Randomized Methods in Complexity

Prof. Nitin Saxena
Department of Computer Science and Engineering
Indian Institute of Technology - Kanpur

Lecture – 14
Graph Products

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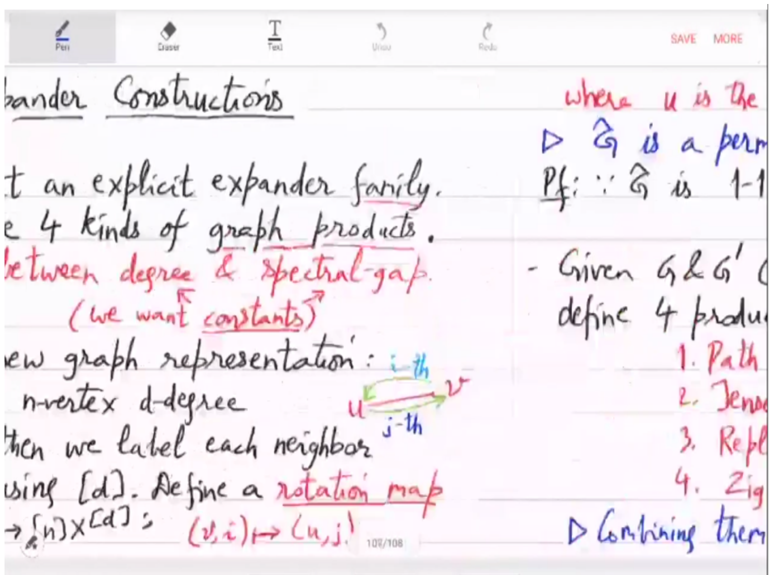

We have started this topic of explicit expander construction. So, for that we defined a rotation map on a graph, so it is basically a map from [n]× [d] to itself, n is the number of vertices, d is the degree of the graph. We will map v, i to u, j if the i-th neighbor of v is u and the j-th neighbor of u is v.

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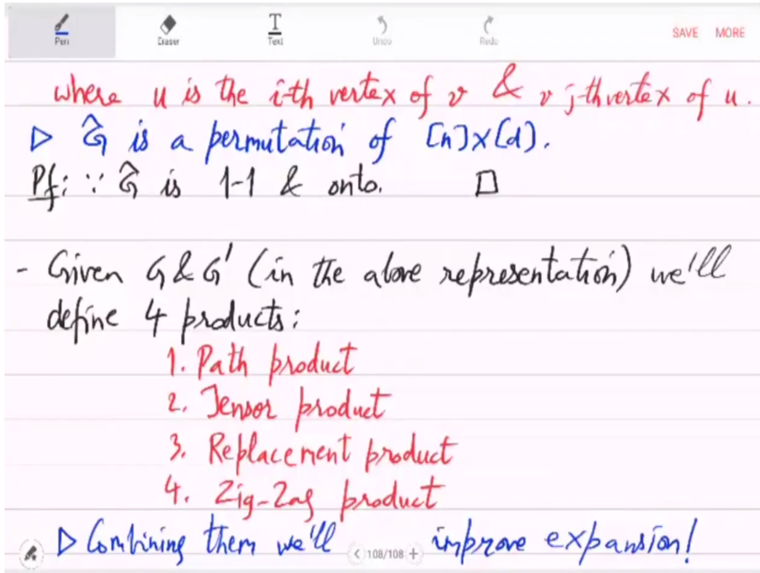

So, this is a permutation on [n]× [d]. This seems to be a simple thing, but it will be very helpful in later graph products that we will define and we call it $\hat{G}$. It is a permutation. So, we will now see 4 ways to multiply graphs.

1)Path product.

2)tensor product,

3)replacement product

4)zig-zag product.

So, they will have different properties.

They will either increase the graph size or they will increase the spectral gap or they will increase the degree of the graph and a combination of these will give us the right balance which is explicit expander high spectral gap and degree constant and an infinite family of graphs.

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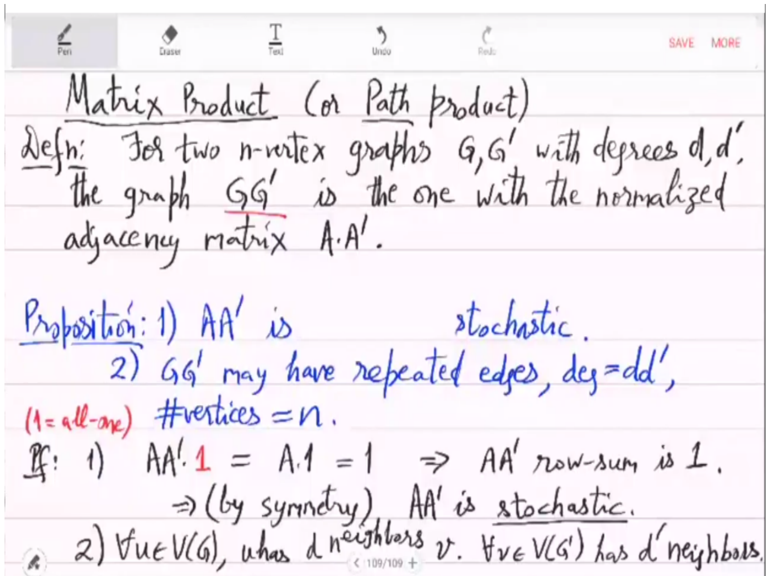

So, let us start with the matrix product. This is the easiest most natural one. Its graph interpretation will look like path product. So, what you do here is for two n-vertex graphs G, G' with degrees d, d' the graph product is written as GG' that is the notation of the matrix product. This is the graph with the normalized adjacency matrix AA'.

So, A is symmetric, stochastic and $A'$is symmetric, stochastic. You just multiply them and this is called the path product of the two graphs. So, the first property is that AA' is stochastic. Actually, it is not really symmetric, symmetricity is missing. It is only stochastic. AA' is stochastic and the graph GG' may have repeated edges and degree $=dd'$ counting the edges even when they repeat.

So, repetition will be visible in A' when you see the entry. So, for e example instead of entry 1 /d you might see 2 /d. So that means that there are two edges, so that is all what repetition means. And in that sense if you count degree will be dd' and number of vertices is what you started with. So, GG' is an n vertex graph, degree has increased. So proof is straightforward.

So, A' is stochastic because if you look at its action on this $\overline{1}$ vector right, entry is 1 / n each entry. So, since A' was stochastic you will get $\overline{1}$ and then since A is stochastic you will get $\overline{1}$. So,$AA'.\overline{1} = A\overline{1} = \overline{1} \Rightarrow AA'$ row sum is 1. And similarly, by symmetry you can show that column sum is also. So, both row sum and column sum they are 1.

And second is for every vertex in G number of neighbors of u, so u has, maybe I should have just combined the proof, let me do it here. Let me say that this is the all one vector, so 1=all-one, let me work with this. So, then $A'1 = d'1$ and then when you apply A on this all-one vector you will get d. So, you will get dd' all one. So, this will show you, well that is not really true, no sorry it is quite true.

So even with all-one vector, actually with all-one vector already this is 1 and then when you want to look at the degree what you should do is yeah so $\forall u \epsilon V(G),\ u\ has\ d\ neighbors.\ \ \forall v \epsilon V(G')\ has\ d'\ neighbors.$. So, when you look at A' what is happening? So, from u you are going to a neighbor following edge of G and then you will follow an edge of G' to go to a neighbor of v and G' which are d' many. So overall, the number of neighbors of u will be dd' in $GG'.Neighbors\ of\ u \epsilon V(GG')\ are\ d \times d'$

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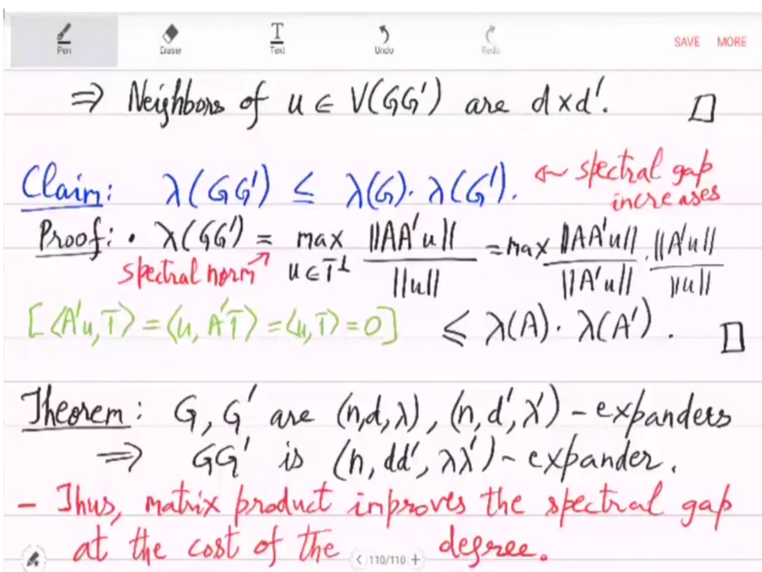

So that is it, that is the proof okay. So, degree is dd' in GG' and vertices and vertices we have not changed. So more interesting property is about the spectral gap which is that the spectral gap of GG' decreases[$\lambda(GG') \leq \lambda(G).\lambda(G')$]. So, what is the saying is that the spectral norm or the second largest eigenvalue of G which was less than 1 before and same for G' now when you multiply you will get something even smaller.

So, spectral norm actually reduces, spectral gap increases. So, at the cost of degree increase you can increase the spectral gap. So, this is a very good product and why is that happening? So intuitively if you look at expansion since you are increasing the degree so you are able to cover more ground in one step. So, just because of that you would expect the expansion to improve that is what this inequality is expressing mathematically and the proof is equally simple.

So, $\lambda(GG')$ is by definition what? Not by definition but we had proved this property at least this connection that it is the amount by which the matrix shrinks a vector orthogonal to 1, right. So, $\lambda(GG') = max_{u \epsilon \bar{1}^{\perp}} \frac{||AA'u||}{||u||} = max \frac{||AA'u||}{||A'u||} \frac{||A'u||}{||u||}$ okay and why is this product interesting?

Well because $A'u$ if u is orthogonal to 1, $A'u$ is also orthogonal to 1. So, $[< A'u > = < u, A'\bar{1} > = < u, \bar{1} > = 0] \leq \lambda(A). \lambda(A')$, right., so that is it. $\lambda(A) = \lambda(G)$ by definition you can talk about the graph or the matrix.

So, you get that $\lambda(GG') \leq \lambda(G). \lambda(G')$. This follows simply because of the spectral norm. So, what we have shown in terms of expanders is this beautiful theorem that if $G, G'\ are\ (n, d, \lambda), (n, d', \lambda') - expanders \Rightarrow GG'\ is\ (n, dd', \lambda\lambda')\ expanders$ because the degree grows and also this lambda multiplies, so it decreases, it is in the right direction.

Thus, matrix product improves the spectral gap at the cost of the degree. So, there is a trade-off. If you start with the spectral gap very small, then you have to do this many, many times, so your degree will keep on increasing. So, if you started with a general graph which has just no expansion guarantee just connected graph, then you know that spectral gap is around inverse poly.

So, to make it constant you will have to then apply this again and again and logarithmic many times and then the degree will become non-constant. So, this is not a perfect solution, it just is giving you a trade-off but not an optimal trade-off.

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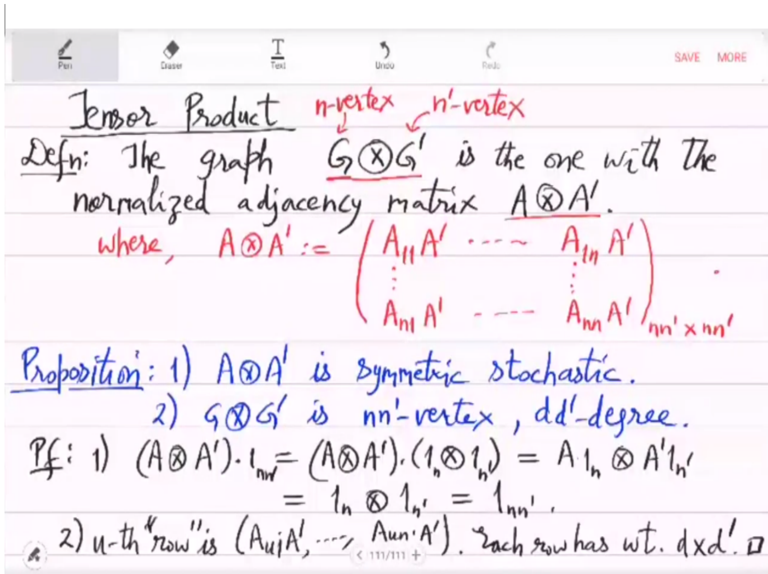

Second product that we look at is tensor product. So, this tensor product does not play a deep role in the construction. Its only job is to give you an infinite family. So, basically it will grow the size of the graph without changing the expansion so that you can produce of infinite family. So, the graph $G \otimes G'$is the one with the normalized adjacency matrix $A \otimes A'$.

So, the graph product is this notation and matrix product is this. So, if you do not know what is tensor product for matrices here it is. Also, we are assuming here this is n-vertex like before, but this we can now allow to be different n' vertex where $A \otimes A'=\frac{A_{11}A'\ldots\ldots A_{1n}A'}{A_{n1}A'\ldots\ldots A_{nn}A'}$ definition is basically what you do is you look at the matrix A and in every entry you put this big matrix A'. So, here is the picture.

So, this is the block matrix representation, so you have n ×n matrix A and in every entry you have put A' so the dimension goes to nn' both sides that is it. That is the tensor product for two matrices and the matrices do not have to be related at all. Dimensions are different. In fact, in general they may even be rectangular matrices but here we look at square matrix and then you get a square matrix tensor product.

So, the properties of this is $A \otimes A'$ is symmetric stochastic. This is the graph that you will get will be undirected graph symmetric metric normalized adjacency matrix is A' which is symmetric and the graph has $G \otimes G'$ is nn'vertex and the degree is, you can see from the definition degree is dd'. So, what is the proof of this? Symmetric you can see easily from the definition.

Stochastic you consider this where this all-one vector is of; I mean the dimension of this is nn' so that you do the multiplication. $(A \otimes A').1_{nn'} = (A \otimes A').(1_n \otimes 1_{n'})$ So, nn' once you can see as a tensor product of as I said you can look at the definition also for rectangular matrices. So, it is $1 \otimes 1$ different dimensions.

And this product you can actually see from the definition in red that you will get $A1_n$ which is $(A \otimes A').1_{nn'} = (A \otimes A').(1_n \otimes 1_{n'}) = A1_n \otimes A'1_{n'} = 1_n \otimes 1_{n'} = 1_{nn'}$. this shows that row-sum is 1 also column-sum is by symmetry 1, so this is stochastic and second property is even is easier and then prime vertices is clear. Why is the degree dd'? So that you can again see from the definition.

So, consider the u-th row$(Au_1A', \ldots\ldots, Au_nA')$. So, the number of these Au_1 to Au_n which are non-zero, this is d and for every matrix that is put there you can see that every row or u-th this is kind of the block row and each row here has weight which means number of non-zero entries in a row dd'. So, you can see that degree dd' and vertices is clearly grown by definition to nn'.

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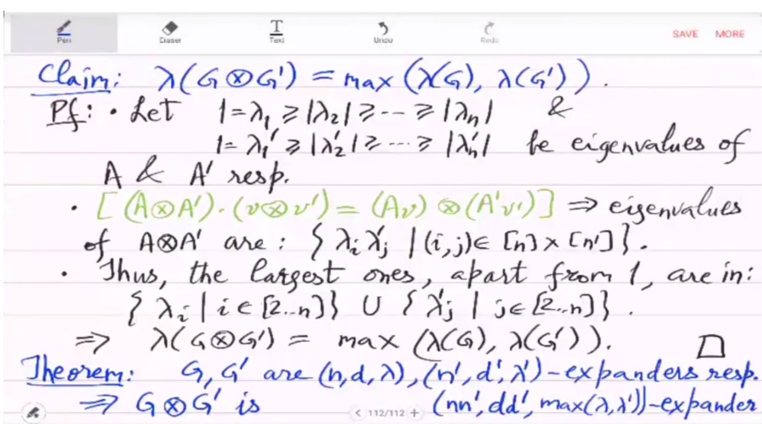

Now let us analyze the second largest eigenvalue$\lambda(G \otimes G') = max(\lambda(G), \lambda(G'))$ so why is that? So, this trick that we used $1 \otimes 1$ right, so now that is the trick you have to apply that trick on eigenvectors. So, eigenvector of A eigenvector of A' you take the tensor product and show that that is an eigenvector of $A \otimes A'$.

$Let\ 1 = \lambda_1 \geq |\lambda_2| \geq ...... \geq |\lambda_n|\ and$

$1 = \lambda_1' \geq |\lambda_2'| \geq .... \geq |\lambda_n'|\ be\ the\ eigen\ vaues\ of\ A\ and\ A'\ respectively$

. Now observe that $[(A \otimes A').(v \otimes v') = (Av) \otimes (A'v')]$. This you can see from the definition of tensor product. So, this means that the eigenvalues of $A \otimes A'$: $\{\lambda_i \lambda_j' \,|(i, j) \epsilon [n] \times [n']\}$.

Because you take the i-th eigenvector v_i and the j-th eigenvector v_j' respectively A' and then by this green formula you can see that $v_i \otimes v_j'$ is an eigenvector of $A \otimes A'$ and on the RHS what you will get is $\lambda_i \lambda_j'$as the eigenvalue. If you want to do this formally you have to prove that there are no other eigenvalues, but that also you can prove as an exercise.

These are the only, I mean since it is an nn' dimensional matrix if you exhibit nn'eigenvectors there cannot be anything else. So, these are the only eigenvalues. These are exactly all the eigenvalues and thus the largest ones apart from 1 are, so all the lambdas here are≤1, right. So, when you look at the absolute value the maximum will come from λ_i's and λ_j'.

So, you can just have to consider this set $\{\lambda_i | i \epsilon [2 .... n]\} \cup \{\lambda_j' | j \epsilon [2 .... n])$. So, λ_2 is the maximum in the first set lambda, sorry λ_j'. So, λ_2' in the second set that is what you get that $\lambda(G \otimes G') = max(\lambda(G), \lambda(G'))$. It is max of λ_2 and λ_2'absolute values, fine. So, what this gives you is if $G, G'\ are\ (n, d, \lambda), (n', d', \lambda')$ expanders respectively.

Then tensor product $G \otimes G'$ has vertices $(nn', dd', max(\lambda, \lambda'))$expander. So, the size grows, expansion does not change, but the problem is degree also grows. So, in this also the degree is growing.

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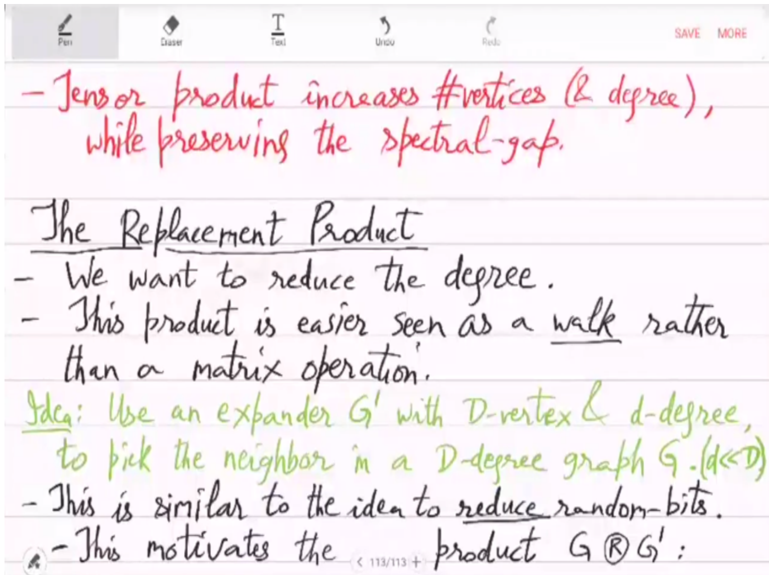

So, we learned that tensor product increases number of vertices and degree while preserving the spectral gap. So, this is not helping an expansion but it is only helping in giving a family but still there is a flip side which is degree is also increasing that you do not want, so that is a problem. Both these products the degree is increasing, so we have to do something about the degree increase.

When the degree increases, we have to apply another product or transformation so that a degree reduces, so what will that be? So that is a more complicated product called the replacement product. So, we want to reduce the degree that is the goal. So, what the replacement product will do is seen better in as a walk instead of matrix product or instead of matrix analysis. So, this product is easier seen as a walk rather than a matrix operation.

What is the idea? So, the idea is that as the term vaguely suggests when you have lots of neighbors in a vertex that is the issue of high degree, right, that a vertex has lots of neighbors which neighbor to go to that you decide using another expander on that many vertices. So, if you have big D many neighbors, then you consult an expander with big D many vertices but much smaller degree.

So, using that expander you do a walk, you move to that other expander do a walk and where you end that will decide what neighbor to pick. So, you are replacing your neighbor choice by a suitable expander walk. So that is the basic idea. Use an expander G' with D-vertex and d- degree to pick the neighbor in a D-degree graph G. So, you have a graph G whose degree is quite large D.

So, when you want to pick a neighbor there to do a random walk in G, you transfer control to another expander G', completely different expander G' which is on vertex D many with a much smaller degree d. There you do a random walk and where you end that becomes the neighbor you picked in G okay that is the plan. And we will pick d $\ll$D.

So, this we are posing as a walk in two graphs, but ultimately this will actually give you a new graph. When we unfold this, we will get a new graph that we have to analyze, but this is the basic idea. So, this is similar to the process or the idea to reduce random-bits. It is a similar idea. So, this motivates the product G®G' that is the notation.

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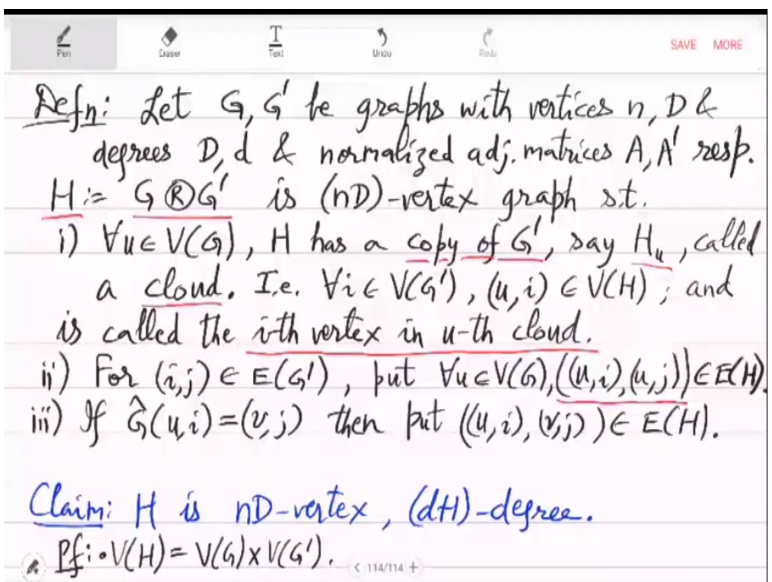

So, let us define it formally now. So let G, G' be graphs with vertices n, D and degreesbig D, d and this normalized adjacency matrix A, A' everything respectively. So, for G is n vertex D degree matrix A graph and G' is a D vertex small d degree A'matrix graph. So, the replacement product matrix H is defined as $H:= G®G'\ is\ nD$-vertex graph such that for every vertex $\forall u \epsilon V(G)$, something like the tensor product for every vertex in G you put this G'.

Copy of G' say H_u called a cloud. That is in formula $\forall i \epsilon V(G')$, $(u, i) \epsilon V(H)$ and is called the i-th vertex in u-th cloud. So, in every vertex in the graph G you put G' call it the cloud, and in the u-th cloud you have the i-th vertex. So you the vertices are now (u, i). So the vertices are like $V \times V'$ that is the vertex set. What is now the edge set, how do you connect? How do you draw edges in H?

So, in the cloud the edges will follow what G' has. So for (i, j)ϵE(G')so put $\forall u \epsilon V(G)$, $((u, i), (u, j)) \epsilon E(H)$ okay that is an edge in the u-th cloud. This is just following naturally the edges of G'. And more interesting is how do you connect two clouds that will follow the edge set of G, how do you connect? The obvious naive thing would be to connect every vertex in a cloud to every other vertex in the second cloud according to G connection.

But that is not good for us because the degree will be too much, we also want to control the degree. So, what we will do is we look at the rotation map here. So, if$\hat{G}(u, i) = (v, j)$ which means the i-th neighbor of u in G is v and the j-th neighbor of v in G is u, that is the rotation map of G, then $((u, i), (v, j)) \epsilon E(H)$you connect these two vertices okay. So that describes all the vertices and all the edges in the replacement product H.

Now the claim is that H is nD vertex () degree which is a big improvement on D. The degree has suddenly fallen. Why did it fall? So intuitively in the definition if you look at condition 2 that is where we added lots of edges or lots of neighbors but that was dependent on G' so it is only d. So, we have actually reduced the degree a lot.

If l d is much smaller than D, then this is a degree reduction and vertices have obviously grown multiplicatively. So, size of G times size of G' so that is the proof. So, $V(H) = V(G) \times V(G')$.

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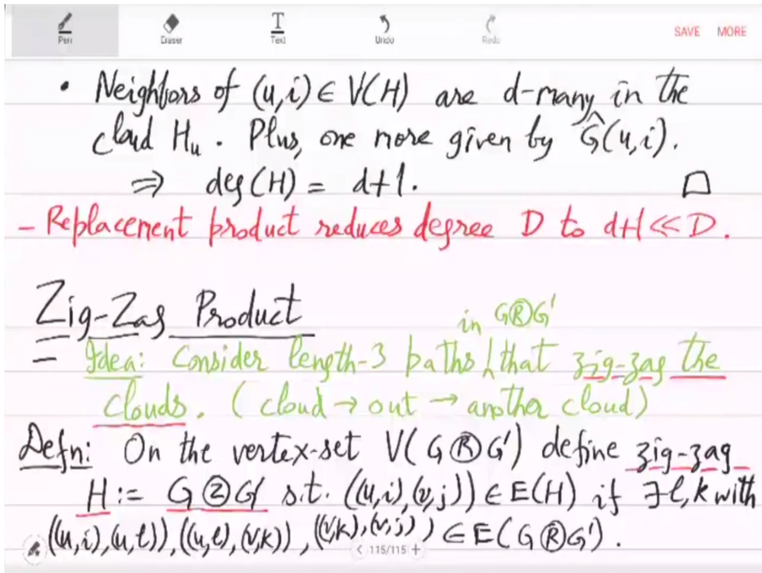

And second is that neighbors of $(u, i)\epsilon V(H)$ are d-many in the cloud H_u and is there any other neighbor in H? Yeah, there is just one more which is given by this $\hat{G}(u, i)$ So, this means that degree $(H) = d + 1$.

So, this replacement product reduces degree $D\ to\ d + 1 \ll D$ that is the job that is the purpose of this product. So, the first two products they blow up the degree multiplicatively, this third product reduces the degree. So, now we have three operations which are kind of balancing each other out when taken in the right combination. And the right combination will be the final product which is called zig-zag product.

So, in the zig-zag product we will now combine these three. In zig-zag product now what we will do is we will travel once within a cloud, once outside and then another time in the cloud. This is why it is called zig-zag, so you zig in the cloud, zag outside, zig inside again. So that is the idea. Consider length-3 paths that zig-zag the clouds. So, $(cloud \rightarrow out \rightarrow another\ cloud)$. So, formally we have to describe the graph, the G zig-zag product G' graph.

What are the vertices? What are the edges? Vertices are the same as for replacement product. So, on the vertex set $V(G Ⓡ G')$define the zig-zag product H :=$G Ⓡ G'$ s.t($(u, i), (v, k))\epsilon E(H)$ now $\exists l, k$ such that essentially you go from i to l inside the cloud of u, then you go from l to k outside the cloud and then you go from k to j.

So, I should have called this j. So i to l in the cloud, l to k outside the cloud and in the next cloud k to j that is what I will write now. It will look complicated. So $((u,i),(u,l)),((u,l),(v,k)),((v,k),(v,j)) \epsilon E(G Ⓡ G')$ the replacement product. So, length-3 paths in the replacement product is what we are using.

It is a bit better connected than the replacement product graph because you are looking at 3 length paths there. So, notice that we did not do the 1λ analysis, spectral gap analysis for the replacement product. Now we will do it for zigzag and it will be pretty good. So, what is the number of vertices that is the same as n times D. What is the degree? Notice that because of these two cloud edges so this i to l and k to j, you will get now degree to be d times d, d^2 degree.

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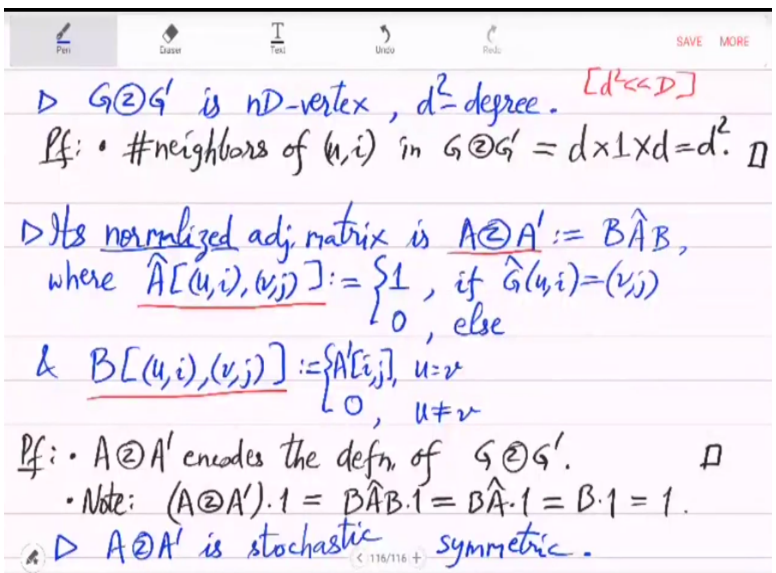

$G Ⓩ G'\ is\ nD - vertex, d^2 - degree$

So, basically neighbors of (u, i) in $G Ⓩ G'$=d and for each such neighbor when you pick there is only one out, right, so that is not an option, not multiple option. And then when you go to the next cloud there you will have again d neighbor, so this is d^2. So, we will assume d^2 also to be very small compared to D.

So, for vertex growth multiplicative in D, the degree is still pretty small. It carries the properties of replacement product, but now we have to do the lambda analysis the spectral gap analysis that is quite complicated here. So, let us first write this in matrix form. So, it is normalized adjacency matrix is denoted like this $A Ⓩ A'$. You can imagine this matrix B as the intracloud step and $\hat{A}$as the intercloud across the cloud step.

So, it is $A Ⓩ A' := B\hat{A}B$. We have to define these matrices. So, $\hat{A}[(u,i),(v,j)] = \{1, if \hat{G}(u,i) = (v,j)$

$=0, else$

. &$B[(u,i),(v,j)] := A'[i,j], u = v$

$= 0, if\ u \neq v$

. Proof is quite simple once I give this statement because I mean look at the just the definition of $\hat{A}B$ and the definition of zig-zag product keeping in mind always the replacement product.

So, $A Ⓩ A'$ encodes the definition of $G Ⓩ G'$t and also note that$(A Ⓩ A').1 =B\hat{A}B.1$ Now what is B times 1? From the definition of B you can see that when you look at the(u, i)-th row number of neighbors is exactly small d for every row, right. So, you have $(A Ⓩ A').1 =B\hat{A}B.1=$ $B\hat{A}.1 = B.1 = 1$

Other thing is that this is symmetric matrix, you can see from the definition it is symmetric. So, this actually means that let me finish the proof here and then state this property that this is $A Ⓩ A'$ is stochastic symmetric.

So, this is why we have already said that it is a normalized adjacency matrix so that is also true. So, we have a nice expression for this zig-zag product matrix and what we will do next time is we will do the spectral analysis so what is the spectral gap.