Riemann Hypothesis and its Applications
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Lecture – 30

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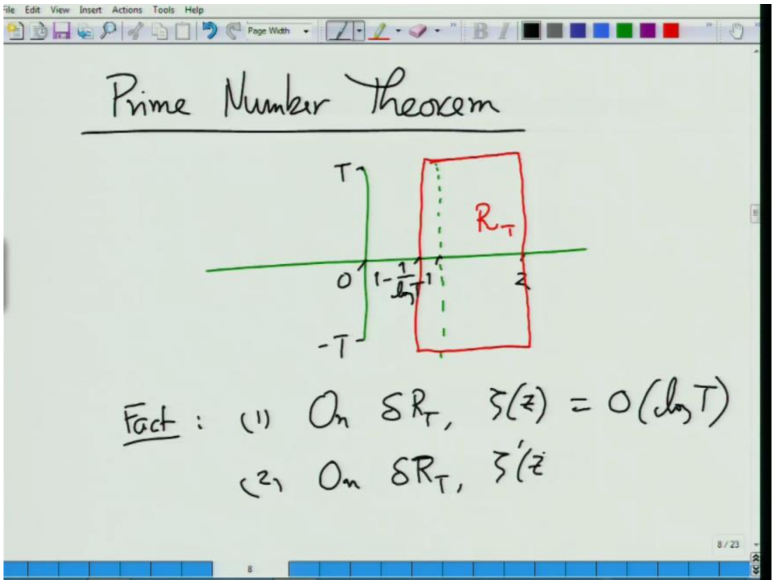

Today we are going to prove prime number theorem, and this essentially follows from everything we have done so far plus the (( )) sum problems we are solved. So what did the problem which we remember one of the (( )) sum problem was that if you consider if we consider a rectangle here, (refer time: 01:00) one minus one over log t, we write a name R T. Zeta z was a order log T, and zeta prime z is order log square absolute values.

Student: Zeta z minus one by z minus one.

Professor: This is on the boundary, oh the question, (( )) the question was about the boundary or overall.

Student: (( ))

Professor: Overall?

Student: Something for overall step two things.

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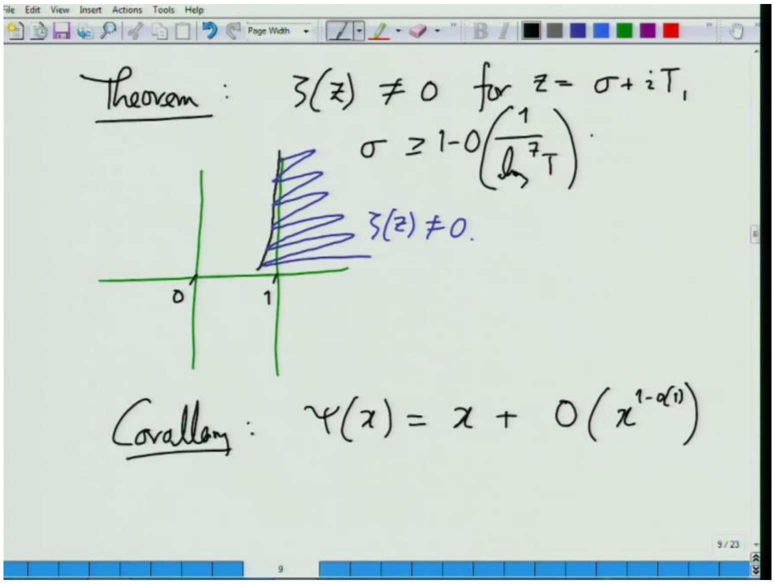

Professor: And the step two was it, so once we calculate overall boundaries pretty straightforward. So these are the two ah results that will make use off. And so now come to the theorem statement that zeta z is not 0 for… Log to the thing seven also, so what is this say in terms of the picture that we have. So we know that zeta is not 0 on the right of the line, there is equal to 1 right. And what this theorem is saying is that for little less sigma, just little less that less becomes lesser and lesser as we increase T. Whereas T goes to infinity, this is converging towards one ok. So it is something like a curve like this, such T goes to infinity, it reaches ones that this is the curve. And then in this region, entire region, zeta z is not equal to 0. So no 0 of the zeta function lies in this region which automatically translates to the prime number theorem.

What is the corollary of this that psi of x is x plus order how much? So if the 0 where at only at sigma equals to half then it would be error would be square root of x times log square, but they are not away from arc, therefore, much closer to one. So what will come out is something like x to be one minus small o 1, something very very small there, inversely proportional going very detain towards 0, but going up. But I will not do that part, because this is sufficient to eventually my (( )) all of this.

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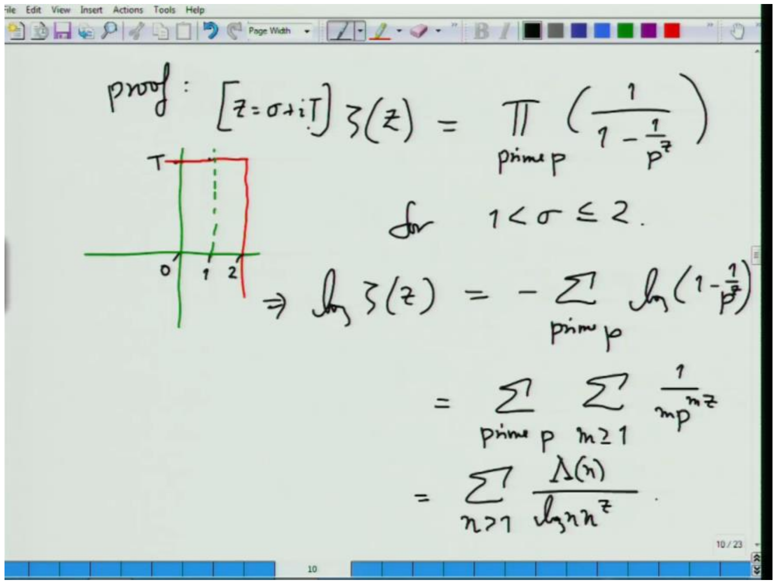

So let's do the proof of this. So we will just look at the K with at the height T, you have this ah you have the curve or rectangle on the boundary of which we know how this zeta behaves. So let's first do something very nice which is let's start with the definition of zeta z at (( )) will always assume z is sigma plus I T. So we fix the height T, and that is the real part ah between two and coming towards becoming little less than 1 (( )). To bigger with this we will vary it start from two and go towards to this side – towards one, not quite reaching one. So in that region, we know what zeta z is, what is the original definition, product over all prime P, which is one over one minus one over p to the…

For one less than sigma less than… Take log on both sides, now expand this log also around one, then you get ah get another series there, which is m greater than equal to one I think, and this one by log of one minus x (( )) two by three because how it goes, m P to the m z. And let's combine this and write them as ah which I can, so this is running over all prime P this is running over all prime m essentially. So ah this will run through all numbers n except for number 0, 1, 1 will not occur here. All other parts are there, this is the but this tell this is running over all prime power.

To get rid of prime other numbers is taking lambda in here, which is 0 whenever and it is not a prime power, so that takes here all that. So when n is prime power this is log of so if the n is p to the n this is log p. So this becomes n to the z, and what we do about this, see when n is p to the n, this is p to the n z, what is log n, m log p and what is lambda n log p. So log p and log p is cancel, we get one over n, so that's n alternative expression for log of zeta when real part is greater than one. Now we will little bit more work.

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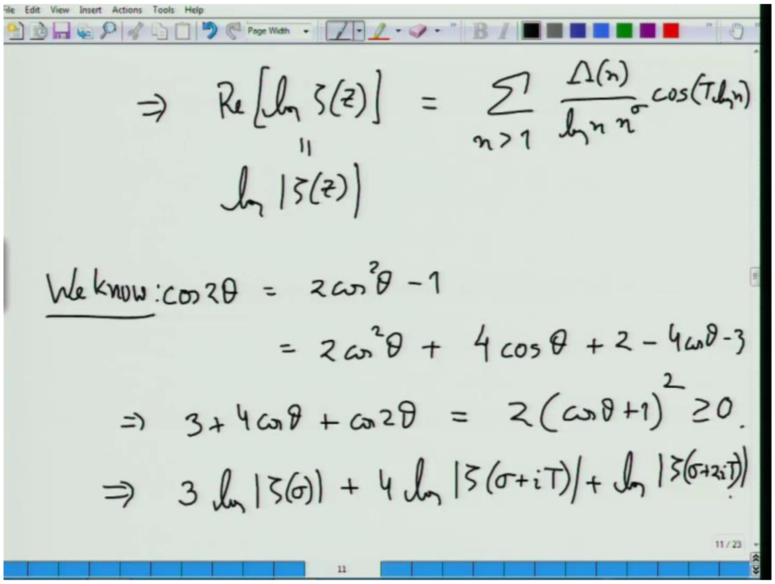

Real part of log zeta is what, looking at this n to the sigma, so I can write z is sigma plus I T. So let's we have to keep in mind for the entire proof. So I can break this n to the z and n to the sigma plus n to the I T. And so this is ah will be cos of [noise]n to the I T is e to the I T log n. (refer time: 12:00) And when you go denominator to the numerator becomes negative, cos of minus also (( )). So this is cos of T log n. By the way, this is also equal to log of absolute value of the number is real part of log of…

Now we use the simple trigonometry identity, which is what is that, involving cos two theta cos theta, cos two theta is a two cos theta minus one. I just want to write in some kind of way, because it is positive. So this involves cos two theta and cos theta and one (( )) perfect square. How do we do that, so we get cos square theta by that, now I need to get the rest of it. So make this perfect square two cos square theta plus what else you need, plus 4 cos theta plus two minus four cos theta minus three, so this makes sense ok. So this means 3 plus 4 cos theta plus cos two theta is two times cos theta plus one whole square, which is greater than equal to zero. This is the key point that I want to show.

Let me explain what I am trying to do. If you look at this expression, this is greater than equal to 0, log of n is greater than equal to 0, because n is greater than one. N to the sigma is greater than equal to 0, sigma is between 1 and 2. This is the only quantity that can become negative. So I want this expression make to make this (( )) expression always greater than equal to 0. So what do I do ensure this, well the simplest way to add something to this, so that this cos becomes bigger than greater than equal to 0 always, and that is the trick which we use. So using this, what we can say is three log zeta of sigma plus 4 log zeta of sigma plus I T plus log zeta of sigma plus two I T is greater than equal to 0.

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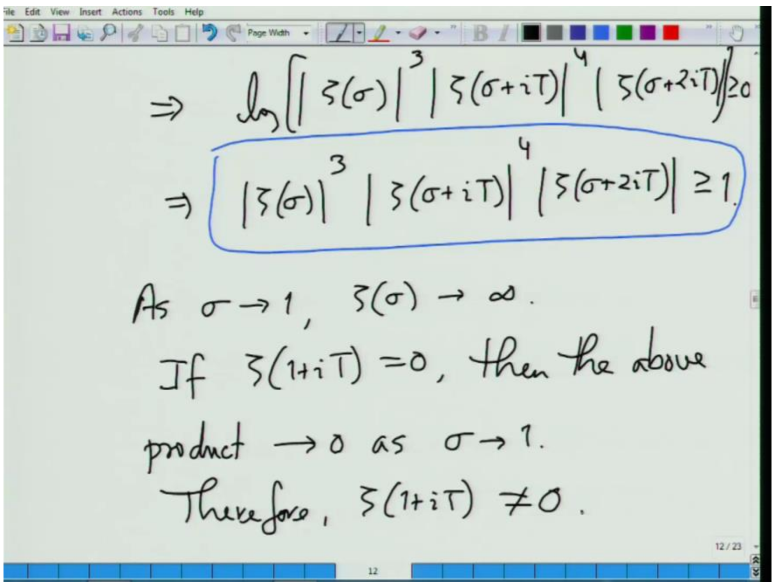

And this gives us, so now these are all logs, and sums of logs, so obviously I can rewrite this is the log of absolute value of zeta sigma cube, whole cube, (( )) zeta of sigma plus I T to the four is greater than or equal to 0. And this of course to that the product of absolute value this quantity is greater than equal to one. And this is very important identity for us, because now we can use this two argues, essentially argue of the prime number theorem. And what this is saying is sigma is approach we can this between one and two. So I can take it towards one. But looking at the right hand side of this, this product is always greater than equal to 1. So we can take it as close to one as possible, it will stay product will straight greater than equal to one. In fact, then where use the continuity actually take it 1, and the product will stay greater than equal to 1, even if sigma is equal to 1.

Because zeta function as a analytic continuation, at sigma equals 1, so when the I take the analytic continuation take those values, this product has to remain equal to 1. Now suppose, one more thing, how would this product greater than equal to 1, that is 1 thing. As sigma goes towards to 1, zeta of sigma goes to infinity, so that will make it bigger and bigger, so that will try to satisfy this.

And as longer these two, characters remain bound it, it is going to shoot up and problem is solved. The problem may occur, when zeta of sigma plus I T is the 0, rather zeta 1 plus I T. In that case, that 0 will try to cancel out this, this divergence ok. Now who will win, that is the next question. One is going to (( )), one is going to send it to 0, that is also easy to decide who will win? You can see look at the expression and tell me who will win.

Student: (( ))

Professor: The order, the order of 0 and order of pole, what is the order of pole at zeta equal to 1. 1? So zeta cube, will have order three. So it will be I mean zeta 1, all zeta of sigma around one will behave like one over sigma minus one and cube of this behave like one over sigma minus one cube. Now if this is a pole I am sorry if this is 0 at sigma equals one, to sigma one will be factorial of this as sigma tends towards one. And this is fourth power 0 of order four so that 0 of order four cancel out this pole of order three, and this still be less 1 0 of order 1. So these two will together still go towards 0.

So now comes a third character, now this third guy, they still known tend it to ah infinity or at least keep it above one, provide it, this also a pole of order at least one. If it is not a pole order of one then this whole things goes to 0. But does it have a pole of order one, no zeta function is only one pole, which is the sigma equal to one, nothing nowhere else. So this will be bound it. This is does together with the pole of order three, this is the order of at least four. So this product will go towards the 0, if zeta 1 plus I T which is not possible (( )).

Therefore, T was after (( )). So it says that on that line sigma equals one, the zeta function always two, there is no 0 of the zeta function. So this at least already shows part of that theorem, that is not quite show the full form theorem, because theorem says that actually small region to the left of that line also where there is no 0. But at least on that line, there is no 0.

Now to show that we can go slightly beyond this line, well we use the ah this the two things are will not. Starting from this expression, so we will we already argued from this expression, continue from (( )). We have zeta cube, zeta of sigma cube and the fourth power of this and this product is greater than equal to 1.

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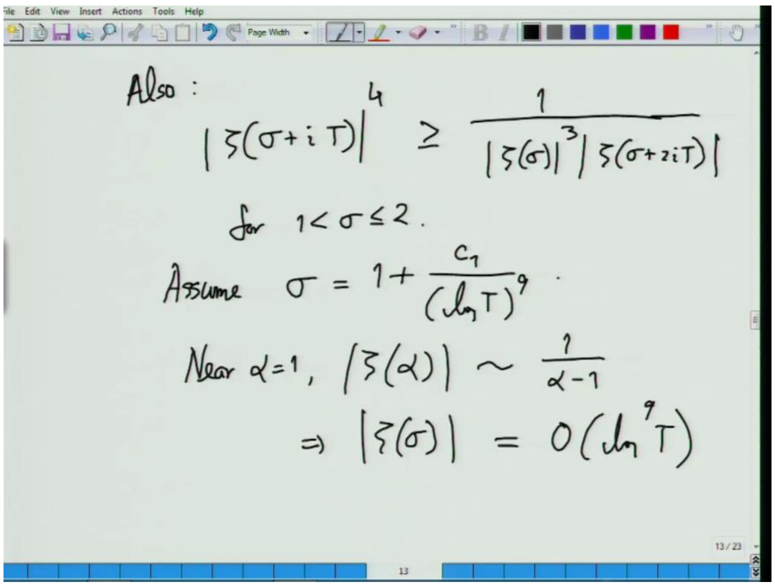

So let’s rewrite this. I am interested in on the standing the absolute value of this, this is greater than equal to one over cube (( )). So now let’s assume sigma to be just a little more than one that is it, may be some constant c one divide by log of T to the nine. So this is just little more (( )), no problem this inequality is (( )). At this value of sigma, what is the zeta of sigma. While zeta of sigma, zeta of alpha, at least the absolute value behaves like 1 over alpha minus 1, with the residue of one there. This plus a another Taylor series if you have (( )) which sort of go vanishes away. So for this sigma is really very close to one, so we can assume that this is plugging the value of sigma, because I already fixed the sigma. So zeta of sigma will be so this minus one, one over this, so this is about order log to the 9.

Now the next so this gives the estimate of this. How does this was, I think this was (( )). Over this, this is where I am going to use, the result is proved on that line, the entire line actually ah the value was ah zeta of z was order log T. Z is sigma plus I T and sigma being anywhere between one, little less than 1 to 2. So I 2 T will still be order log T, in an absolute value.

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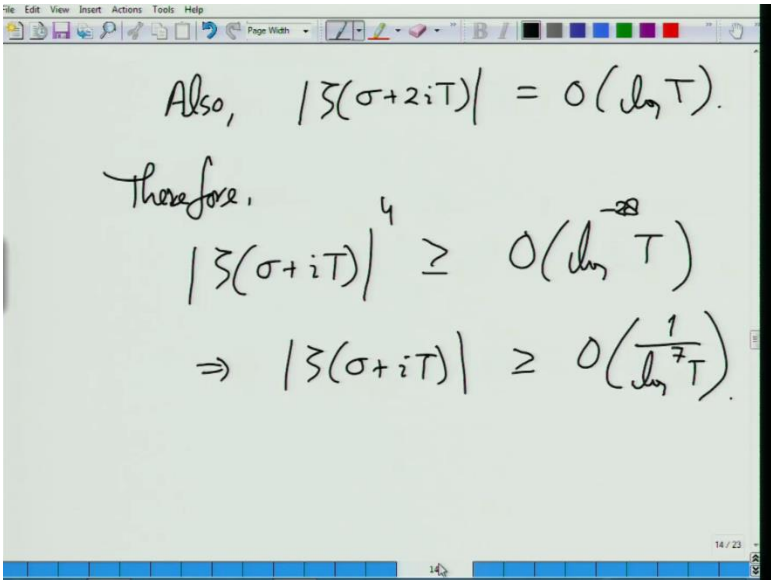

So zeta of sigma plus 2 I T this is bounded by order log T. Therefore, this is greater than equal to just (( )) everything here. This is bounded this is an upper bound on this, the log T is upper bound on this, so inverse of that these are the lower bound and multiply this out, log to the nineteen cube 27 plus one more is order log to the 28 T, this is fourth power. And this gives what, minus 28? Where minus 28? Just a minute, log to the minus 28. So the point at that sigma this is at least one over this.

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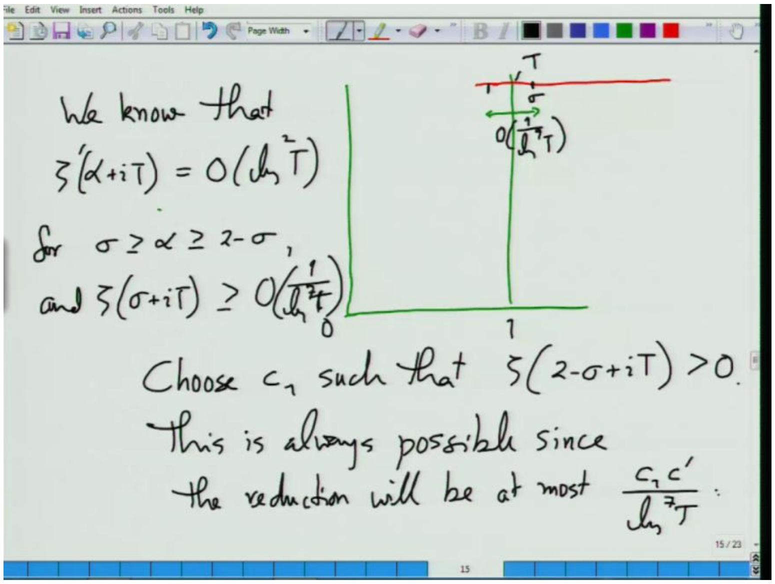

Now this goes to the next (( )), unless look at very closely, this is 0, this is 1, this is T and this line little here, looking there to look at these two points. This is sigma, little more (( )). So at this point, I know that zeta is ah at least greater than equal to one over log to the seventeen. I also know that throughout this region, the derivative zeta prime alpha is order log square T. For sigma greater than equal to alpha greater than equal to 2 minus sigma. Sigma is little more less more than one; if the 2 minus sigma, this is the same quantity less than one. And zeta prime sigma plus I T is greater than equal to order 1 by log 7 T. So the function which changes from a certain value and with it certain derivative along this way, as you travel along this. Question is can we attain the 0 in this region, the answer is no.

What is the distance between this. This distance how much is this. The sigma is one plus one over log T to the nine. So this is about two (( )) right. So this is the order one by log to the nine T. And this is bigger than log to the one over log to the seven T. So in that tiny window has more on this. In the worst case, you are always shrinking, there was a derivative, which will always taking it down with that speed. And you start with the certain number of here and go (( )) the derivative, how can you get?

Student: Derivative into the distance

Professor: Derivative in the distance, essentially that is the lowest we can get, so that is basically ah log square T into log order log to the ninety which is one over log to the 70. Now which earlier matter of fixing the, adjusting this constants pick initially, so this constants is up to (( )). So we pick it up really really small constant so that ah this I mean this distance constants is so small then you multiply this with this, it is still order one over some constants divide by log to the seventy, (( )) and that is smaller than this. So the reduction in quantity will be at most this. So c prime is the fixed constant, c is up to this (( )). Let’s show that in this we can start from here, and go all the way to this without finding the 0, and that is the prime number theorem, says that number of primes assumed practically approach x y log x.