Computer Graphics
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Lecture - 36
Curve Representation

Welcome everybody to the lectures on computer graphics. Today we are going to discuss descriptions based on curves, curves representation and of course it will be followed by the method on representing nonlinear surfaces or nonplanar surfaces. In solid modeling we have seen the methods by which an object is defined by set of polygons, triangles or quadrilaterals and even for nonplanar objects like spheres, cones, cylinders which are considered as primitives in OpenGL. We know that you can use a sweep representation to construct the wireframe diagram which represents a curved object using a piecewise approximation; a small planar patch represents a small part of a curved object

But it is essential in certain applications that such piecewise approximation may not hold good. It does not provide you a very natural modeling of the surfaces which are curved or nonplanar where there are sharp bends either in 2D or in 3D surface where there are large changes of curvatures of a surface, the gradient change is very fast, it is almost impossible to have a very good approximation or using planar patches or using polygon or modeling. So the reason needs to represent nonplanar objects with the help of curves in 2D and surfaces in 3D

We will look at curves to start with today. Different forms of representing curves which are not only use for modeling surfaces but also could be use for drawing curves. For drawing curves, often you would be joining points with the help of straight lines but if the set of points which define the functional behavior of a system or an experimental data which we have observed y is a function of x or z is the function of xy 2D, 3D or higher dimensions it is in fact impossible to have a good representation of that curve or a surface in 3D or a curve in 2D with the help of linear interpolation. So it is often necessary to obtain a curve in a model act using a curve. The behavior of the functional form of the plot in the data present in a plot should be present in the form of a curve. So we will look at the curved representation.

And broadly I could say that the representation of a curve could be a non-parametric form or a parametric form Y equals f X is a non-parametric form or a parametric form which I will say X(t) or Y(t), t being the parameter we know this parametric form of a line which we have used for various applications such as clipping and intersections. Of course we can have an explicit form in the form of Y equal to mx plus b or an implicit form of the form f(x, y) equal to 0. This implicit form we already have views in cases where we use the Bresenham's algorithm or also in the case of clipping as well. So I can say an explicit and implicit form or a non-parametric parametric form we will keep on interchanging between the two but typically most of these curves representations follow the parametric form. We will see that the parametric form is the most common representation of the curve which is used in most of these cases.

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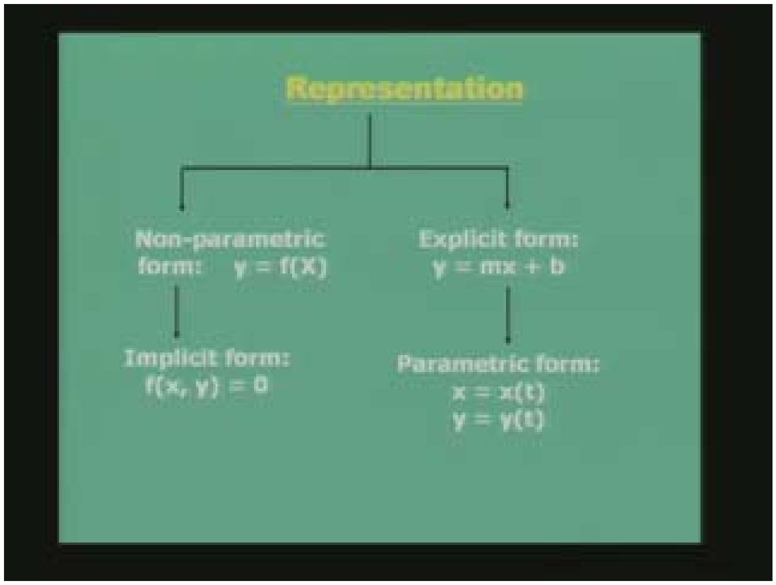

Before we go on to discussions of various parametric form at least we know a few simple examples of curves representation. Let us say take the equation of a circle or an ellipse which we have already used. So far in 2D we have already used the equation of a circle on an ellipse and those points lie on a plane. But a curve could exist in 3D as well. In fact the circle could exist in 3D. The points on a circle lie in a plane it is a two dimensional distribution of point but that plane can be visualized to lie anywhere on the 3D either on a plane or a circle then we have also have a curve entry. So, we know those special functional forms of those curves which produce a circle or an ellipse. And we will see special more such special forms of curves today. To start with, you get used to these expressions of curves for parabola or a hyperbola or an ellipse or a circle.

Of course circle is a special case of an ellipse. So if you have equations of parabola, hyperbola and ellipse in mind you can actually generate various types of curves by varying these parameters of three different curves. And all these three different curves are again special types derived from one particular generalized curved representations or curves or what are called the conic sections.

Let us introduce that word conic sections in this lecture today. A conic section is obtained through the word cone. You take a cone and you can obtain a conic section. You take a cone and intersect that cone with a plane. Take a planar surface and intersect a cone. Now the relative locations of the cone and the plane will dictate the intersection you obtain. But whatever intersection you obtain you will get a curve. The intersection of two surfaces will be a curve.

In 2D intersection of two lines gives a point and in 3D the intersection of two surfaces yield a curve. That curve could be in 3D or the distribution of these points on the curve could be on a two dimensional plane. It should be in a plane because you are using a two dimensional plane to chop off the cone and the resultant intersection of the contour which you get of the conical surface and that which intersects the planar surface results in a conic section. So that is the idea of a conic section intersection of a cone with a planar surface which you will see. We will first look into a second degree implicit representation of a curve and then move on to conic sections. And I must inform you a prairie that after quiet a few series of lectures where we discussed various types of algorithms on computer graphics we are back to mathematics again.

If you remember, of course when we discussed of two dimension transformations, three dimensional transformations, three dimension viewing we had lot of equations mainly to do with matrix manipulations. Of course the same will come here but we will probably come with second degree or third degree equations here. I request you to copy these equations and then try to derive some of these derivations I have used in the slides. I have left some equations in open for you to practice and try out as an exercise. Please practice these equations yourself and draw these figures which will come on the slides. You need to practice these equations otherwise you will not be able to master both the concepts of visualization of these curves. And of course we will be coming across surfaces at the end of this remaining lecture series. You must practice these equations to get a strong feeling and control over this set of representations of curves. So let us look back into the slide. The second degree implicit representation is often given by this expression.

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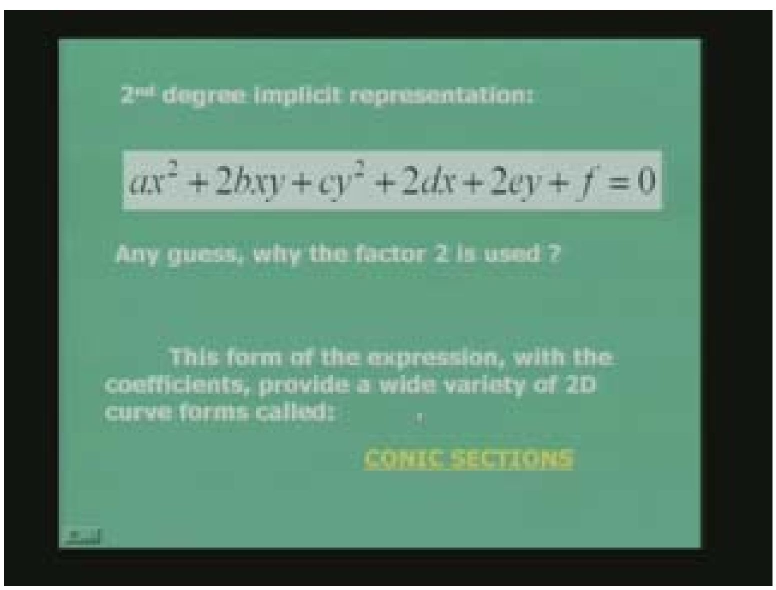

You see, this is a second degree equation because you have square terms both in x and y and of course you have a binomial term here xy. And you look at the coefficients you have seven of them a b c d e and f. And this is an implicit representation of a curve something similar to f(x, y) is equal to 0 with seven parameters. The only point which you should start thinking is there is a factor two here which is added to b d and e.

I could have written this expression in the form of ax square plus bxy and then cy square dx plus ey plus f is equal to 0. That means just through out the factor two. And it occurs in three of those seven terms and you keep guessing why. But the answer will not be in this slide but after a few slides in this lecture today that is towards the end and we will see why this factor two use you. But you can also represent a second degree implicit form without this factor two. Nobody stops you from doing that. But there is a reason, a mathematical reason and a logical reason also. Later on we will see why this factor two is used. So you can keep guessing why there is a factor two used along with a coefficient b d and e only but not with a c and f. So this form of expression with the coefficients provides a wide variety of 2D curve forms which are called the conic sections.

We discussed about these conic sections just now and this is the mathematical expression of a conic section given on the top of this slide. And this form of the expression with the coefficient which means if you keep these six coefficients a b c d e and f varying, six of these coefficients are present in this second degree implicit form and you keep varying these coefficients in the implicit representation you will get various types and forms of curves of course in two dimensional because you are talking of x and y. So in a two dimensional plane or two dimensional access system you will have various types of curves and they all fall under the broad generalized category of what are called the conic sections. This is what we will call as the conic sections and we just discussed sometime back on what is meant by conic sections. So that is the implicit form and they are called the conic sections. So let us look at some special forms of conic sections.

I did mention sometime back that the three special cases of conic sections are parabola hyperbola and ellipse. So we will discuss three of these and we will discuss three special cases not with respect to those six parameters of the second degree conic which we just discussed now but we will bring that later on. So do not confuse the coefficients a to f, six of those which we discussed in the previous slide for the generalized second degree implicit representation for a conic which was the form. But we will see those special forms and the ranges of values you give to those six parameters. You will have special curves like the parabola, hyperbola and ellipse.

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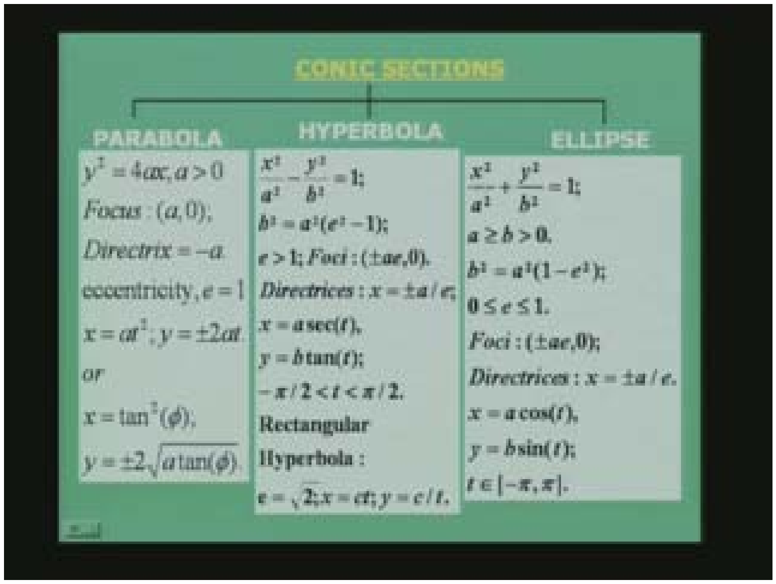

But we will look into the geometrical class of equations of parabola, ellipse and hyperbola. These are again given in any fundamental books and geometry on computer graphics. So we will look into those expressions and the parameters used now to describe these three different curves are not related. Right now do not relate these with the parameter which we have discussed. So come back to the parabola. This ‘a’ is different from the ‘a’ which we discussed earlier. I repeat, this ‘a’ has no relation with this parameter ‘a’ in the second degree implicit representation. This is just a generalized form of a conic. So let us first discuss the special forms and the ‘a’ is called the focus of the parabola. We will come into the figure of the parabola later on. But we will first see these three special forms of these special types of conic sections or special class of conic sections.

Parabola is an expression, y square equal 4x you all know that and a must be positive number otherwise y will be imaginary. We will know that hence focus of the parabola is a, 0. We introduce the term directrix here which is given as minus a. we will define this later on using the figure. So it is a point x equals minus a is on a point on the x axis or you can visualize this directrix to be a vertical line x equal to minus a. x equal to minus a will be a vertical line which is parallel to y axis and it will be towards the negative x axis and that is called the directrix. We will see that in the figure. So, that is the directrix, so we know what the focus is and also the directrix are symmetrical. Remember, focus is a point but directrix is a line. You also define the term eccentricity in this context which is a very important property of all the special conics e which is equal to 1 in the case of a parabola.

We will see thus eccentricity e will not be equal to 1 in the case of hyperbola or ellipse, in the parabola it is equal to 1. We will define what eccentricity is. It is the ratio of two different distances. We will see what this eccentricity term is. These are different forms of writing the parabola also.

This is a parametric form of writing the parabola. So with the parameter; a given here x equals at square t is the parametric form, a is the co coefficient of the parabola. And so if you substitute these two equals on to y square equals to 4x you will see this is satisfied. It can also be represented in another trigonometric form using this particular expression x and y tan square fi. Of course fi has to be within certain range and this is also an expression of the parabola. We will see the other forms and then look into the figure for the other case of a parabola. This is the equation of a hyperbola.

We will see the difference between the two and then look into the figure also. So there are two parameters in the expression of the hyperbola a and b as given here. We had only the coefficient a for the parabola but now there are two coefficients a and b for the hyperbola. This is the expression and it could be also written on a expression form like this in terms of eccentricity, in the form of eccentricity it could be also written in this corresponding form where eccentricity is more than 1. The relation between a and b is given in terms of this eccentricity here where e is more than 1. So the coefficients a and b are related using this expression.

There is two focus, so we have a foci for a hyperbola at plus and minus a into e. At plus and minus a into e we have the focus of the hyperbola. And the directrix is given by again lines, vertical lines but there are again two of them. You remember, in the case of parabola we have x equals minus a but in the case of hyperbola we have x equals plus and minus a by e. So you can use the same expression since e is equal to 1 so it does not matter I can simply write minus a here in the case of a parabola because a by e with e equals 1 will give you minus a. So directrix is basically plus and minus a by e in the case of hyperbola and the focus will represent minus ae and 0. So this is the parametric form for the hyperbola as you had this in the case of a parabola at square and 2at as the parametric form for a parabola we have this or even this let us say for the parabola in terms of tan fi we have in terms of the secant which is 1 by cosine term and the tangent of t. So t between the parameter a and b be the coefficients of the parabola you can use parameters a and b but I have used coefficients a and b and t being the parametric form, the parameter parametric form of the expression t varies between this range minus pi by 2 to plus pi by 2.

A special case when the eccentricity e which is more than 1 as given in the case of a hyperbola becomes equal to root 2 then it becomes a rectangular hyperbola which is equal to root 2 in the case of a rectangular hyperbola and x and y are given in the parametric form also as c multiplied by t or c by t. c is an arbitrary constant and this hyperbola becomes a rectangular hyperbola. We will see this special case when we look into the figure of what is a rectangular hyperbola with respect to a normal hyperbola given by these expressions here or the parametric form given here. So I hope the concept of focus, directrix and eccentricity is getting clear. Of course we have to identify what is the expression of eccentricity here which is more than 1.

We look into the third form which is an ellipse as given here. The nature is actually almost similar to the case of a hyperbola where we have the minus sign replaced by the plus that is the change. We have the ellipse and of course a special condition is that a is greater than equal to b and b will always be positive, a and b are always positive values and the relation between a square and b square is almost similar.

Since now the eccentricity in this case for an ellipse unlike the hyperbola where we had the eccentricity more than 1 it is now less than 1 it is a fractional number lying between 0 and 1. So these are very interesting information where we have in the case of a parabola where we will use the letter e, the variable e mentioning or specifying the eccentricity of this special type of conic section. So e is equal to 1 in the case of a parabola, it is more than 1 in the case of hyperbola and it is between 0 and 1 in the case of a ellipse.

For an ellipse the fractional number is between 0 and 1 so when it is equal to 1 we have a parabola and if it is more than 1 it is a hyperbola. So these are the three ranges. It is never negative because the eccentricity is a ratio of two distances we will define that term later on using a figure. Right now we will go a head and finish the differences in terms of trying to look at the focus eccentricity e and directrix for these three curves. So the focus is plus and minus a into e in the case of an ellipse as it was a by e, in the case of a hyperbola it was only minus a and in the case of a parabola it is plus and minus ae.

And of course it is on the x axis. So focus is here, focus is a it is ae plus and minus ae in the hyperbola and it is the same in the case of ellipse and hyperbola. Directrix is a by e the same as the hyperbola. Directrix was a since a is equal to 1 the formula for directrix is actually the same. So these are vertical lines at plus and minus a by e but in the case of parabola it is only at minus a. This is the parametric form for the ellipse so cosine t and sine t you can see that from ellipse it can degenerate very easily to a circle when a is equal to b and t is within the range minus pi to plus pi so you have a full period for the ellipse which comes back. Let us look at the figures now for the parabola and hyperbola first.

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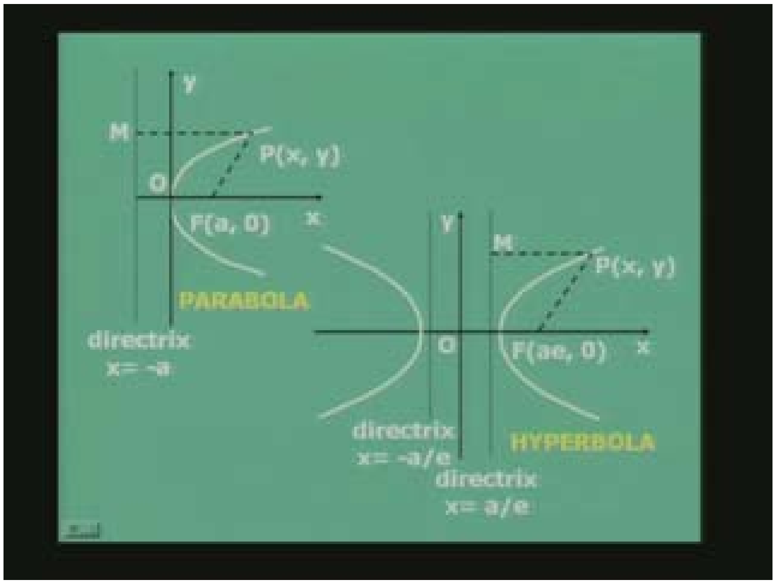

Starting with the parabola as I was talking about the focus was at F which is a, 0. So this distance is a. This is any point on the curve say P(x, y) it could be anywhere on the curve. And the vertical dotted line is a directrix which is at x equals minus a. If you look back you remember the directrix is at minus a as given here for the case of a parabola. So if you look forward for the parabola the directrix is at minus a. And this is the origin O, this is a point M which is the perpendicular draft from the point P on the directrix line. On the directrix if you draw a perpendicular line from P that project the point P on the directrix you get the projection of P as M on the directrix line. So the eccentricity term which is defined as e is equal to 1 here, of course it is not very clear with respect to this figure I must admit here that most of these curves are not purely computer simulated.

Some of them will be computer simulated but I have tried to draw them by hand sketch so some of these are qualitatively looking very good but it may not be quantitatively accurate so I must inform you this. So, for a parabola the eccentricity is defined by the ratio of this distance that is between P to F and P to M. I repeat; from PF to PM that the ratio of the distance of a point on the curve to the focus divided by the distance of the point from the curve through the directrix. So the ratio of these two distances is the eccentricity e and since we are talking of a ratio and distance is always a positive value so the ratio also will be a positive number it will never be negative.

Of course it could be less then 1, it could be equal to 1, it could be more than 1 and based on these three conditions we have three different curves. As for example the case of a parabola these two distances are same and hence the eccentricity e is equal to 1. So e is equal to 1 in the case of a parabola we have seen, it is more than 1 in the case of a hyperbola, ellipse it is less then 1 but greater than 0 it is a fractional number. So remember these three ranges of the values of e. I hope you have noted down very carefully these three terms from the previous slide itself.

And remember, the eccentricity we have just defined now with this help of this figure. It is definitely going to be the same. It is the ratio of the distance of the point on the curve to the focus divided by the distance of the point from the directrix. That is the definition of eccentricity if the parabola here and this is the case for a hyperbola. And we have two directrix as given earlier at plus and minus a. Two of these vertical dashed or dotted lines at x equals a by e here and x equals minus a by e here is the origin and this the point P again on the curve on the hyperbola and you drop the perpendicular on to the M. So if you take a point P here on the negative x axis you can drop and you will get another point MPF. So this is the distance which is be more than 1 here there is P to F divided by P to M so that is the eccentricity here.

The focus is now ae because since e was equal to 1it was a, o, now it is ae, 0 and ae is more than 1 we know that so this is the case of a hyperbola. So this is how a hyperbola qualitatively will look at. I must again admit that this not a computer simulated one. But one can easily do that using any simulation tool box, you can use simulings or matlab tool box or any other thing to or any other plot command to generate points for a hyperbola, parabola or an ellipse and plot that. So qualitatively the curves will look like this. Let us look in to the last one which is for the case of an ellipse. It has two focus here like the case of a hyperbola so it is at ae, 0 and minus ae, 0 so these are the two focus of the ellipse here and this is the point P(x, y) 0.

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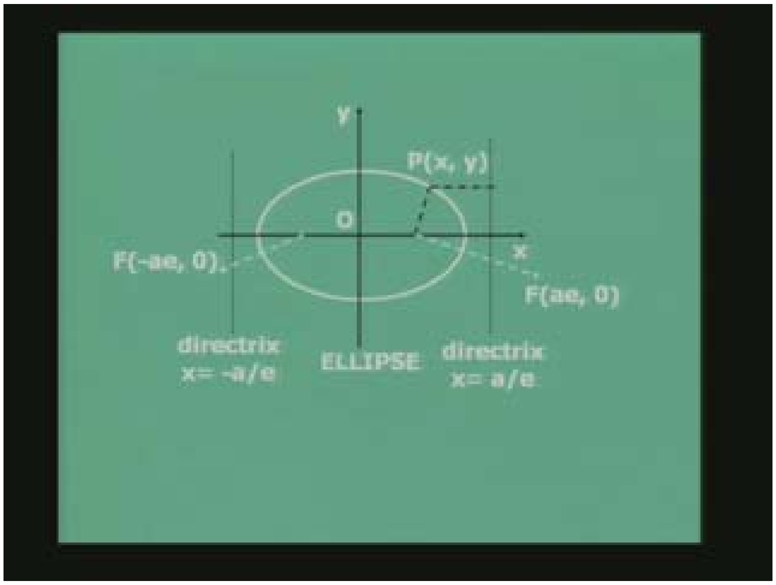

Remember, the definition of e terms that is the ratio here will be such that this intersection is at a, 0 here, for the ellipse it is 0, b at this point and a, 0 at this point and the ratio of distance will be less than 1 but more than 0. So this is the ratio of this by this distance P to F and P to M here among this point is M and P to F and P to M will be between 0 and 1 which is a fractional number. So this is the case of an ellipse. We will go back and compare this figure with the case of a parabola and hyperbola again. This is what you have for a parabola and hyperbola. Focus ae, 0 or a, 0 directrix plus and minus a by e the same concept hold goods here.

There are two directrix for an ellipse, the two focus ae, 0. Parabola of course you have only one directrix and one focal point and the eccentricity is defined by the ratio of these two distances of a point from the curve to the focus and projection of the point on the directrix. This is what we have as the three special curves under the category of conic section. In the three special classes of curves as parabola, hyperbola and ellipse the difference is mainly in the eccentricity which distinguishes them as termed here. But of course we have seen in the previous slides also that the expressions do vary in terms of the Cartesian coordinate form or the parametric form of the expressions of an ellipse hyperbola or parabola.

So let us look back into the slide to compare the three different and special classes of conic the parabola, ellipse and hyperbola once again. And we will look back into this slide to find out the difference in both the Cartesian coordinate form then the focus which is ae, 0 where e is equal to 1 in the case of a parabola, e is greater than 1 in the hyperbola and e is between 0 to 1and it is a fraction in the case of an ellipse. The directrix is at ae, minus ae here single directrix in the parabola, two directrix here ae plus minus a by e the directrix is same expression in the case of hyperbola and ellipse and of course the parametric form here. You can either use any of these or this in the case of a hyperbola using a trigonometric form 1 by cosine and a tangent and a cosine and a sine in the case of an ellipse. So these are the three expressions for the conics. Look back into the figure, these are the figures again for the parabola and hyperbola the directrix and focus are the main area of interest here.

We should know how to compute the eccentricity by the ratio of these two distances and in case of an ellipse as well here. So once you have seen these three forms of the special class of conic sections we will move forward to talk of the general form of a conic once again. But I will probably give you a home assignment right now to derive the polar coordinate of the equation of a conic, take this as a home assignment. If you look back into the slide I will give you the answer, try to derive this and take it as a home assignment. The polar equation of a conic given in this particular form or theta form where L is the distance of eccentricity e multiplied by the distance between the points F and d.

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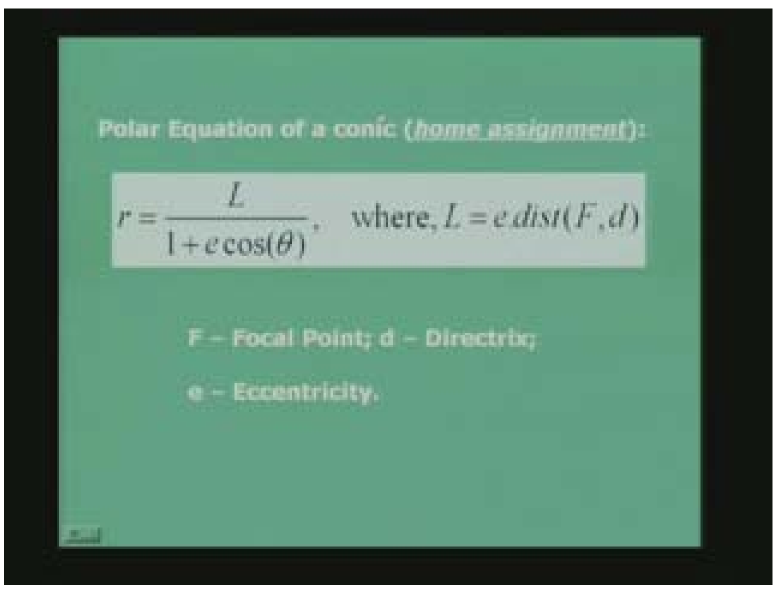

F is the focal point, d is the directrix, d is a point on the x axis we know that and so F, d distance e is what L is And so this is the result which I want you to derive, L divided by 1 plus e cos theta. I will give you a hint to derive these expressions. Once you pickup this hint and follow, it will be easy otherwise you cannot easily derive this from the Cartesian coordinate form. Assume the focus to be at the origin and that will help you to derive the expression of this polar equation of a conic in Cartesian coordinate form. So, coming back to the second degree general expression of second degree implicit representation of a curve and if this conic section passes through the origin then of course the parameter f goes down to 0.

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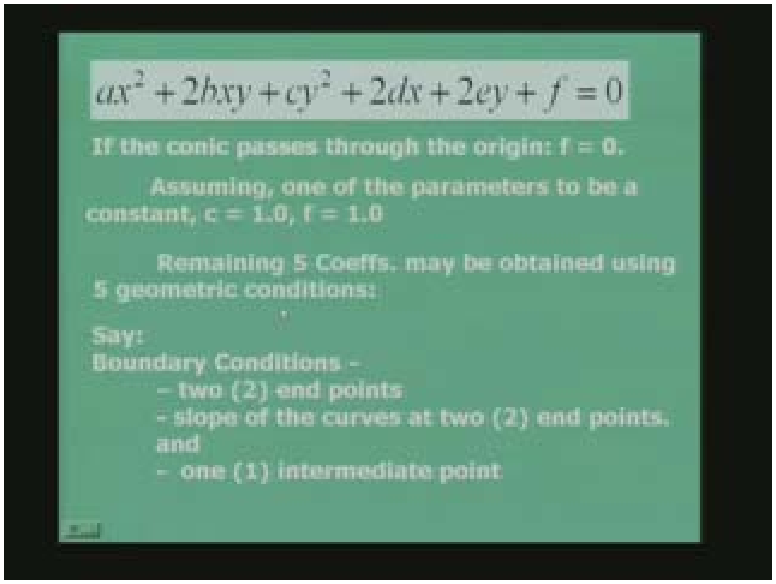

Remember, this a and b has nothing to do with the ab which we discussed as parameters for our ellipse parabola or hyperbola. These are the six parameters of the second degree generalized expression of a conic. And we now we will discover if you have not understood why we need these factor two in three of these six conics. But you can easily visualize that if this conic passes through the origin then f is equal to 0 because if you substitute x and y equal to 0 on to this expression you will easily derive that the factor f or the coefficient or parameter f should be equal to 0.

Assuming one of the parameters to be a constant say c is equal to 1 or f is equal to 1, whatever but of course f can be also visualized to be 0 either by assuming the conic section to pass to 0 or we can move the curve such that the curve passes through the origin or the origin can be moved to a point on the curve. We have done these cases when we were drawing lines, circles and ellipses using Bresenham's integrity algorithm. So moving the origin to the point on the curve or bringing back the curve to the origin does not make a big difference. So you can always do that and normalize the parameters and shift the curve back and we know the translational functional form.

So assuming one of the parameters to be a constant we actually will be having five coefficients left and those have to be derived or obtained using five geometric conditions. Either one of the c and f can be assumed to be a constant or known either f to be equal to 0 or f could remain as is equal to 1 or c is equal to 1. You can actually do these by dividing this entire expression by any one of these parameters. Try to do that, take this entire expression and divide this by either c or f you will get rid of one of these parameters very easily. So you will be left with five out of the six parameters. So we are talking of these five coefficients. So there are various ways by which the six can be reduced to five. Assume the curve to be passing through the origin so f is equal to 0.

The other thing could be that you can take one of this to be constant, c or f to be equal to one. The third way of visualizing this is to divide this entire expression by anyone of these coefficients again c or f. So you will be left with only the remaining five coefficients out of the six. And looking back to the slide these remaining five coefficients out of the six may be obtained using five different geometric conditions. And say the boundary conditions can be obtained by these special conditions of boundaries where we talk of two end points. So you assume that the two end points of a curved are known, x_1y_1 the curve is traveling from a point x_1y_1 to a point x_2y_2. This has happened in the case of a line. When we drew a line using Bresenham's also the line moved from a point x_1y_1 to a point x_2y_2.

Of course in the case of a ellipse or a circle which we drew with Bresenham's we knew where the curve was starting from a,0 we were moving in the second octant or first quadrant in the case of an ellipse, circle and using symmetry we generate the point. So we know where the starting point is. We know in most cases especially in the case of a curve we assume the finishing point to be known otherwise the curve cannot be defined. So the two end points are the first two boundary conditions. The two end points are known as the boundary conditions and we also assume that the slope of the curves at these two end points is also known.

Therefore, we not only know where the curve starts and where the curve ends, it could be in any form but we know how the curve starts and how the curve ends like in the case of an ellipse, hyperbola, parabola or circle. If you take a part of those curves and say this is our arc of our interest we will not only know the starting point and the finishing point of a circle, ellipse, parabola, hyperbola, whatever curve we are talking about but we also know how the curve starts and how the curve finishes.

When we talk of the term how I mean the derivative or the slope or the tangent to that curve at those starting and finishing points is also known. I repeat; I not only know the starting and the finishing point of the curve but I also know the nature by which the curve leaves the starting point and finishes at the end point. That means the tangent to that curve at the starting and finishing point is known or the slope of the curve or the derivative of this curve is known. It is something similar to trying to give an analogy in the physical world, we know the trajectory by which an aircraft will takeoff and the trajectory by which the aircraft will also land in an airport. That is pretty well defined and all the aircraft which is flying in the world have to follow certain trajectories of descending terms of its slope or derivative. Of course it is not a pure curve sometimes it could descend in a linear form but at the end it could have a slow moving curve form and tangentially touch the point at the air stream where it lands and of course it must also take off in a particular manner from the air stream at the starting point.

So we have a starting and a finishing point and we also know the derivative slope or the tangent to those curves at those two points. Those are the first 4, 2 plus 2 is equal to 4 boundary conditions and one more boundary condition we can visualize is by assuming an intermediate point. That is why we have the five geometric conditions to obtain the five different coefficients of these generalized conic. And we know how the six parameters or six coefficients have been reduced to five coefficients and we now use five geometric conditions to obtain these five different coefficients of the generalized conic.

What are those five boundary conditions? Please try to recollect, the two end points, coordinates of these two end points x_1y_1 x_2y_2 in 2D you can visualize this curve in 3D also. $x_1y_1z_1$ to $x_2y_2z_2$ the curve could be in 3D that is also possible. So we know those two coordinates of the two end points or the starting point and the finishing point of the curve.

We know the slope of the tangent as to that is how the curve will start and finish. The slope of the derivative say the curve is coming out like this so that is a tangent to the curve and the way it finishes at the end point. So we can also define this, we will come up with that figure illustrating the slope or the derivative or the tangent to that curve at the end point and finishing point. So two end point coordinates and two of the derivatives at the slope and end point so four of them and fifth point is obtained by an intermediate point in between because we not only should know where it starts and where it finishes but we should also know some point in between where it is going to traverse. So, if you change anyone of these five geometric conditions either the starting point or the finishing point or anyone of these two tangents at the start and the finish or even the intermediate point if you vary what will happen the curve will change.

The nature by which the curve appears or looks like will change and in mathematical terms the coefficients will vary. The coefficients which define this curve in terms of this five or six also will vary. Look back into the slide; this is the general way by which these coefficients are obtained. So there are six from this we reduce to five by allowing the origin to be at the conic or the conic to be at the origin or assuming one of the parameters to be a constant and we are left over or we have five of these coefficients which may be obtained using five of these geometric conditions, two slope of the curves at the end points or the tangents or the derivatives of the curves at the two end points and one intermediate point.

I hope this is clear and when we look forward we will see special forms of these conics also. We also have to find when these a b c d e f results in these three different special types of conic sections like the ellipse, parabola, hyperbola etc. And of course when you know an ellipse you can also say circle under that category. Generalize conic; if you look back that is the expression. So we have still not discovered or found out why these two are necessary.

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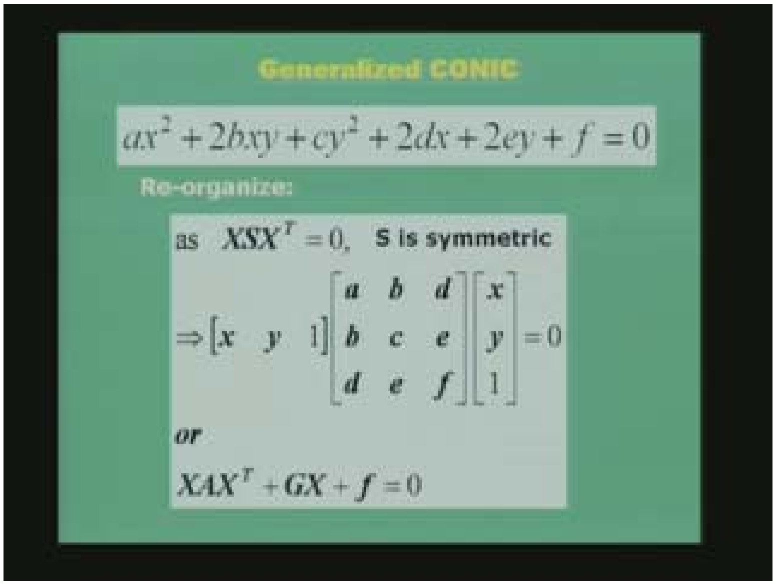

So we will discuss about that right now. We will re-organize these expressions of the generalized conic in a matrix form in this nature. We will re-organize this expression in the form XS, X transpose is equal to 0 where XS is a symmetric matrix 3 into 3. X is given by this particular form.

Since it is a 2D x y one vector so X is nothing but a homogeneous representation of a point in 2D that is X here and the S is a symmetric matrix with the coefficients a b c d f obtained from this quadric expression here and is plugged into this matrix 3 into 3. So you can see here, I leave this as an exercise for you to visualize that if you break open this expression you will get this quadratic form. You can almost visualize that if you multiply these three matrices you will get this one particular expression.

You can take this as a home exercise or you can solve this one as the lecture proceeds to visualize how these matrix form of the expression could be either derived from the second degree or the second degree expression could be obtained form the matrix form.

Now if you carefully look back the S matrix which is given here is a symmetric matrix. There are two of these terms b d and e, I repeat two bs, two of these ds and two of these es and this has to be there because S is a symmetric matrix a c f are the diagonal terms. So the non-diagonal terms or the off-diagonal terms in this symmetric matrix must duplicate itself to give the symmetric nature of the matrix. And if that is so you require two of these off-diagonal terms which are b d and e and hence that is the reason why we use that factor two in that quadratic form. That is the answer to my question which I asked at the beginning of the class today, why this two could be used.

You may not use two. Nobody forces you to use two but if you do not use two you can still write the expression second degree of the generalized conic in this matrix form but you can imagine what will happen.
What will happen? That symmetric matrix, the off-diagonal terms will have to be divided by a two term. So we will have a b by 2, d by 2 and e by 2 in those off-diagonal terms. So to avoid that divide by 2 term in the matrix on the off-diagonal terms of the symmetric matrix X, to keep the diagonal terms as it is without a divide by two terms we add the factor two in the second degree expression for the generalized conic 2b 2d and 2e to ensure that you do not have to divide by two term in the symmetric matrix.

Look back, see if you eliminate this two here that means do not use 2 in these three terms that means just use bxy dx and ey you can still right in this form, re-organize in this form and come up with the symmetric matrix again. But you will have a b by 2 at these two places, you will have a d by 2 in these two and in this you will have an e by 2. So to avoid that divide by 2 factor in the off-diagonal terms of the symmetric matrix S you introduce the factor 2 here. That is the answer to my question. I am not sure whether all of you got this answer before I gave the solution I am the reason behind, it is not a solution it is a reason behind using the factor 2 here. So this is the expression which is used for the generalized conics in matrix form. You can also write this generalized conic expression in this form particular XA GX and f.

I leave this as an exercise for you where this f is the same as this, GX will be involved with d and e and a will only have the matrix a c and f and it will be a diagonal matrix in fact.

Let me visualize what will happen is, now it will be again a symmetric matrix because you will have a c f will be 0 you will not have this factor f because it is coming out. So I leave this again as a home exercise to write the expression of a, given G as your e d and e terms will go into the matrix G. The f is this term f, so if that is so what is going to be your a matrix 3 by 3 when S is given here.

If S is given as this I have already given some part of the solution where f will vanish, d and e will not remain. So the a b c terms will remain in fact and you will have 0 0 and you may have a singularity one term here. So please try to write this matrix a, it will again be a symmetric matrix but it will not be in the form as given by SCL. So the two different ways by which we can write the generalized conic second degree implicit representation of a conic in the form of a matrix either XAX X transpose is equal to 0 otherwise XAX transpose plus GX plus f is equal to 0. Look back, either you can write in this form, or you can write in this form. But we will stick to this form because it looks highly compact and quiet easy to handle.

Special condition is when the second degree generalized conic becomes a parabola. Out of those six parameters when b square equals ac that equation, which is that equation which we are talking about? We are talking about this equation. This generalized second degree implicit representation of a conic becomes a parabola when b square equals ac that is very interesting, it becomes a parabola. If b square is less than ac then the equation represents an ellipse and you can almost guess that if b square is greater than ac the equation represents a hyperbola. So these are the special conditions of the second degree conic. So relation between b a and c or b square ac dictates which of these three forms it is going to take. So the linear term that is the d e and f does not have a significant role to play with. In fact if you go back and look into this expression here you can visualize that if we switch off a b and c that is put a is equal to 0, b is equal to 0 and c also equal to 0 you will be left with this term in the generalized conic.

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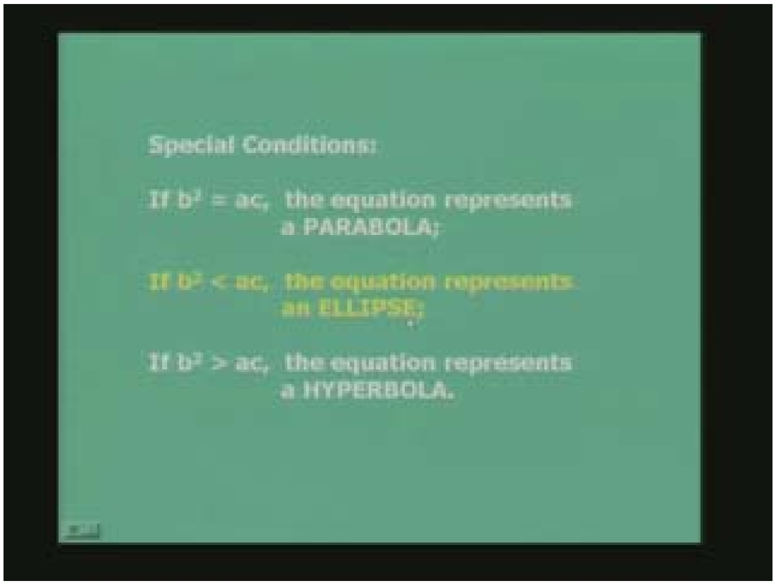

What will this expression give you? Any guess? When a b c is equal to 0 this expression will give you a straight line, it is a linear form. It is a linear form of the generalized expression of the conic so that is a linear term. So d and e and f are responsible for the linear part of the expressions.

The second degree terms a b and c are what are going to dictate as to what is going to be form of your conic. So bring this back, b square equals ac is a parabola, second degree term relations again b square less than ac is an ellipse, b square more than ac is a hyperbola. We move on to second degree space curves in 3D which can be represented by explicit non-parametric representation as this were x is equal to x, y is equals f x and g of x it could be any functional form f and g second degree or third degree or even linear forms of x.

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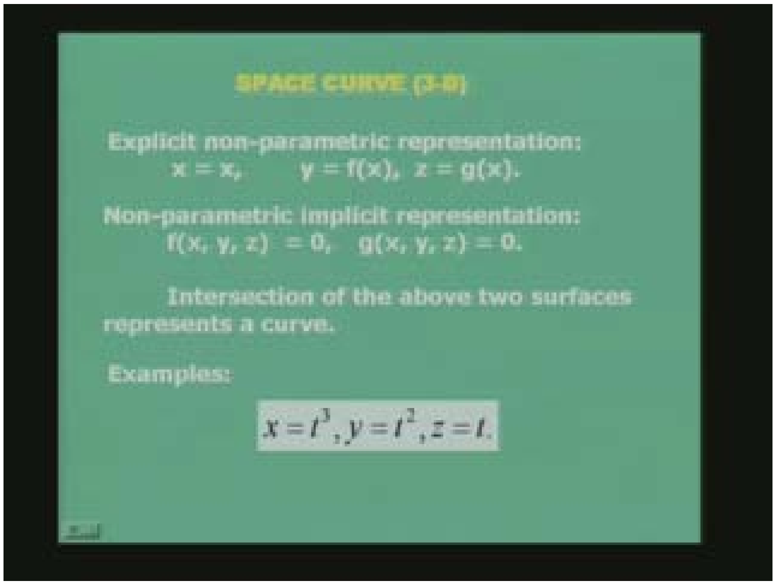

You can almost visualize that this will represent a plane if f and g or a line or a plane if it is a linear form f and g. You can have non-parametric implicit form, f x y z equals 0 and g x y z is equal to 0 that is also possible. This is an interesting case non-parametric implicit representation f x y z or f x y equal to 0 g x y equal to 0 in 2D or f x y z equal to 0 and g x y equal to 0 in 3D.

Now if these f x y z and g x y z are linear expressions of x y z you can visualize that these two are going to be planes. We have seen this equation of a plane ax plus by plus cz plus d is equal to 0. So this functional form f and g if they are linear they will represent a plane or they will represent two different planes f and g or two different planes will intersect to form a line. So the straight line can be represented in parametric form or it can be represented in an implicit form as intersections of two planes, two lines intersect to form a point in 2D. I discussed about this earlier and now two planes in 3D intersect to form a line in 3D.

If these two functional forms represent a non-linear surface, it could be a cone, it could be a cylinder, it could be a sphere or any other arbitrary surface depending upon the coefficients, six of those we have seen in the x y but you can have more of those when you have three variables x y z. So if you have two arbitrary surfaces nonplanar curved surfaces in 3D, intersections of those will produce not a line but it will produce a curve. A typical example is, try to visualize an intersection between a sphere and a cone. Assume a sphere and a cone in between, so you intersect those two surfaces. You will come up with a conic section also but it will be any general curve depending upon the relative positions of the sphere and the cone.

We discussed about a conic section between intersection of a cone and a plane. That intersection gives you a contour which is called a conic section. But two different quadratic surfaces or second degree or curved surfaces as they are called popularly they will intersect to form a curve. You can visualize any sphere, ellipse, ellipsoids or even cylinders and cones intersect, any two of these will intersect to form a curve. That is what is the representation of a curve in implicit as these two equations when they are simultaneously satisfied will have a curve in 3D. That is what we are talking about, intersection of the above two surfaces represents a curve.

Examples, this is the parametric form. The x t cube t square and t we are talking about a non-parametric representation, explicit non-parametric representation or parametric representation in the form of t cube explicit where of course since in this case you can put x equals x, y equals x square and z equals x cube can be visualized as the parametric form, this is an explicit parametric form. And parametric space curve can be represented as x t f t and z t with terms of t and I will give examples as the curve on the seam of a base ball. If you see x y z and the theta is the one which is varying in this particular range there are two other parameters lambda mu a and b where a and b are the parameters.

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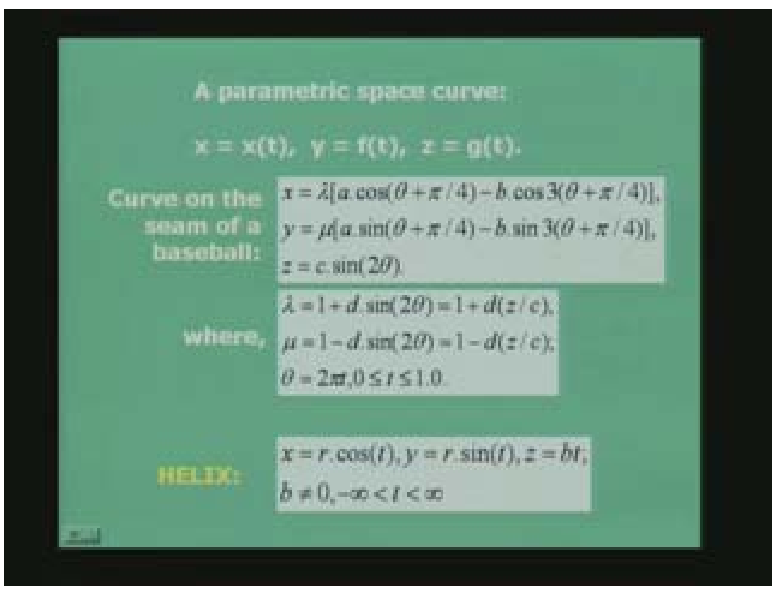

We can define lambda and mu in terms of another parameter d theta and c but this is highly complex. I will tell you what this curve on this base on the seam of a base ball means. That is not a very popular game in our country but it is very popular in the west. Talk of the curve on the seam of a base ball which is thrown in a very fast speed, the tennis ball which is used to play in all tennis also has a similar structure. So the curve which is the seam of the base ball will have structure which will be on the surface of a sphere but it is quiet a curve and that can also be represented in the parametric form and it is not very easy.

I ask you to refer into a book, I do not have the example of that particular curve right now in the presentation but you can look into book by the Rogers and Adams and other books which talks of these illustrations of the curve on the of the baseball which will be represented in x y and z in this particular form lambda and mu as again a function.
So a b c and d are the parameters of the curve in this case or the coefficients and theta where is in this particular range two pi t where t varies between 0 and 1. So if you vary t from 0 to 1 theta will vary from 0 to 2pi and you are basically tracing the curve on the seam of the base ball that is very interesting.

Interesting complex formulation but interesting vary t from 0 to 1, theta varies from 0 to 2pi and you are tracing the curve only. Helix is a very interesting example. This is something like the polar coordinate form or cos theta type but we use t which varies from minus infinity to plus infinity and as you vary that you can see cosine and sine and this is a circular on an ellipse. So it is basically a circle, I can put r_1 and r_2 and make it an elliptical structure but helix has a circle base and the z varies linearly, b is not equal to 0 it is a positive number or negative number.

We are talking of a helix like a spring type of structure so when you vary that x y follows a circular pattern but z varies. So when you vary t, x y varies in a two dimensional plane like a circle and z varies linearly. It could goes up or goes down based on the positive or negative value of b will dictate whether z values are going up circularly or coming down. You can visualize this to be a very larger spring structure going up or down. We will introduce the parametric cubics curves before we wind up the lecture today which will in fact open up a wide scope of discussion of various curves where we talk of x t, y t, z t as parametric cubic curves.

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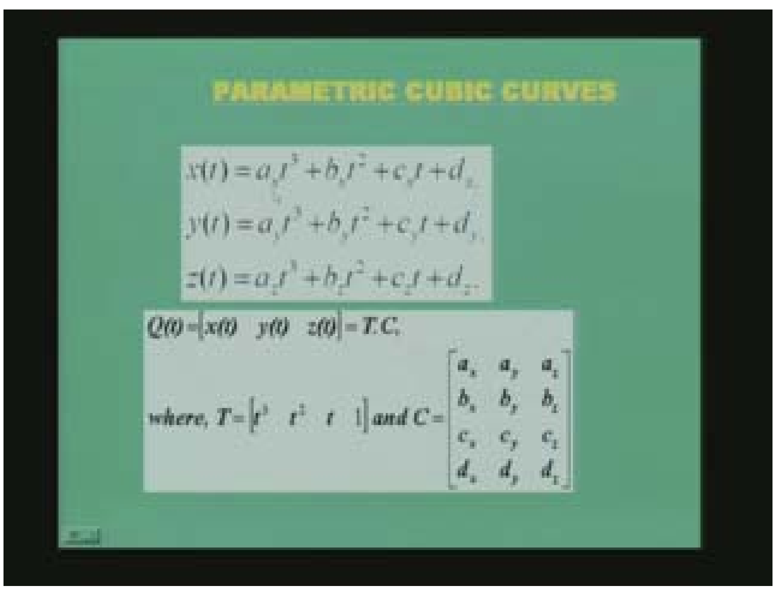

So, each of these expressions have a linear second degree and a third degree terms. So it is a cubic expression of t as a parameter and a b c d are the four parameters in x y and z.

The subscript indicates the parameters in the x y and z domain. And you can rewrite these expressions or put them together into the matrix form as Q of t indicating the parametric cubic curve expressions x t y t z t as T multiplied by C where T is given as this particular matrix, it is a vector of the parameters t cube t square t and 1 and C as this particular parameters. So you can write this as your parametric cubic curve expression.

Parametric cubic curve expression is what you can write in this particular form. You can actually transform it into a slightly different form. If you look back, if you go forward keep this in mind and if you have copied T and C it is very easy as you can see this expression ax ay az to dx dy dz forms this parametric matrix C. You go ahead and in general we write Q t which is same as the x t, y t and z t in the form of T M and G.

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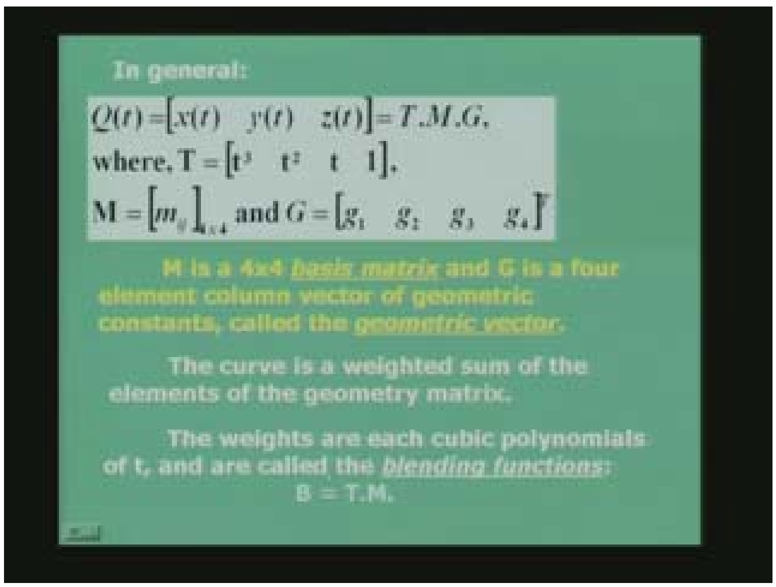

So you break C into two parts M and G and T remains the same. M and G is a matrix, mig of 4 into 4 matrix and G is a 4 into 1 matrix where M is a 4 into 4 basis vector matrix, M and G is four element column vector or a geometric constants called the geometric vector. So M is called a basis matrix M and G is called a vector of geometric constants or it is called a geometric vector. This curve is now hence a weighted sum of the elements of the geometry matrix or the geometric vector. So the weights are given by this basis and G is what controls the geometry of the curve.

The weights are each cubic polynomials of t you can visualize as T multiplied by M which is called the basis matrix and are called the blending functions B. So the various ways by which these curves are represented, T multiplied by M gives a matrix B which is also called the blending function. So, this blending functions will act on the geometric vector given by G is, you must write this expression yourself to get control of these. So the various terms we introduce in the class today are; we discussed basis matrix, we discussed about a geometric vector G and we discussed about a weighted sum of these elements and then we also called of a blending function B which is again if you look back

B is the product of T multiplied by M that is a blending function which blends on the geometric vector G. So B multiplied by G gives you a curved shape. So depending upon various types of this basis vector, geometric vector and blending functions you can have different types of cubic curves, parametric cubic curves of this parameter T is one which is all the parameters and you vary M and G or the blending function B which is a product of M and G to get various types of parametric curves in space curves or in 3D. We will start from here on in the next class where will have extensions of this parametric curves in different forms and we will see, there we come up with cubic splines.

This length itself took our special class of curves called Bezier curves based on certain geometry these curves will take different shapes such as cubic splines, Bezier curves and also a combination of these which leads into what are called the B splines. So these are the curves and of course towards the end of the last lecture we will talk about various types of surfaces which are extensions of curves in 3D.
Thank you very much we will stop here.