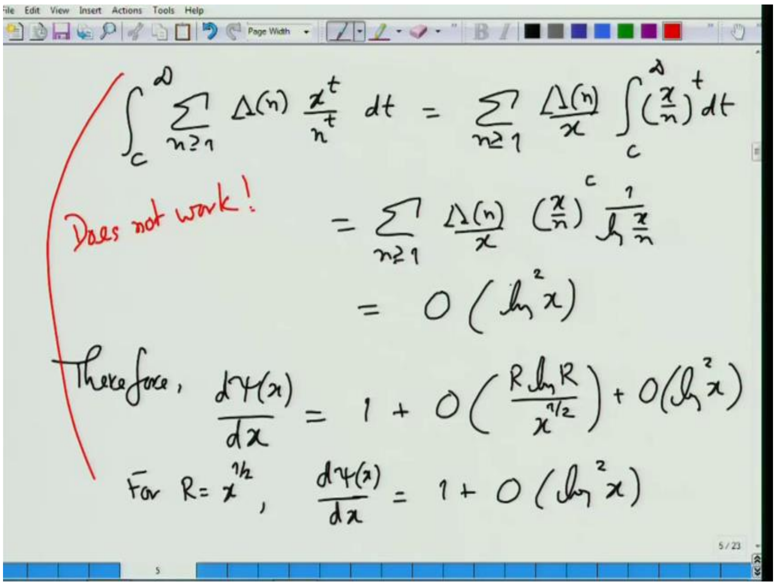

This is of course, this is also wrong whatever I have done is simply, so what is going to work.

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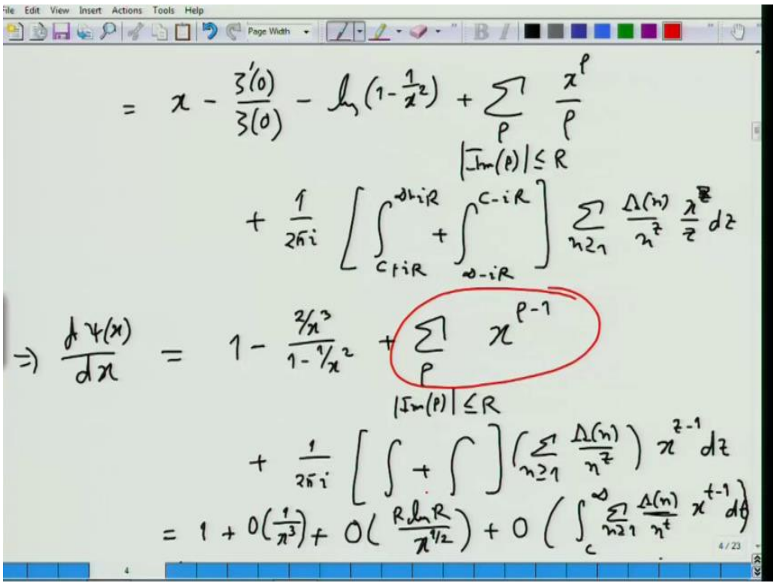

This seems to be I do not see how you can approve this just look at this you have to take the absolute value you can take the square root x and then R log R is just come out. You cannot avoid it and the moment it does come out R is always keeping it to be square root x. Can we can we play round there or probably not, if you play around with the value of

R then psi x itself is going to change. You know that, but how does it matter that is a very good point what is stopping us from choosing a better value.

You see this equality holds for all values of R, so let us choose a value of R which optimizes the error, but here here we can say fine we can choose value of R which optimizes the error, but what about there. There is no R, because R is gone when you take absolute value here R simply goes out and then you end up with log square x.

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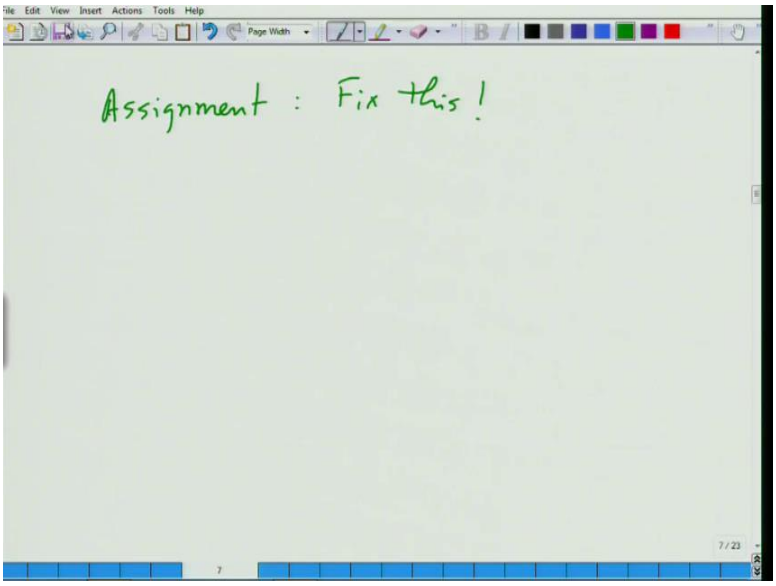

Another is assignment problem, there is clearly a fix it is just that we have not been able to find it and there is a simple fix it is not anything complicated. Maybe you can well you cannot disprove Riemann hypothesis, but may be you can disprove this connection between this Riemann hypothesis and prime.