

Algorithms for Big Data
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Lecture – 35
The Cat and Mouse Game

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The slide is titled "The Cat and Mouse Game" and features the IIT Madras logo in the top right corner. The main content is a bulleted list of conditions for the game. A blue cloud-shaped box contains a challenge: "Show that $O(m^2n)$ is an upper bound on the expected time before the cat gets his meal." The slide number "17" is visible in the bottom right corner.

- A cat and a mouse
 - start their own independent random walks
 - at the same time,
 - but different nodes
 - in an *arbitrary* graph on n nodes using m edges.
- If they meet, the cat will eat the mouse.

Show that $O(m^2n)$ is an upper bound on the expected time before the cat gets his meal.

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So, before we explore this cat and mouse game, let me actually motivate the reasoning for this problem, at least one reasoning. So, if you are, if we have two random walks exploring a large graph, you kind of interested, you know when will they meet next? So, this is, this problem is, may be these two random walks are walking in this graph and they are exploring in. When they meet, they have to perform some action and we want to understand how often they will meet. So, that is a kind of the motivation for this problem. But of course, you know, being good (Refer Time: 01:00), you know instead of facing, reposing the problem in, in that manner, we will try to post it in a little bit more fun way.

So, in this we treat this as sort of a game between a cat and a mouse. They start their own independent locations and they do their own independent random walk. They start at the same time and then we ask, I mean, that this is an arbitrary graph that they are both exploring. And you know, when the cat and the mouse meet as a, as there is a, when they

both, while they are doing a random walk, while they, if they both come to the same vertex, then the cat will eat the mouse.

So, the question we are interested in, want to post in the following way. We want to show that $O(m^2 n)$ is an upper bound on the expected time before the cat gets its meal, basically, the time the cat will reach a vertex where the mouse will also be present.

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The Cat and Mouse Game

- Hint: Consider a new graph G^* where each pair of vertices in the original graph forms the set of vertices.
- Add an edge between nodes (u, v) and (u', v') in G^* iff $(u, u') \in E \wedge (v, v') \in E$.
- Notice that a random walk in G^* that reaches any vertex of the form (u, u) corresponds to the cat getting its meal.
- Clearly, the expected time to reach such a vertex is upper bounded by $O(m^2 n^2)$ as the graph G^* has at most m^2 edges and n^2 nodes.
- We are now left to show a tighter bound of $O(m^2 n)$.

Now, to understand this problem and find a way to get a handle on this is what we do is, consider a new graph and we call that graph G^* and this graph, there are more vertices, there are actually n^2 vertices. Basically, every pair of vertices in the original graph forms the set of vertices. In fact, every pair where each of the pair is drawn with replacement, so you, you, so even pairs, you, even, even have pairs of performed u, u , where it does not, we do not need both the vertices to be distinct.

So, that means, we really have n^2 vertices in this new graph G^* , but how do we place the adjacent in this new graph. So, you, so you consider two vertices, u and v and u' and v' in this new graph G^* . Now, there will be an edge fitting these two vertices if and only if (u, u') was an edge in E and also (v, v') was an edge in E .

So, in others intuition is following. Now, first if you look at any such vertex, the first one gives you the current location of the cat in the original graph G and v kind of gives you

the current location of the mouse in the original graph G . So, so that is the intuition and you can immediately notice that a random walk in G^* that reaches any vertex of the form (u, u) corresponds to the cat getting its meal, why? Because well, both the cat and the mouse are in the same vertex, so then the cat will eat the mouse.

Now, we just saw a little while ago and rather, we reminded ourselves little while ago, that the expected time to cover the entire graph and therefore, also reach a vertex of the form (u, u) is going to be at most $m^2 n$ because this graph G^* has at most m^2 edges and we saw that it has n^2 nodes, but we need to show something a little bit tighter.

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The Cat and Mouse Game

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Diagram: A graph G^* is shown with nodes (u, v) and (s, s) . A path of length n is indicated between these nodes. A handwritten note $O(m^2)$ points to the graph. A blue cloud contains the text: "Show that $O(m^2 n)$ is an upper bound on the expected time before the cat gets his meal."

If you recall what we need to show is, that the, the cover time is $m^2 n$. So, we need to shave off a factor of n . So, how do we do that is the question. So, let us, let us think about why that might be the case. I leave you with one intuition.

So, now let us look at this graph G^* here. Now, wherever be your starting point (u, v) , you first argue, that there is a path of length at most n that will then lead you to a vertex of the form (s, s) and now we know what is the, the time, the expected time to go from here to here when you are doing a random walk and similarly, you know the expected time to go from here to here and so on.

You can add them all up, this whole thing will be of length n , each one of them, each one of these components we need to argue is going to be at most O of m squared and therefore, we should be able to get the final result. So, that is the sort of the technique that we will need to follow in order to be able to prove this (Refer Time: 06:31).