Computer Graphics
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Lecture – 37
Curve Representation (contd…)

We continue the discussion on representation of curves and then we will move on to descriptions of surfaces. So we were discussing nonlinear equations and specifically in the last class we discussed generalized conics and special cases of conics which lends itself to ellipse, hyperbolas and parabolas. Towards the end of our last class we then moved into parametric cubic curves and we introduced the matrix basis equation, the basis matrix vector and the geometric vector and we also introduced the term blending function. So we will continue from there onwards today.

So, if you look back into equations of parametric cubic curves as given in the slide here where x t, y t and z t are cubic polynomials of t here you see you have the x, y and t each is a separate function of t, ax and then the bys and the bs and the cs and the ds are the corresponding coefficients. We can write this as a form of vector or matrix equation form of Q is equal to T multiplied by C where T of course is a vector of the parameters and C is the corresponding coefficients so this is what we can write. So with is background if we move ahead in general we will say that Q t is a product of matrix multiplications in the order of T M and G where T was as defined earlier as given here four element vector t cube, t square t and 1.

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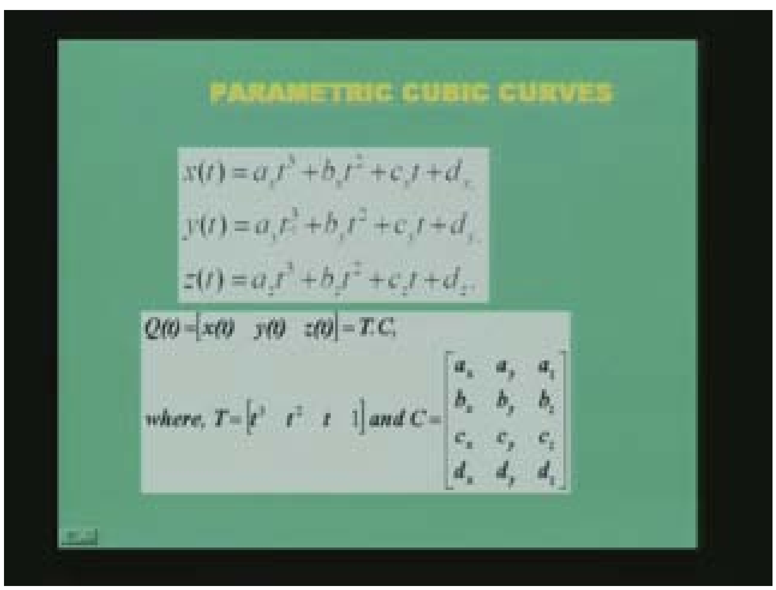

The C in the previous slide is broken into two parts; M and G, and this convention will follow almost throughout the series of discussions which we are going to have today and also in next class where we talk of M as a four cross four element basis matrix and G is a four element column vector of the geometric constants called the geometric vector. So you have the basis matrix here in the geometric vector and the parameter vector T. So T M and G there are three corresponding terms to represent the curve. T M and G is what we should remember.

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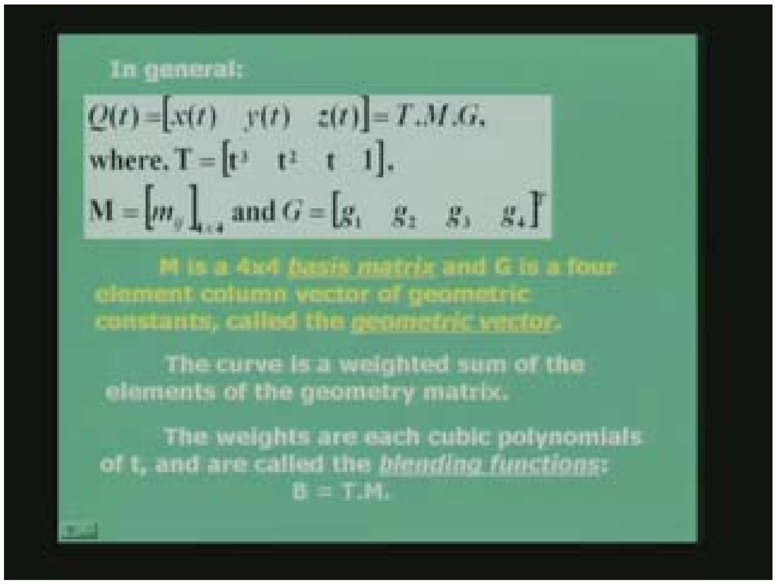

Of course there are various ways by which this M could be broken but we will stick to the scenario and T M and G are the three components of the matrix vectors which are used to represent cubic curves in this case but it could be of any order also.

Before progressing further with the equations we had seen a few equations in the last class and today, of course we are going to see more with few illustrations as much as possible.

I must mention here that you must not only copy these equations today but also practice them. That means you must try to derive the formulations which are projected to you through the slides today, yesterday and in the next class which you are going to see further.

The other factor which I must inform you also is that when we talk of curves we are not only trying to fit a set of x y points or x y z points 2D or 3D data on to a curve as experimental plots to demand. But there is another application in computer graphics where curves mainly have a very significant role to play in the process of computer animation where you need a set of points or set of polygons or an object or even a set of objects to follow a particular trajectory.

And the trajectory could be any order of any degree which could be represented by the standard general conic form, it could be a curve on the form of an ellipse or a parabola, it could be a three dimensional trajectory.

In three dimensional space if you want an object to move away and provide the users a visualization of objects let us say the satellites and planets move in elliptical orbit so you need equations of an ellipse in that case. So these are certain other applications of curves.

Of course when we talk of solid modeling we do use a set of curves together and bind them to form a surface which we will see later on in this class and also in the next class we will see how surfaces are represented as extensions of curves. And we have special form of surfaces like spheres, extension of a circle from an ellipse we can talk of an ellipsoid and of course we have cylinders and paraboloids and we will see hyperboloids and things like that later on. But we must have the basis of the curves. And the mathematics behind the 3D of curves is very strong enough to understand and the extension of these for a surface.

When we talk of these curves it could be 2D or 3D it does not matter and we talk of a matrix element vector representation. Now, why we generally take parametric forms is number one and that too in the form of basis element matrix vector, there are two reasons for this. We know the advantages of parametric form of a curve or a line. I do not think I need to retreat that again but just if we look back we use parametric form of lines and curves for intersections here. We favored the use of parametric curve or parametric representation of a curve because as you vary t you are moving along the curve. So, in terms of trying to move form one point to another either the application demands animation or for any other reason parametric representation lends itself to traverse along the curve that is in number one.

Advantage of using parametric representation for curves, the other part of it is the matrix form. We will find that this representation which we have just seen now we will go back after a short while where Q of t is represented as a matrix multiplication of three matrices. Some of them of course are column vector, some of them 4 into 4 it could be 3 into 4 or 3 into 3 depending upon the order of the curve, the number of points on the curve and so on which are specifically the boundary conditions used to compute the parameters of the curve. So we had seen these Q is equal to T M and G and typically you need to have matrix multiplications computed to compute any point on the curve given in a parameter T

This is not a disadvantage; this is an advantage because there are efficient graphics, hardware graphics and accelerators available on application specific graphical interfaces. We talk of high end video graphics chords installed on PCs or high end work stations starting from silicon graphics on one side to any other stations which support graphics. Specific graphics chords which support OpenGL standards such as end media will have efficient hardware support to do these computations efficiently in terms of z buffer rendering matrix multiplication 4/4 is a matrix multiplication which is hardware encoded.

So, if you provide the corresponding syntax and say please take couple of these matrices or may be three or four of them and apply the sequence of multiplications that computation is going to be very fast on the graphics hardware than implemented using a typical matrix multiplication software library. That is also fast but there is no comparison when we put these computations into these graphics hardware.

These are the reasons we will follow throughout these lecture series today and also in the next class and the parametric representation of curves mostly and also we will say this nature of the matrix multiplication. So, coming back to the slide we talked of this Q of t which is the multiplication of three of these matrices T M and G as given here. T is the vector of the parameters of higher order terms, m_{ij} is the basis matrix and G is the geometry vector. Or the curve we can visualize is a weighted sum of the elements of the geometric matrix or the geometric vector which are similar to the boundary conditions.

And the weights of each of these cubic polynomials of T which are mostly in the matrix M lends itself up to what are called the blending functions. So if you say G is pre multiplied by a matrix B which is T dot M as given here we call this B as a blending function.

Blending is something like smoothing where you try to interface two different curves together to form a larger complex curve. It is necessary that although I said sometime that we have to use and allow curves to represent objects and contours and plots because sometimes piece-wise approximation of lines or polygonal planar surfaces or highly curved surfaces it becomes a very crude approximation.

You cannot have a very good approximation of a highly sharp bend curve or surfaces which has a sharp rather small radius of curvature to be approximated by simple polygons. So we have to have curves and that to in many applications a single curve whatever format you choose you will look into special forms of cubic parametric curves of splines and Bezier curves which are highly efficient to fit to certain points. They are also not able to handle curves which oscillate quiet rapidly.

Sharp transitions frequently occurring, we might say that it may not be in some sense periodic. But periodic type signals with sharp curvatures of surfaces and contours it is often difficult to fit a single curve. You can try to force that with a higher degree polynomial use some interpolation extrapolation techniques and try to fit but it will always be a very coarse approximation.

So what often is done is, we will see later on that a highly complex curve is also broken up into smaller segments and each of the segments are represented by these various types of curve representation. And it is necessary to bind these sections of curves at the end points which are typically done by the help of these blending functions which control in nature and the manner in which these curves will start from the starting point and finish at the ending point.

Of course we have not seen the figure yet we will see that very soon. I was talking of this example in the previous class where I asked you to visualize that a plane must take off in a certain way and land up in certain angle in an air strap. It cannot rise up or fall at any arbitrary angles larger than or greater than something or even less than something because that could cause a problem.

So we are talking of trying to fit certain trajectories and smooth it with the help of blending function. That is an example of mathematical blending function here. The T dot M is a blending function here. Keep this idea in mind Q is equal to T M G is a representation which we will come back to later on.

We start with the first of the popular representation called the cubic splines which are used to represent curves and the expression is given in the following manner; Parametric representation P of t; summation of four terms B of i is what are called for the time being you can say these are some points, control point which control the manner in which the curve will take its shape. And of course you can see that t is raised to the power two and since i is equal to 1 to 4 the maximum highest order of t will be three and hence the term cubic spline.

t varies from the range t_1 to t_2 and Pt is the position vector of any point on the cubic spline segment. So, when we vary t form t_1 to t_2 you are basically moving on the curve or P of t is a position vector pointing to any point on the curve from the starting point when t is equal to t_1 and you reach the finishing point or end point of the curve when t equals t_2.

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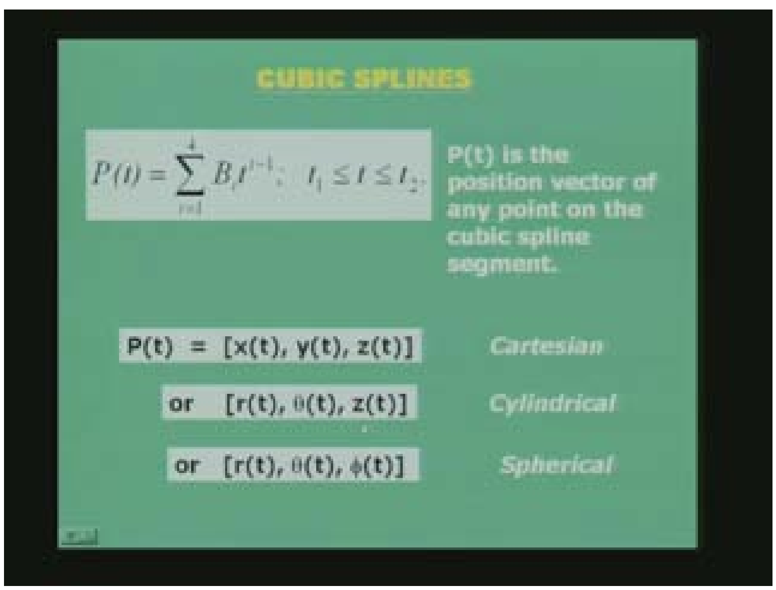

We will see the role of the B_{is} later on. But before that, we will have a look into the P of t which itself could be a three element vector in Cartesian coordinate form x y z. You can use homogeneous representation but since x y z are the three elements which are required I am putting those or you could use cylindrical Cartesian coordinate.

Cylindrical coordinate form R theta and z or even the spherical form or theta fi. So whatever may be the case this cubic spline holds good.

But of course you cannot straight away mix up any of these two representations and start using in your application or theory. You must stick to one of the forms of Cartesian cylindrical and spherical and then write this expression for cubic splines. We break open the elements and we stick to Cartesian coordinate form and B_i is will now have the three components B_{ix}, B_{iy} and B_{iz} for the x y and z coordinates respectively. t varies in the same range. The indices and the t to the power i – 1 is the same. So the B which we had in the previous slide is now broken up into three components. So we will have 3 into 4 is equal to 12 coefficients for the B_{is}.

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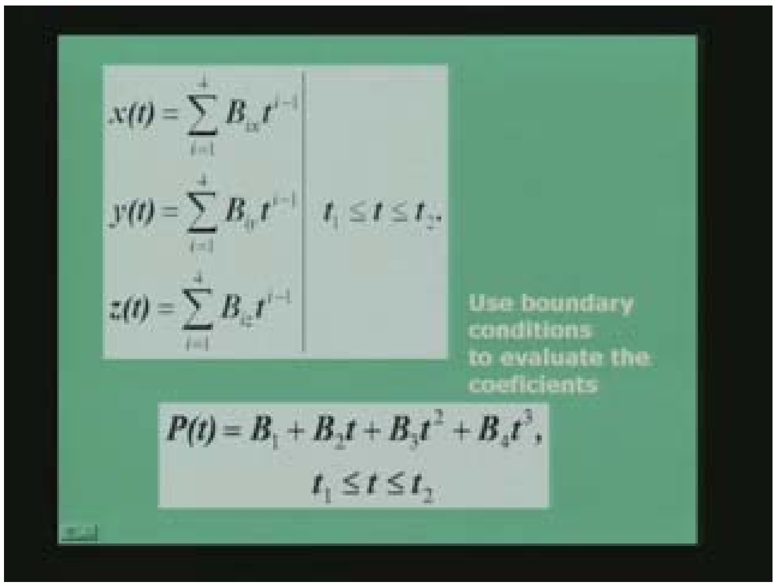

So you can visualize P of t now if you break open and write it in series form with i equals 1 where we have t to the power 0 which is B_1, B_2t, B_3t square and B_4t cube, t varies from again to t_1 to t_2. And each of these elements B_1, B_2, B_3 and B_4 that is any B_i for i varying from 1 to 4 will also have three components I will say B_{ix} B_{iy} B_{iz}. So 4 into 3 there are twelve coefficients of these parameters which control the shape of the cubic spline. That means if you change anyone of these B_{is} you will have a different type of a curve. But the problem to do with cubic spline is just to compute this B_{is} because you know this range from t_1 to t_2 where the t could vary.

For the time being you can visualize that the t_1 could be 0, t_2 could be some value of capital t. So you are varying from 0 to T say the t value, the parameter and you need to fit the cubic spline over the set of points and you need to compute these B_i. So the question comes is how to compute these B_{is}. We will see here how to use the boundary conditions to evaluate these coefficients.

Remember, there are four of these B_1, B_2, B_3, B_4 which are B_{is} and each of the coefficients B_{is} will have individual three components along x, y and z respectively. So basically although there are 12 coefficients but we will visualize each of these B_{is} as a vector, if you take B_1 it will have B_1x, B_1y and B_1z. So, each of this B_{is} is a three element column vector. So we will visualize B_{is} as column vector and whatever we write as a equation we will assume that or we must consider that we are writing a vector equation not a scalar one and it has to be broken up into three components x y z to obtain the individual components of the B_{is} along x, y and z respectively. I hope you are able to follow this point that if B_1 has three components B_1x, B_1y and B_1z then B_4 also will have B_4x, B_4y and B_4z respectively. So let us see how the boundary conditions are used to evaluate the coefficients.

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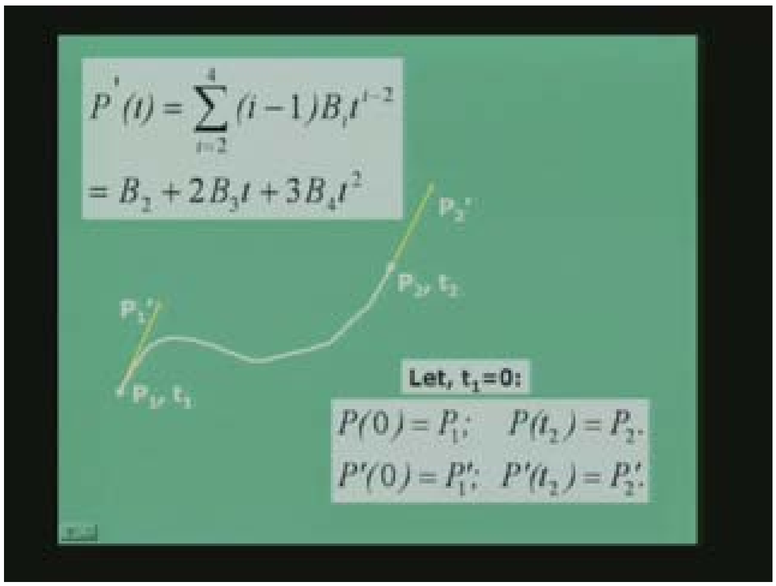

We took a derivative, let us take the derivative of this P of t, if you remember which was the P of t expression was sigma B_i t to the power i – 1. If you remember the expression, I can say it again; P of t was sigma B_i t to the power i – 1. So if you take the derivative what happens? So i – 1 will come out and this is the nature which you have. So this is the expression which you have and after taking it, this indicates that you have taken the derivative.

So del P of del t, this P prime indicates derivative of P with respect to t. If you do that with respect to any of these expression as given here do not worry about this x, you just assume that P of t is B_i t to the power i – 1. This is what if you take along x, y or z the combined vector equation in the differential domain will be something like this. And if you see that the B_1 term will vanish because when we assume that we are talking about this in fact the range runs from i is equal to 1. I must admit here that i is equal to 2 here. i is equal to 2 to 4 that is why it gives you 2, 3 and 4 the three coefficients because the B_1 term will vanish after the differentiation because it is a constant point.

So if you know that this is the expression which you obtained and if you are satisfied that this is the expression which you obtained after differentiation and this is the curve.

Let us say this is an example of a curve again. I must admit here that it is not computer simulated but a hand drawn curve where it runs from a starting point P_1 to a finishing point P_2 here. So at P_1 the parameter is t_1 that is the starting parameter value t is equal to t_1 and at P_2 at the finishing point or end point P_2 the parameter t is equal to t_2. We also assume that you know or it is known due to some boundary conditions.

You know the boundary conditions at P_1 and P_2 due to environmental factors which influence the curve. That means the experimental plot dictates that at P_1 the slope of the curve should be in this direction. That means the tangent of the curve is given by this P_1 prime which is nothing but the derivative P prime at t is equal to t_1. And P_2 prime is also the tangent at the place where the curve touches the point P_2.

So P_2 prime is the tangent or the slope of the curve when t is equal to t_2. So if you know that and try to put these boundary conditions into these expressions for the derivative of this cubic spline P prime t and also as informed you earlier let us assume without any loss of general t that t_1 is equal to 0 and t is equal to t_2 at the finishing points. So there is no harm done by assuming for time being that t_1 is equal to 0 because we are talking of only this curve and we are not trying to fit this curve beyond or for values less than t_1.

And we are also interested in values of t more than t_2. If that is a case then we can safely say that the t varies from 0 to 2 instead of t_1 to t_2. So we put a condition t_1 is equal to 0 and if you put that and visualize the expression of P of 0 that means P at the values t is equal to 0 it will be the point P_1, P at the value t_2 is the P_2. So first two lines are very clear and the first two equations given in the first line here and the derivative at 0.

If you look at the derivative at 0 of course it will give you the P_1 prime which is this one and the derivative at t_2 two is nothing but the P_2 prime. So we know that this is P_1 prime and this is P_2 prime. So we have already seen these four boundary conditions for the cubic splines. And how many numbers of unknowns are there? There are four unknowns B_1, B_2, B_3, B_4 and four conditions. So you should be able to evaluate all these unknown coefficients B_{is} in terms of the boundary conditions which are given.

What are the boundary conditions four of them? Remember, there are four unknown so we need four boundary conditions to evaluate these four unknowns which are the four coefficients of the cubic splines. And the four boundary conditions are the starting point coordinate is known and finishing point coordinates are also known and the slopes or the tangent to the curve at the starting point and the finishing point are also known.

So there are four, I repeat again, starting point coordinate, starting point slope, ending point coordinate and ending point slope. So there are four boundary conditions which we can use to obtain this four unknown coefficients B_1, B_2, B_3, and B_4 for the cubic spline. So, if you look back into the slide, here are the four boundary conditions once again and the coordinates P_1 and P_2 are the giving the two boundary conditions at t is equal to 0 and t equals t_2 we assume t_1 is equal to 0 so that is what we substitute here. The derivative at t is equal to 0 and t is equal to t_2 is also given to us. That means P_1 prime and P_2 prime is known to us.

Now if that is so, if you try to substitute P_1 of 0 on to the expression here you see a very interesting phenomenon in the sense that P prime at 0 will be B_2 because these two terms will vanish when you substitute t is equal to 0.

I am talking about this boundary condition that is the slope at the starting point is known to us at the parameter value t is equal to 0. If that is so, if you substitute here P prime 0 which is this one, if I substitute t is equal to 0 here then these two terms will vanish. So this P_1 prime will be nothing but P_2. So one of the coefficients are already available to you which is B_2. You can apply a similar logic here as well where P of 0 is P_1. What is P of 0?

If you remember the expression for P of t it was B_1, B_2t then B_3t square and then B_4t cube. So if you see that expression not the gradient expression, I hope you have already copied the original expression of P of t. If you have originally copied the expression of P of t then if you remember and substitute t equal to 0 what do you get?

Three of those four terms will vanish. You will be left with only one term which is B_1. So it is very interesting to know that out of those four boundary conditions two of them directly give you the first two coefficients which are B_1 and B_2. So, first two coefficients B_1 and B_2 are known. They are nothing but now if you look back into the slide they will be nothing but P_1 and P_1 prime.

For the two boundary conditions P of 0 and P prime of 0 are nothing but P_1 and P_1 prime and they give that should be equal to B_1 and B_2. So the solution first of all will be B_1 should be equal to P_1 and B_2 will be equal to the P_1 prime is nothing but P_1 is the coordinates of the starting points, slope at the starting point.

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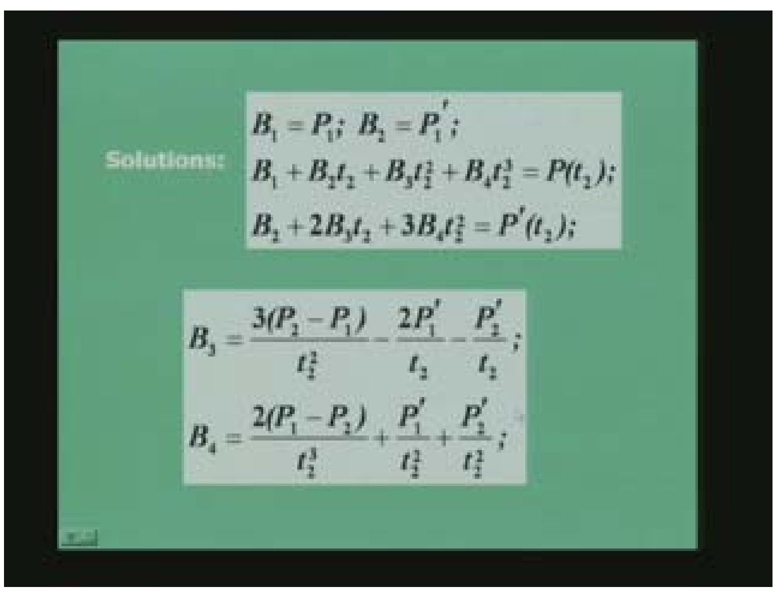

Those two lends itself to the first two coefficients B_1 and B_2. And if that is so, then you look back to the other two boundaries conditions which are given here. P of t_2 is this point coordinate and P prime of t_2 is nothing but the slope or the tangent at that particular point, the finishing point or the end point P_2 so you can substitute that in the expressions of P of t_2 which is given as this. This is the original expression of the cubic spline and we have also just seen the derivative in the previous slide. So just substitute t is equal to t_2 and that gives you the P prime of t_2.

Now B_1 and B_2 are known so you can substitute this B_1 B_2 here. So with the second and third equations here there are two unknowns which are B_3 and B_4 and then there are two equations. So, in the last two equations there are two unknowns B_3 and B_4 because B_1 and B_2 are already known to you. So please try to solve the equations now to obtain the solutions of B_3 and B_4.

You should be easily able to solve two equations, two unknowns because B_1 B_2 is also known, t_2 is also known and the coordinates of the end point of the curve and its derivative or slope at the end point is also known. So, in right hand side two terms are known, B_1 B_2 is known, the unknowns are here B_3 and B_4, B_3 and B_4 two unknowns and these two equations which are the last two equations and two unknowns.

If you solve it please attempt to do that right now and substitute. You should be able to derive yourself. You should be able to derive it, take it as a home exercise for those who are not able to solve it right now in the class. It is better that if you solve it right now, I am giving you sometime for you to solve it and those who are unable to do in the short frame of time please go back and try to solve it yourself. But I am going to put the answer for those who are able to solve the equations right now. So if you look back we are looking at the solution of these two equations B_3 and B_4 are the unknowns because B_1 and B_2 are known from here itself. Let us look at the solution now. This is what you should get.

Those who have already obtained the solution, please try to verify the expressions of B_3 and B_4 as given here in the slide. And those who have not please copy them right now and please be careful with the subscripts, superscripts that is t square and t_2 cube and t_2 square as you can see here you also have a derivative.

So there are expressions of P_1 P_2 and P_1 prime P_2 prime on the numerator and t_{is} with the square and cubic term will occur at the denominator of the term. So, this is the solution which you can copy and those who have already derived the solution can verify it with that which is given on the screen right now. And for others who would have not finished it please try to do it later on. This is how you obtain the four coefficients P_1 P_2 P_3 and P_4.

B_1 B_2 B_3 and B_4 are obtained from the four boundary conditions we talked about and the solution is given in the slide. So what is the final form? The equation of a single cubic spline segment, if you look now after we substitute back the known coefficients B_1 B_2 B_3 and B_4 back into the expression of the cubic spline you will get an expression in this particular form where the P_1 P_2 are known, they are the two end points and the P_1 prime and the P_2 prime are the slopes or the derivatives at the starting and the finishing point respectively where t is equal to 0 at the starting point and t is equal to t_2 at the finishing point or the end point.

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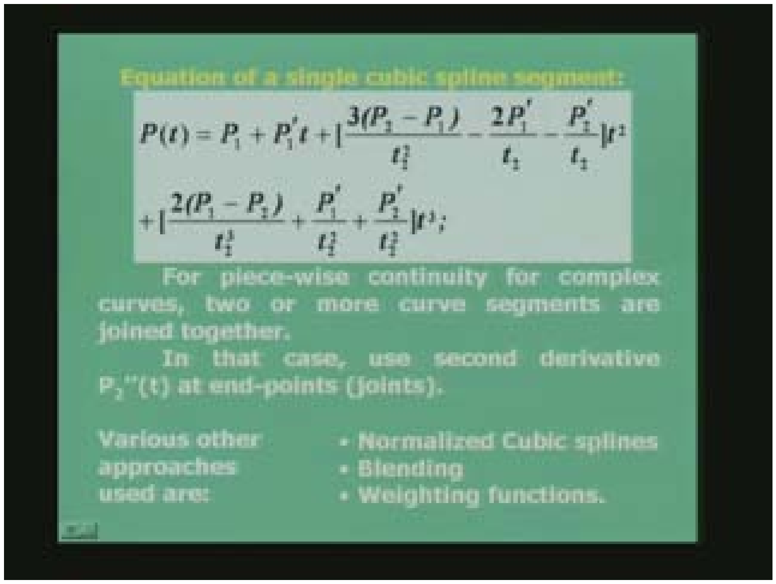

We can also consider t_2 to be 1 if you want to make this equation look simpler. Well, for piece-wise continuity of complex curves two or more curve segments are joined together. We discussed this sometime back before we started to discuss the mathematics of cubic spline or any curve representation when we were discussing general cubic curve representation, parametric representation before we talked about splines.

We said that in general for given two points and the two slopes which this curve will be able to follow a pattern you expect in the data. But we assume that the curve is fairly smooth in between the two end points and there is no radical change of the curve in terms of its derivatives which we expect to happen in a spline segment.

One piece of curve starting from P_1 to P_2 and the derivative starting at P_1 prime and finishing at P_2 prime we have seen that curve segment now. That is of no problem if the curve segments does not vary quiet a lot but if there are abrupt variations it is not possible to fit a curved segment into a set of points to fit the points correctly because there will be lots of variation in the middle and the curve will not be able to appropriately represent those variations or what is mathematically called as fit into the nature of variations or the data points in the plot.

So what is typically done is that you break a complex curve when talking of a complex curve I am assuming that you cannot fit easily, forget a line you cannot get fit a parabola or a hyper bola second degree or not even a third degree expression easily.

You might be able to raise the polynomial order and there are mathematics based on numerical techniques which tries to fit a second order polynomial and regrace it or using a least square PHIGS based concept to fit it on a curve, that is possible but that is not a very ideal phenomena because what this regression will try to do, of course there are concepts based on regression and interpolation both.

Interpolation is good in one respect it is bad in the other. we do not have much scope to discuss that but those who have little knowledge will be able to follow that when you are regressing a curve you are trying to fit as much of the points as possible even a small variation could upset the curve. Whereas when you talk of interpolation you are forcing the curve to pass through the points but it is not guaranteed that it will follow the nature of the curve of the actual variations in between any two points which the least square tries to do. The least square does not guarantee that the curve will pass through the points.

When we talk of a least square regression using any second order third order or nth order higher degree terms when you regress you actually do not force the curve to pass through the points but it tries to fit as many points as possible and tries to pass through as many curves and tries to capture the global nature of variations. Whereas for interpolation you are forcing the curve to pass through the points but in between points in nature of variation will not be that good.

Coming back to cubic spline, so that was to do with the regression interpolation from the point of view of numerical analysis and numerical algorithms, some fundamentals of those. Coming to cubic spline segments if you take a single segment it may not be able to fit a complex curve because if there are large variations you are only worried about this starting point and the finishing point and the tangents in between. And if there are large variations in between you may not be able to definitely fit the curve.

So what you do? If there is a curve let us say even if you take a sinusoid what you would try to do is break it up into two parts or even four parts and force the curve to start from the starting point and finish in the ending point or the finishing point. And if you fit the slopes at certain segments of the curve this spline will be able to fit properly or it will fit a set of points. But you will not be able to do that for the entire curve because it is complex, it has more number of variations, the slopes may be at the starting and end point but in the middle you have problems.
So you split the curve into two or more curved segments depending on a requirement.

Typically what is ensured is you could try to split the curve at points where there is a large change of curvature, you take that as an ending point of the previous curve and a starting point of a next curve. Even if you take two curves and try to join them or blend them or to create a complex curve you would try to select the point of the junction between the two curves to the point where there is a sharp change in curvature and where the single spline segment would not have been able to fit the curves.

But you can force the tangent to the curve to actually fit or follow the nature of the variations but not the central part of the curve. So that is what is done. So, for piece-wise continuity where we break the complex curve into two or more curved segments they are joined by two or more spline segments. In that case what is also done is we use the second derivative which is called the P_2 double prime of t. That means at the second end point or the joint you use the second derivative in that case to fit and ensure that the two curves blend itself or measure together very uniformly because after joining these two or more curves at those end points you should have a smooth effect.

It should not be a case that a lot of discontinuities are visible where what you would try to do is if the slope of the certain curve at the end point is in certain manner and the starting point is in a completely different manner you will have a big discontinuity which you do not want to see.

Although you are using piece-wise continuous representation but you will like to have a smooth transition from one curve to another to visualize a very complex curve with lot of variations, oscillations and fluctuations at various points because it is probably an output of an experimental data observed from a setup. And every setup in a nature has lot of degree about of smoothness in it and that is what we are trying to fit using certain curved representation. But before we move on to this discussion about second derivatives at end point I would like just put one more fact here that there are various other approaches which are also used for piece-wise continuity of complex curves which talks of normalized cubic splines.

I will tell the nature of this later on. we talk of different types of blending functions or weighting functions in addition to this second derivative which we are going to discuss. In addition to that we try to talk of blending functions which will mesh these curves very easily. We talk of different weighing functions or normalized cubic splines which will allow one curve to join smoothly with the second curve, the second with the third and so on. So if you move ahead let us look at this second derivative here this is the expression of the second derivative which you will have. And again I must say that there are two terms of that, I should be running from three to four, there are two terms because in these two expressions B_1 and B_2 coefficients will vary.

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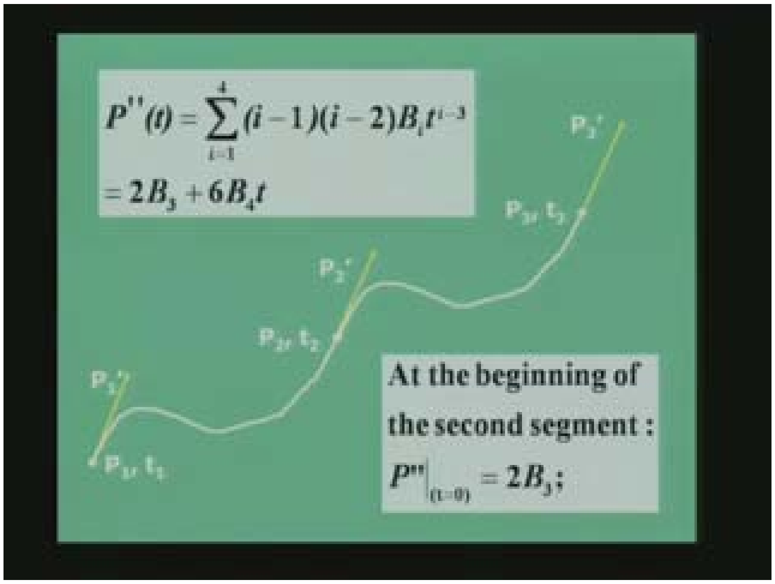

So P double prime should be $2B_3$ plus $6B_4$ times t. This is a two term which you will get after the second derivative. You have seen the expression of first derivative already so the second derivative. And at the beginning of the second segment, so this is the first segment of the curve which runs from P_1 to P_2 and what you have done is you put a second curve which runs from the point P_2 to P_3. So if t_1 is equal to 0 the t runs from 0 to t_2 for the first curve and for the second curve it runs from 0 to t three it is a hand drawn curve. And the three points in consideration are P_1, P_2 and P_3 which are known to us. We also know the derivatives of these points.

So we have a first curve running from P_1 to P_2 both the slopes are known and the second curve which is joined together at P_2 runs from P_2 to P_3 where we know the derivative of the curve at P_2 and the derivative of the curve at P_3 also so that is P_3 prime. So at the beginning of the second segment which is here, if you substitute P double prime t is equal to 0 here you will have $2B_3$. So you have this $2B_3$ which should be the second derivative of the second curve at the starting point.

So if you look here, when you use the second derivative you can use the two boundary conditions that is at the starting point and the finishing point. If you look back into the slide I would request you to substitute t is equal to 0 in this expression so that is very easy as this term will vanish and you are left with $2B_3$. Also, if you take this first curved segment because that is what we have written for the starting point t is equal to 0 for the second segment.

If you look at the first segment where t is equal to t_2 we can substitute this t_2 here and say my second derivative of the curve at this point will be $2B_3$ plus $6B_4$ t_2 for the first curve. So you can mesh these two second derivatives which are going to be the same at that junction point P_2 and it is the finishing point or end point of the first curve and the starting point of the second curve.

That point P_2 is now the end point of the first curve and the starting point of the second curve so that is what you need to visualize. So $2B_3$ which will give you the second derivative of the starting or the beginning of the second segment here is the same as the second derivative of the first curve at the end. So that will be $2B_3$ plus $6B_4$ t_2 which will be also equal to $2B_3$. So if you take these two relations and go ahead and with little bit of matrix manipulation,

I leave it as an exercise for you to find out that if you join two curves the generalized equations for any two adjacent cubic spline segments. If you say that there is a first segment P_k of t and a second segment $P_{k \text{ plus } 1}$t, in the previous example if you visualize k is equal to 1so you had a P_1 and a P_2. There were two curves but you can now visualize that you have a sequence of cubic spline segments one after another which are joined together. So, you have a P_1 P_2 and P_3 and so on. P_{k-1} P_k, $P_{k \text{ plus } 1}$ up to some P_n. So there are n different cubic spline segments which are joined together in succession to create a complex curve that is how you implement piece-wise continuity. We did that when we were talking about polygonal modeling for complex shape starting from sphere, cylinders, cones, extra.

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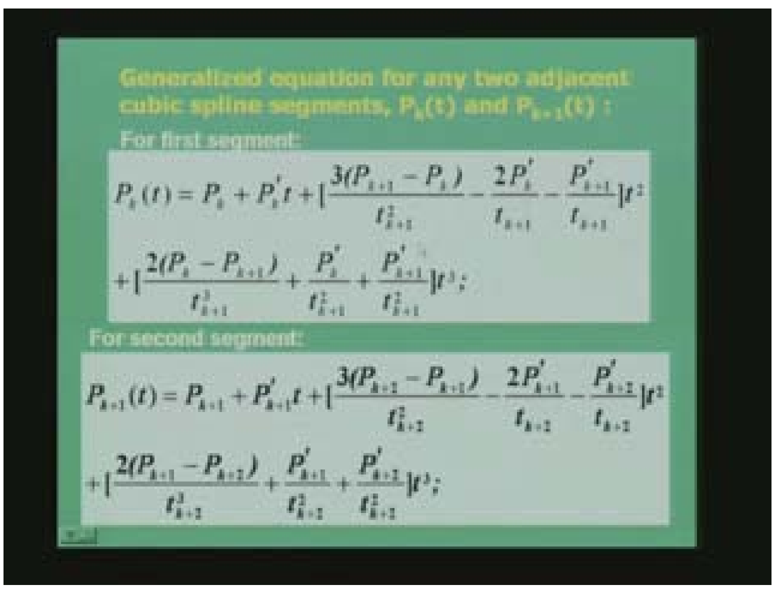

So we do that same concept in 2D curves or 3D curves where the variation is quiet complex and it is not smooth anymore with respect to the third degree variations which cubic splines can handle. So what you do, you break it up into smaller segments and each of these you try to fit a cubic spline segment P_1 P_2 P_3 and so on. $P_{k \text{ minus } 1}$ P_k, $P_{k \text{ plus } 1}$ up to P_n.

So we are looking at just any two adjacent segments which are to be meshed together with the help of their second derivative because the first derivative of each of these points helps you to obtain the B_{is}. We will see how this second degree can be superimposed to adjust the B_{is} such that you have perfect blending or meshing between any two adjacent segments.

And if you do that between any two segments and carry it over from P_1 P_2 through P_k, $P_{k\ plus\ 1}$ up to $P_{n\ minus\ 1}$ to P_n the same concept will hold good and you will have a smooth interpolation or transition from one curve to another. So, you look for the first segment and derive using the knowledge which we just discussed. If you go back this is what we were discussing that the second derivative expressions here as a function of t.

So substitute t is equal to 2 for the first segment and t is equal to 0 for the second segment and a little bit of manipulation will help you to obtain, this is the form which you have for your first segment curve and this is what you have for the second segment.

As you can see that the nature of these expressions are similar and we are talking of scenario where the first curve moves from 0 to $t_{k\ plus\ 1}$ and the second curve moves form 0 to $t_{k\ plus\ 2}$. The starting points and the finishing points of the curve of the first segment are P_k and $P_{k\ plus\ 1}$. Whereas for the second segment it starts from $P_{k\ plus\ 1}$ which is also the point or the end point of the first segment $P_{k\ plus\ 1}$ which is the starting point of the second segment and it ends at $P_{k\ plus\ 2}$. So these are the expressions which you will have for the cubic spline segment two successive ones which we can evaluate. If you look back into the original expression which we derived earlier this is what we derived. So this is what we derived for our single cubic spline segment. But for piece-wise continuity if you see here the expressions are in fact more or less same and when we look into the two adjacent cubic spline segments P_k of t and $P_{k\ 1\ plus\ t}$. So this is how you blend using the second derivative.

And the second derivative is also used to normalize curves segment. I am skipping a lot of Mathematics here because the whole world of curves, various types of representation from splines, we have to move on towards Bezier curves and then surfaces also and look into various examples.

There are various other types of curves representations and we will do as much as mathematics as possible. But with the available time frame I will leave a lot of things for you as an exercise and to read from the literature which is suggested. So, most of these materials had been picked up from the book by Rogers and Adams which gives a very good mathematical frame work for the elements of computer graphics. But you can also look into concepts given in the book by Foley and Van Dam. But concepts and equations have been borrowed from the book by Rogers and Adams which gives a very good mathematical frame work. So, come back to these generalized equations of any two adjacent segments. And this is the equation of a normalized cubic spline segment.

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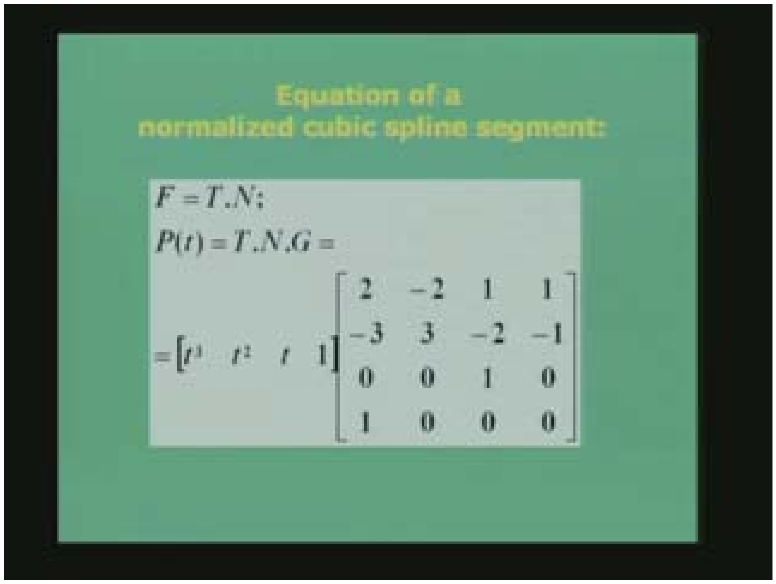

Remember, we discussed about matrix multiplication earlier when we talked about simple cubic parametric curves T N and G. And if you look at T N and G, the T and N is given here G are the coefficients of the geometric matrix vector given by the boundary conditions and the T and the N are given here in terms of the cubic spline segment.

You can derive this very easily from the previous curves. But of course we need a little bit of simplification to be done for a normalized representation.

We move on to a new set of curves called the Bezier curves and that is one of the very interesting phenomena. With the time available to us I will introduce the concepts of Bezier curves and look into a few examples. That is again a different world of mathematics. We have just finished the discussion on parametric cubic curves and cubic splines and move on to another representation called the Bezier curves. And we look into few properties of these Bezier curves and then go into the mathematics to find out how different they are from the previous approaches which we have discussed so far.

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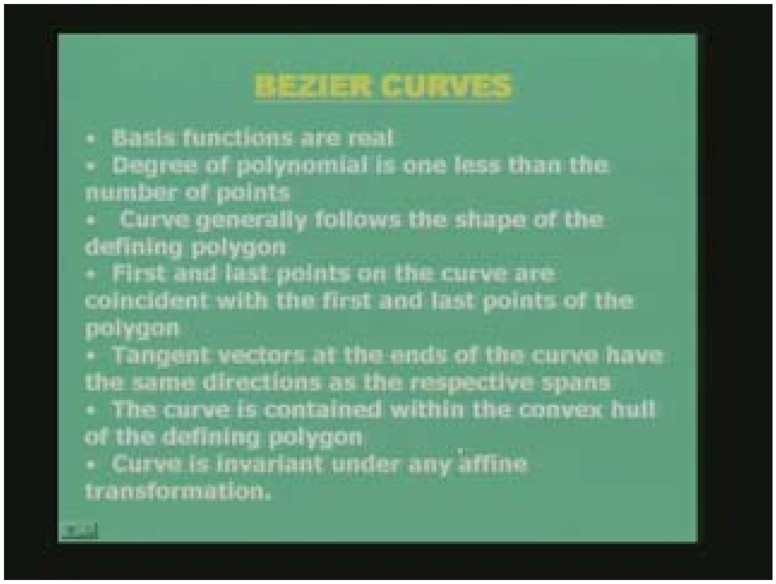

So we move on to the discussion on Bezier curves. We talk of a basis function here which is very real. And we talk of a degree of a polynomial here as well which is one less than the number of points which have to be fitted by the Bezier.

We also say that the curve generally follows the shape of the defining polygon. So we have a defining polygon which defines the curve and the curve has to follow the shape of a defining polygon.

Instead of set of points on which the curve lies we talk of a polygon here. So we will see what does this mean.

The first and the last points, the pair of points which are the first and the last on the curve are coincident with the first and last points of the polygon, this is very interesting. The first and the last point on a curve are the first and the last points of the polygon. The tangent vectors at the ends of the curve have the same directions as the respective spans.
So we talk of first and last points and also we talk of the slope of the tangent vector at the end of the curve. So these boundary conditions which we used earlier in the case of cubic splines are more are less still valid.

We will see how they are used in the case of Bezier curves to derive the coefficients or parameters of the Bezier curve.

But we still talk of the first and last points. Of course there is the concept of polygon here. We will see how that concept is brought in but we still have the first and last point which are also the first and the last points of the curve and they are the first and last points of the polygon, the set of end points to define a polygon. So the first and last point of the polygon will be the first and last point of the curve which are the boundary conditions.

And the tangent to the curve also has to follow the polygonal boundaries in some respect which we call as respective spans of the curve. So that is what the last point we will see that with a figure. After a short while tangent vectors at the end of the curve have the same directions as respective spans and the curve is contained within the convex hull of the defining polygon. So when we talk of a polygon we can always define as a convex hull of a polygon and the curve must be contained within that convex hull.

And the last point is that the curve is invariant under any affine transformation, this is interesting. That means if we change the polygon in terms of its boundary provided you give it a rotation, transition and scale the curve will change its position in the rotation, translation but the nature of the curve remains the same. Even you provide a scale the curve could appear to enlarge, zoom in and zoom out but qualitatively the curve more or less retains its shape.

So, let us look at a few examples based on these conditions which we have seen. And we must understand before, of course the first few points are related to the mathematical part of the explanation of Bezier curves which we might push it to the next class. But we will see what is meant by the shape of the polygon and the first and the last point, the tangent vectors, the curve contains inside the convex hull of the polygon. So we will see how does this polygon define the Bezier curve. These are the set of examples. We will check out one by one, let us take this one. So we will talk of a set of four points which define a polygon and you can easily visualize the convex hull of this polygon.

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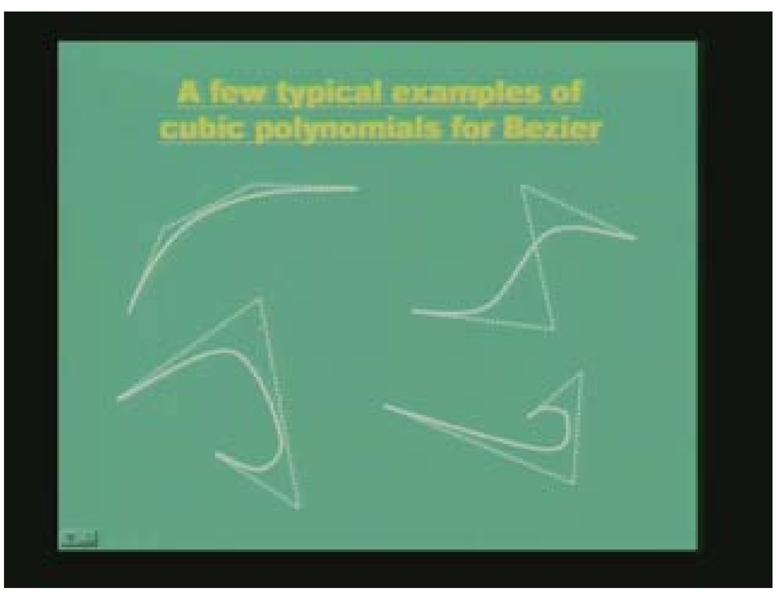

This is the nature by which the Bezier will follow. The Bezier will follow this curve and it will be contained, the first and the last point is same. And it will try to fit these two end points, the other end points as close as possible which are used to define the polygon. Let us take this example, point number 1, 2, 3 and 4 so there are four points again. And this is the first point and this is the last point or the beginning and the end point of the polygon. So that is the convex hull and the curve will also be within inside.

And if you see the tangent to the curve at the start and the finishing point at both these examples they follow the respective span. That means it tries to fit the derivative at the first point. The derivative at the first point will try to follow the polygon edge from the first to the second point and at the end it will follow the edge form the last but one point to the last point.

So here it is from first, second, third and fourth. So from edge between third to fourth point that will be the tangent to the curve at the end. And the edge between the first and the second point will be the slope of the curve at the starting point. The same thing is here. You look at the slope of the curve which is the first edge and the last edge is the slope of the curve at the finishing point. The same is the case here, 1, 2, 3 and 4 is what you have a polygon so it is a convex hull again.

If you visualize the convex hull of this polygon the Bezier curve, again these are hand drawn but you can try to computer simulate them with Bezier tool boxes in any simulation tool box and this is what you will have. This is obtained from the literature that this is what will be shape of the curve.

This will be a very interesting phenomenon where the convex hull of course there could be some errors in drawing but we talk of a convex hull of this polygon the Bezier must be within that convex hull. In fact although truly speaking or a convex polygon which you will get.

But you can define a convex polygon based on the set of points, extreme points and the curve will be within that polygon or the tangent to the curve at the starting point and the finishing point will also be the same as the first edge and the last edge of the polygon. So these are some examples which we have seen as far as Bezier curve is concerned. We will just stop with by introducing the equation of the curve, equation of the Bezier curves.

So let us look at the equation and the mathematics part of it. So, if you look at this slide, the Bezier curve here and the equation of a Bezier curve, the parametric Bezier curve is given by this particular definition.

If you see here the equation is almost similar to the cubic spline except that you do not have t to the power instead of that you have a basis. This is the basis we discussed about, the real basis and we have four points defining the polygon B_0, B_1, B_2 and B_3. B_0, B_1, B_2 and B_3 and the curves starts from the B_0.

And the tangent to that curve should be B_0 to B_1 tangent to the curve at B_3 should be B_2 to B_3 and the parametric Bezier curves which is represented by this particular equation B of i. We will see what is called as this Bernstein basis j n of i of t but t varies from 0 to 1. So this t is equal to 0, t is equal to 1 here and i varies from 0 to n, n is equal to 3 here so there are four points and j n of time this is the expression of the Bernstein basis.

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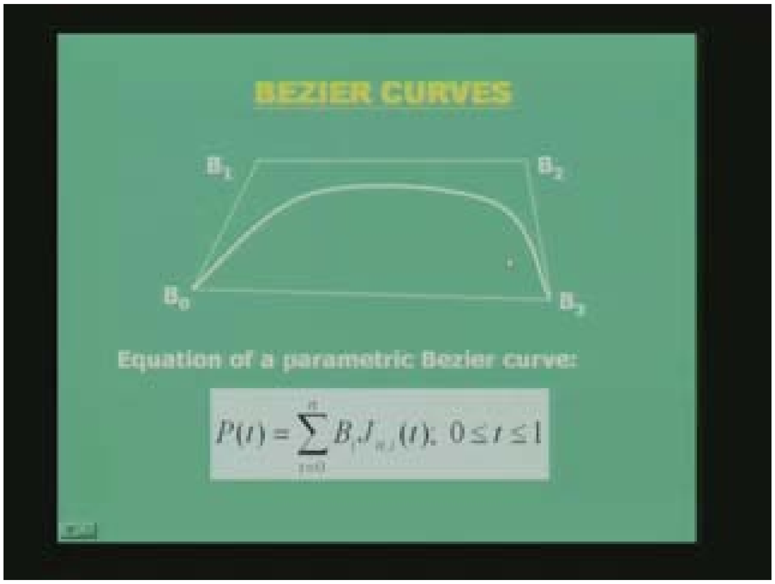

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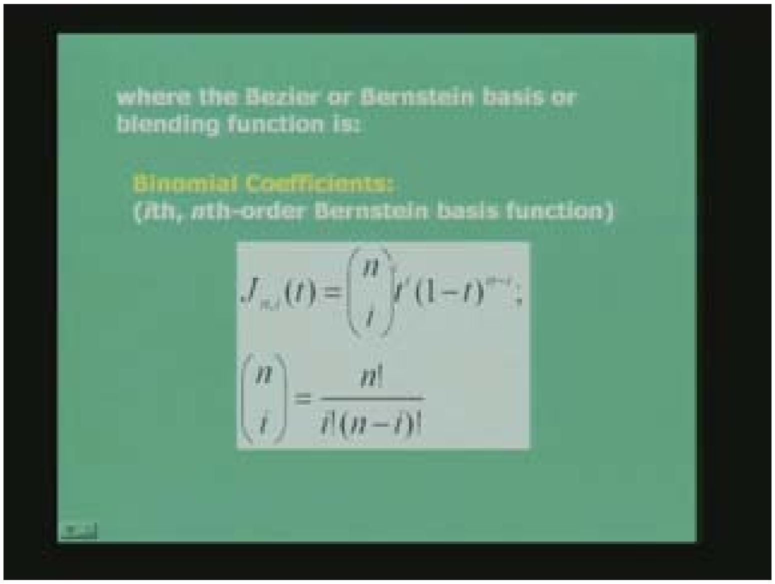

t to the power of i here and 1 minus t n minus i n is what is called the order of the Bernstein basis. So this is called Bezier of Bernstein basis or the blending function. These are also called the binomial coefficients based on the ith and j n order Bernstein basis function is given by this particular equation.

Please copy this, this is basically n computation i is the expression here. This is the expression, you must remember this. Please copy that because when we are discussing in detail about these expressions the nature of these Bernstein basis is going to dictate the nature of the Bezier curves.

The Bernstein basis or the Bezier basis is a very important function which will dictate the smoothness of this curve and the weights will be dictated by the boundary conditions B_{is}.

So we will stop here and we will continue from here onwards in the next class where we introduce these functions again and look at various types of ith nth order Bernstein basis functions. Thank you very much.