Graph Theory
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Module No. # 02
Lecture No. # 10
Menger's theorem

Welcome to the tenth lecture of graph theory. We will recall what we were doing in the last class, this was the theorem we were considering.

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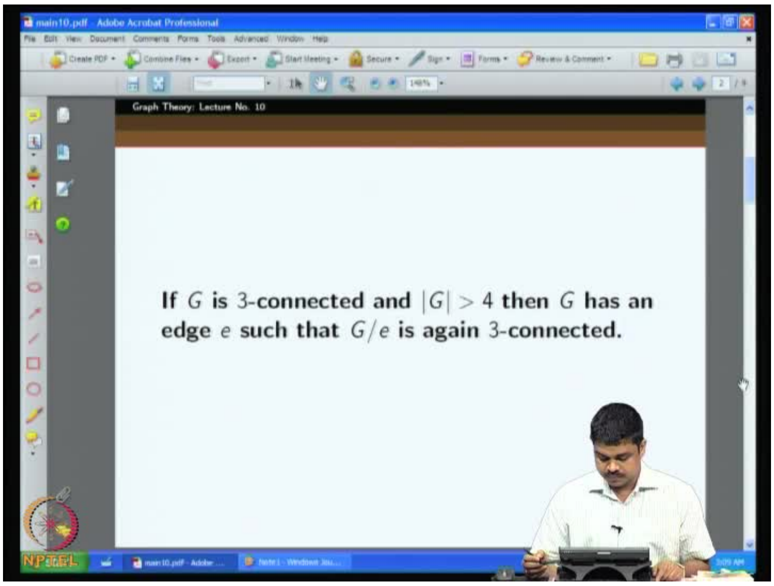

If G is a 3-connected graph with greater than 4 vertices, then G has an edge e, such that if we contract that edge, then G - the remaining graph is again 3-connected, such an edge you can always find in a 3-connected graph with number of vertices greater than 4.

If the number of vertices equal to 4, you cannot possibly do that, because if it is at k 4, if you contract, you end up getting a k 3. If you contract any edge, you end up getting a k 3, it is not 3-connected and it is only 2-connected. So, this condition that the cardinality of the number of vertices is greater than 4 is important.

Now, we had already done half the proof, so we will go back to the main points of the proof and then quickly recall what the main things are. So, the key ideas for that is suppose this theorem is wrong, that means we have a 3-connected graph with more than 4 vertices in it, but every edge is such that if we contract it, you end up getting a graph, whose connectivity is less than 3, that means with the 2 separator - vertex separator or cardinality 2, suppose such a graph exists, we will show a contradiction; we say that it not possible, because then it will give rise to a contradiction.

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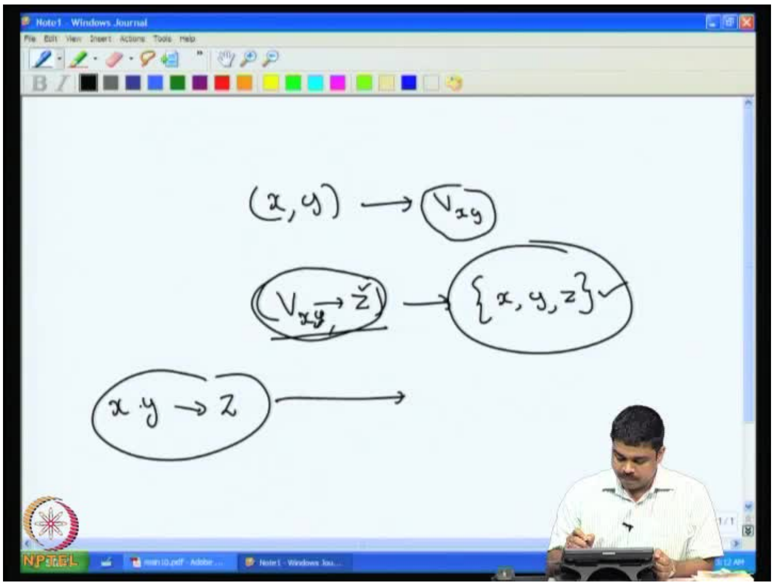

So, the idea was to observe that if each edge say x y, is an edge, if it is contracted, we get this vertex V x y. Then, if it so happen that the resulting graph is not 3 connected, that means there exist the separator of the cardinality 2 in it, then the separator should definitely contain V x y. We have already explained y, this be should be true in the last class and then there should be one more vertex in it. We considered such an edge, such a separator, this separator correspond to the 3 separator, the three vertex separator of cardinality 3, x y z in the original graph. For every edge, we have a corresponding separator, because when we contract that edge, we can get a partner for it, here said. So that x y z together will form a separator of cardinality 3, minimum separator in the original graph G.

Now, we will associate with each vertex x y value, so we can always take the partner, which will give the best value according to this definition. The value was that if we considered the cardinality, so if you remove this 3 vertices, this say three separator x y and its partner, then consider the components that results from removing, by removing this three vertices. Then, we look at the smallest cardinality, smallest sized component. Number of vertices wise minimum component we will look and number of vertices in those components will be the value of this thing.

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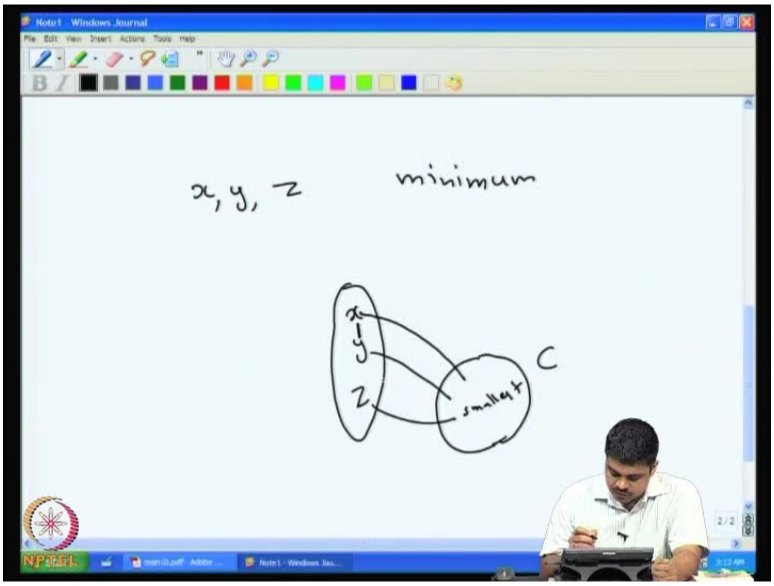

Now, we will we pick up - in the next step, we pick up that edge x y, such that x y and its partner z together has the minimum possible value among all such selections. So, all the contradictions that we are going to show is we will end up, in this case, we will end up with another edge, which will give you an even smaller value, which will be a contradiction. So, this was the proof, you consider this x y, recall this, there is an edge here and its partner is z.

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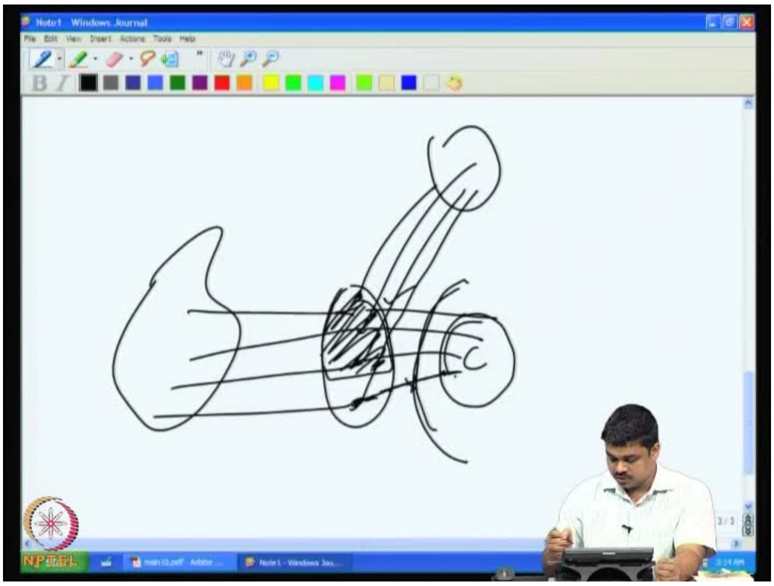

Now, let this be c, be the component of smallest cardinality; this is the smallest component. Now, interesting property of the minimal separator - minimum separator or minimal separator whichever is this. For instance, if you look at this vertex, you will have a connection to this component, the connection to this component will be a connection to this component that we had explained in the last class. Suppose that connection is not there, you could of - for instance, if you had so call minimum separator here, then here is a vertex, which is not connected to component here. This is a component, so there is no connection between this and this component. Then, even by removing this much vertices, I mean removing all the vertices of the separator expect this, we could have disconnected this portion from the remaining graph. Therefore, it is always true that every minimal separator, in particular minimum separator also, every vertex of the separator will be connected by at least one edge to every component like this. It is not possible to have one of the vertices, not to have any edge component.

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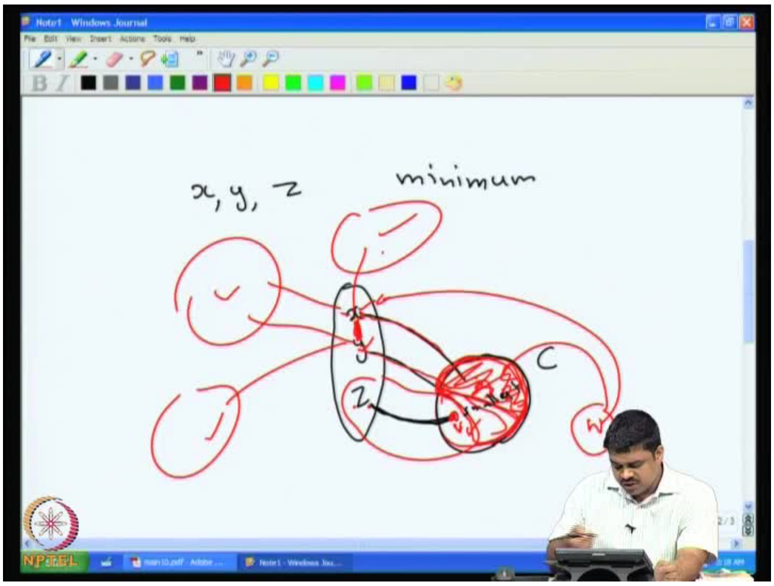

Coming back to this picture here, this z also has a connection here, right. Then look this connection, at least one connection is required for us, so we will take this vertex, let this vertex be called as v. So, we take this edge z v, we will analyze the value that will comfort z v and that will be given to z v. In other words, z v is an edge, so it should have a partner, let us call it w. This w can be from here or here or anywhere, so the other components may be some here, here, here, we do not know wherever; w can be from anywhere, but it is a partner of z v. This z v w, when you remove it from the graph, we will end up with several connected components; what would be the size of the smallest components? Will it be less than the cardinality of c? In which case, we are already, we have already found the contradictions, because we have seen that among all the edges x y had a minimum value and therefore the c supposed to be this smallest. You cannot get any smaller component than c, and then I remove, when you remove z v and w from the graph.

So, it essentially means that, if we can somehow show that I remove z v and w from the graph, a smaller component will result, so then we will get the required contradictions. Of course, we can see that if I remove z v and w, this v is already disappearing from c, so there is a chance that you may get as smaller component. If you want to - if it does not happen what is the thing? Somehow this c should eat up other vertices and become bigger, right. Somehow c should get more vertices attached to it and then become bigger, otherwise, already so will this is showing a tendency to become smaller than the c.

So, one thing can be clear is, this will not split in to more than one component, right, because if it is split in to more than one component, if case c 4, if it is possible that both the component will get some outside vertices and grow, the only way to get outside vertices is to attach to this x. Then of case this y will come along with that, x and y together will go to the same component, then that may catch some outside vertices like this, right, and then grow.

So, only one component can get x and y, therefore only one component from … If we split c into several pieces, only one such piece can get more vertices from outside. It is not possible for c to the split in to more than one component, one piece, because one of the pieces at least will become smaller than the original c, only because if it becomes more than one piece, only one among them can grow larger by taking this x y and more from the outside. Because, the only way to take more vertices from outside is to connect the x y first and then get more vertices, is the thing. It is not possible to connect to x, one component connect to x and get more from outside, the other component connects to y and gets more than outside, because x and y, because there is an edge between them, they have to belong to same piece.

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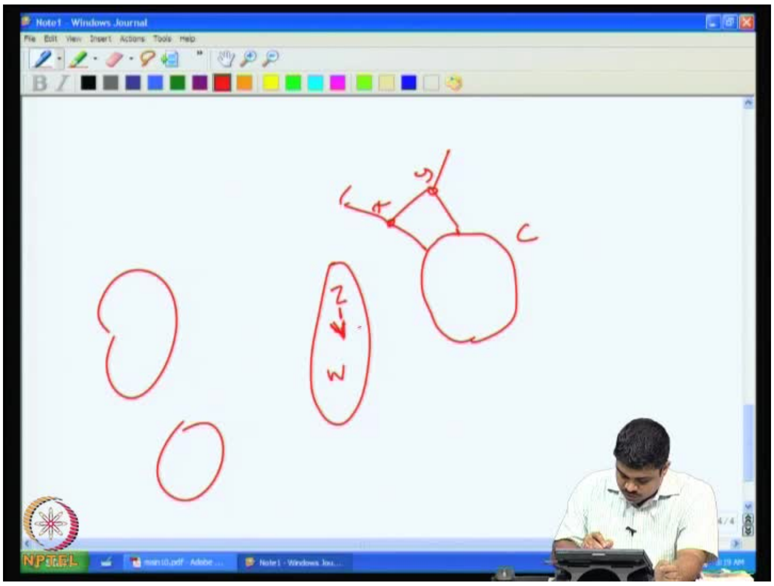

So, that is, if at all they are still there, so therefore … so we can assume that c has to be together there. Now, the only thing we have to worry is… Now, v is removed from c, so definitely it has to get some vertex x or y attach to it, otherwise it is already going to be smaller than the original c. So, it will get x and y attach to this, and now more vertices can be attached to this, but then if you look at the current picture, you can see that, so we have our z v and some w somewhere. And then here is this component, which contains our old vertices of the c, it is possible that it have captured x and y, and become bigger, that it has to, in fact, something more also, and then some other component, one more component at least is there.

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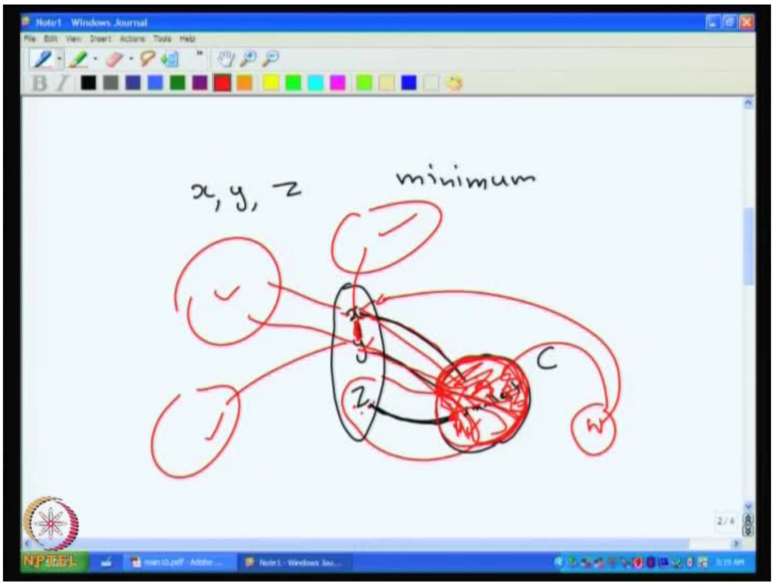

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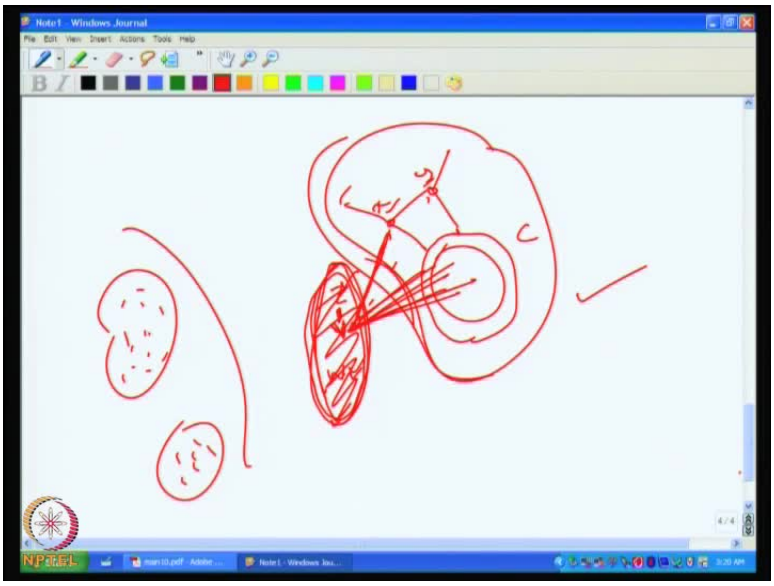

Now, where the neighbors of v are, is the question. The neighbor's v, if you remember, the v is from here, it neighbors can be somewhere here, or it has to be this x or y; z is already we have taken in the separator. So, all the neighbors of v has to be in this collection, say may be here also, x, y and somewhere here. We will not have any neighbor to any of this other components, this component or this component. So, all the neighbors of v is only to one part, which essentially means that so this is not a minimal separator at all, this is not a minimal separator, this is not a minimal separator. So, this is a contradiction, we told that for every edge when you contract, you will get partner, so that is w z v, w is a minimum separator of the original graph (Refer Slide Time: 11:11).

Now, we are saying that this v connected to all this things, and without we said, then w itself will in fact separate this portions from in this portions, because this is only connected to this portions, so that is the contradiction comes in. Therefore, we can conclude that, every edge cannot be such that, when you contract that edge, it becomes less than, connectivity becomes less than 3. So, there should be at least one edge, by contracting, which we can retain the 3 connectivity. This is the essence of the theorem right to the proof.

So, to conclude that is a proof by the contradiction, so essentially we assume that the theorem that is not true, that means for every edge when you contract you can find another vertex, which call as it is partner, such that x y and that vertex z, say its partner, will become together vertex separator of cardinally 3 for this graph G, original graph G. And then, we analyze the smallest component, and then let us say, x y be the edge which gives the smallest component, and then we showed that there will be an another edge by contracting… When we considered the corresponding 3 vertices that will give you a smaller component, which will be a contradiction, this is the essential feature of that.

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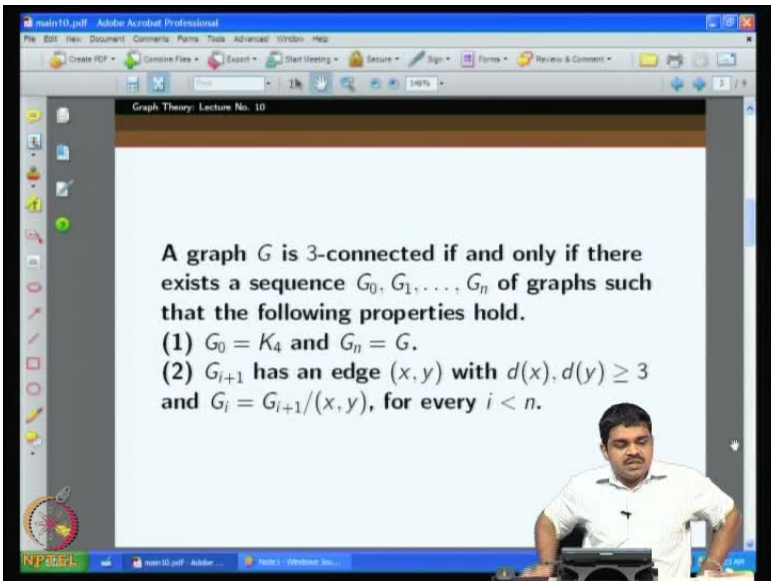

Now, how are we going to use this thing? This is the main statement that we want to prove finally using this. So, if the graph G is 3-connected if and only if there exists a sequence G 0, G 1, G n of graphs, such that the following properties hold. The first graph G 0 equal to k 4, the smallest three connected graph k 4, and G n is our graph, the last graph is our graph. Also G i plus 1 has an edge x y with d of x greater than equal to d of y greater than equal to 3, and G i is equal to G i plus 1; G i is obtained by contacting the edge x y from G i plus 1. See, one side of this thing is very trivial; you give me any trivial in view of the thing that we have already proved, the result we have already proved. You give me a 3-connected graph of case every degree is greater than equal to 3 there, because it is 3-connected graph.

Now, what you do? You can always find out that particular edge, which is guaranty to excess by our previous lemma, which is such that by contract, if we contract that edge, the 3-connectivity is not lost, the resulting graph is again 3-connected. This particular edge if I contract, I will get a smaller graph, smaller by 1 vertex. And then I take a new graph, its again 3-connected, as longer it is greater than - the cardinality is greater than 4, you can again find an edge, such that if we contract that edge, still we will we can retain 3-connectivity, we can contract that edge and we will get a smaller graph, smaller by one vertex. And then, we can find out because it is again 3-connected, we can find another edge with the same property and keep going until we reach the situation that the number of vertices is 4. But, the number of vertices 4 and it is 3-connected means it has to be k 4.

So, therefore, given any 3-connected graph, we can get a sequence of graph G n equal to G, G n minus 1, G n minus 2 up to G 0, which we call as G k 4, we call as G0 that is the k 4. We will be able to reach that, it is very obvious from that earlier lemma we proved. The other way, suppose somebody gave you a k 4 and he started doing it, you pick up an edge x y such that both the degrees of d x and d y is greater than equal to 3, then you do the opposite of contraction, that means if you get a graph such that if you contract, sorry is not like that, if you start from k 4, and if we can find out a graph from that, such that this k 4 can be obtained back by contracting an edge x y, such that both of it is degrees are 3, then you can say that this operation will lead you to a 3-connected graph, this is straight forward statement in fact.

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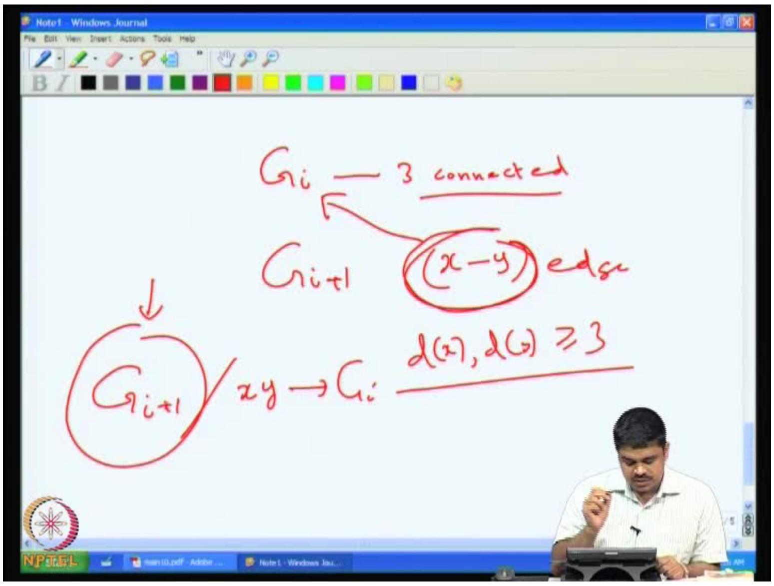

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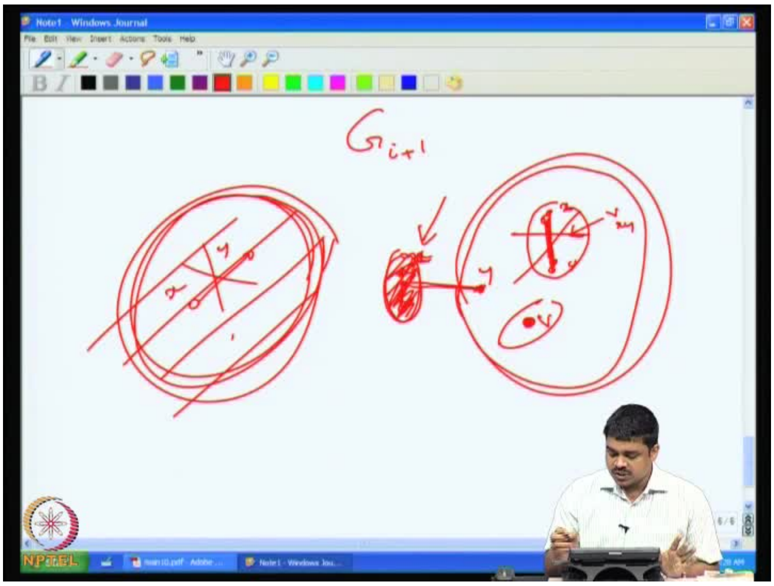

If you want to gregariously prove that, so you take any graph k 4. So, you can say some G i, now currently we have a graph, let this be 3-connected, that means the minimum separator cardinality is 3 and then is more, 3 or more. Now, we found out graph G i and plus 1, it so happens that there is an edge in G i, which is x y edge, such that the degree of x and degree of y both are greater than equal to 3, or at least 3 is there. When you contract this edge, we get this, right. But, in other words, G i plus 1 when you contract this edge, x y edge, we end up getting G i, in that case, I want to say that this is also 3-connected. Suppose, this is not 3-connected, then we will get minimum separator of cardinality 2, so there are two components on this side, so this is the G i plus 1.

Now, you can see that, is it possible to have both my x and y on one side of this thing that means x and y does not belong to this? This is here or here, so what will happen? If I contract this thing, we will replaced it by v x y, and after contracting what we get will be G I, and then, but then G i is not supposed to be to have any separator of cardinality 2, so but then here you can see that this is going to remain as text separator for G i also.

So, it is not possible to have x and y both on one side, because if you contract that we will end up getting a separator of cardinality 2 for G i also, so neither is this possible, nor is this possible right, so both this situations are not possible. The only situation is maybe you can have at least 1 x here, x or y, one of them here, or both of them here, something like that, but then the key thing is see there should be something on this side, say, so y is here. Suppose, there is something on this side and something on this thing, so suppose there is another vertex here other than y, so x I am putting here on this thing, suppose some v is here on this side, then also the same situation. Suppose if I contract, see this two vertices is enough, this V x y and this other vertices is enough to separate V from this part. So, we will get end up getting a separator - minimum separator of cardinality 2 for G i also, which is not possible.

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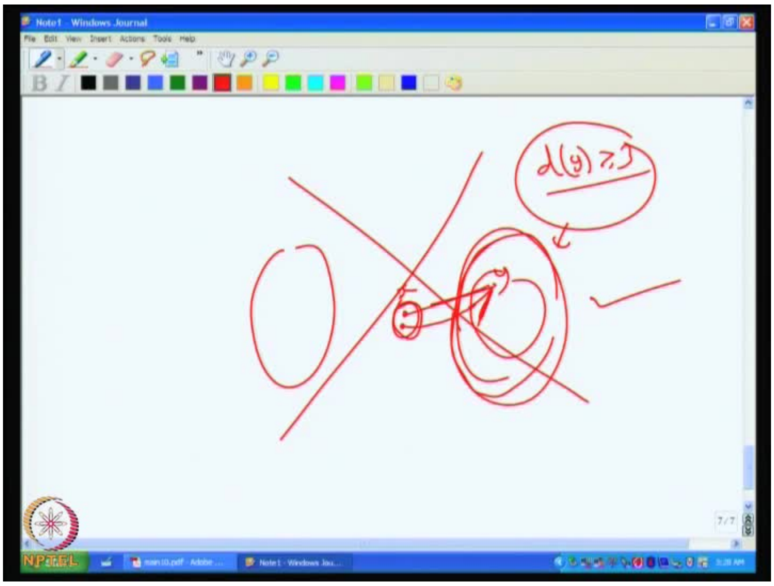

So, the only conclusion that we can make is that the situation is like this, so we have an x, we have some two separator here, so one of them is x, the other is y. And this y is the only thing here, in this side, and then but we also know that the d of y has to be greater than equal to 3, were as the degree 3 edges should go out of this thing, were it did go one can go here, the other can go here, but where will this go? So, there are only two vertices, there is nothing else here, so that is that leads to a contradiction to this assumption and so it is not possible.

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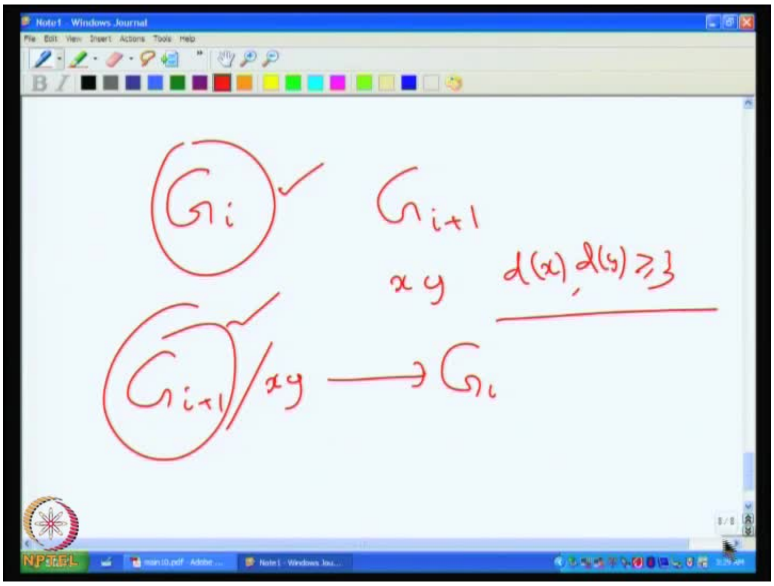

So, we conclude that, so if we start from edge G i and find out a G i plus 1, and an edge x y and G i plus 1 with degrees d x and d y, both greater than or equal to 3 such that G i plus 1 contract when you contract the edge x y and G i plus 1, so we end up getting G i. Then, if G i is 3-connected, G i plus 1 has also as to be 3-connected, so this is what we are told, so that is the other side of it. This is in fact easy to prove when from the using a elementary arguments, while the other thing, that means, given a 3-connected graph, you can get a sequence of graphs, reducing the number of the vertices 1 by 1, and reaching k 4, such that each member in the sequence is 3-connected, is required that lemma, which you proved it; so that is the end of it. So, this is the way 3-connected graph off characterized.

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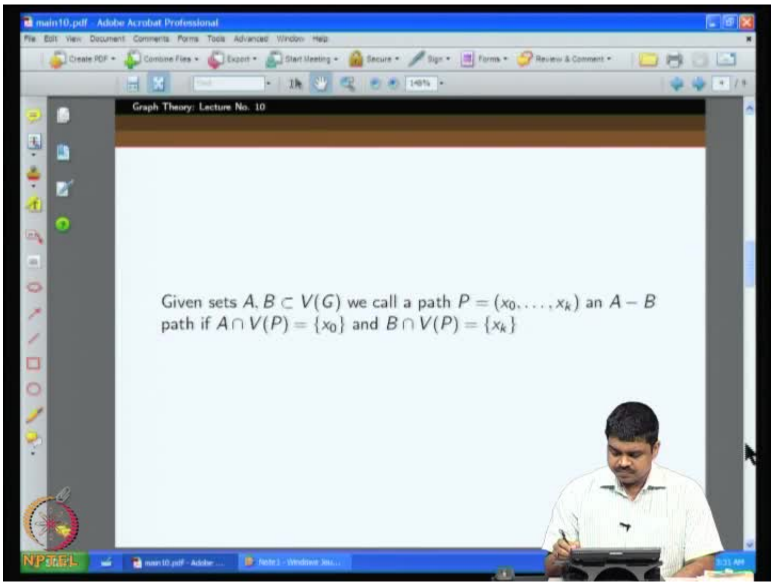

Now, we will go back to the connectivity theorem, the main theorem which we mentioned in the first class. The connectivity the mangers theorem that is what we our next aim is. So let us see the… We need some definition, here given… So, we need to convert the notions into a little different, because into a slightly different way, so we are going to prove the merger's theorem in a slightly more general setting, following in cline hard distills book. So, we consider the sets A B, sub set of the vertex z, we call a path P, so the let us say the x 0, x one, up to x k, which this is called an A-B path. If the first vertex of this path belongs to A, that means x 0 belongs to A, and the only vertex from the path that belongs to a is x 0, that means the start is from A really, and it will never come back to A.

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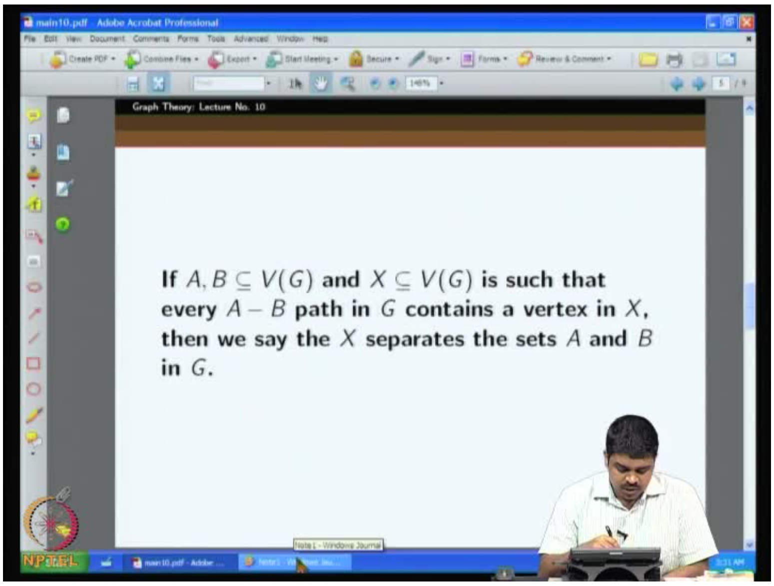

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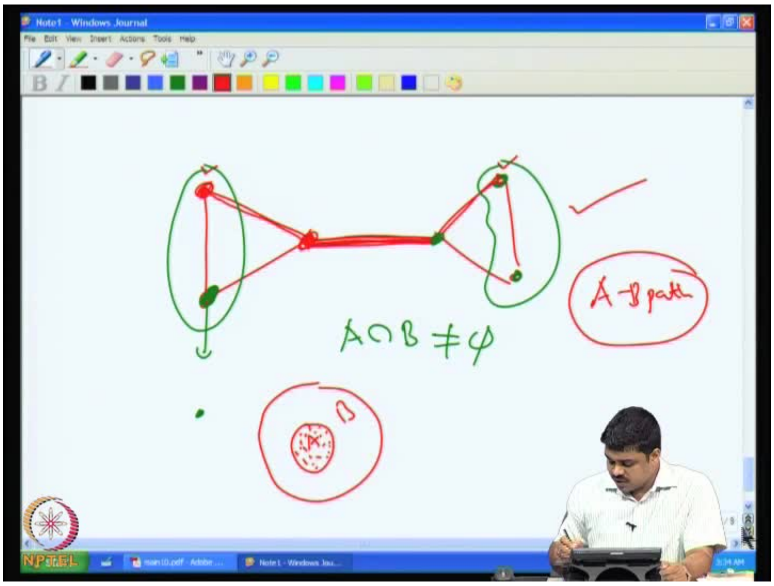

Similarly, the last vertex to the paths belongs to B, and the only vertex of the path that belongs to B is the last vertex namely x k. It is reaches, ones it reaches B, it does not move further, right; this is the A-B path, another notion we want is A B separator. See, when you see A-B path, before going further, let us look at some interesting things about A-B path. So, here, when I mention A-B paths, this A an B do not think that A and B has to be disjoined sets, or so for instance, I can take small graph. So, this is a small graph, here I can this can be a, this can be B, so this can be B, this can be B; this is A and this is B.

Now, an A-B path can be like this, because it starts off from this thing and reaches here. Now, it is also possible that so you can have… this A is this, and B can be this and also this, that means you can have a common vertex between A and B. A intersection B need not be phi, in that case, this vertex can be considered as an A-B paths. This vertex itself is the single ton path which contains only one vertex. Similarly, it is possible that our B contains this and A contains this, then this edge I can be considered as an A-B path.

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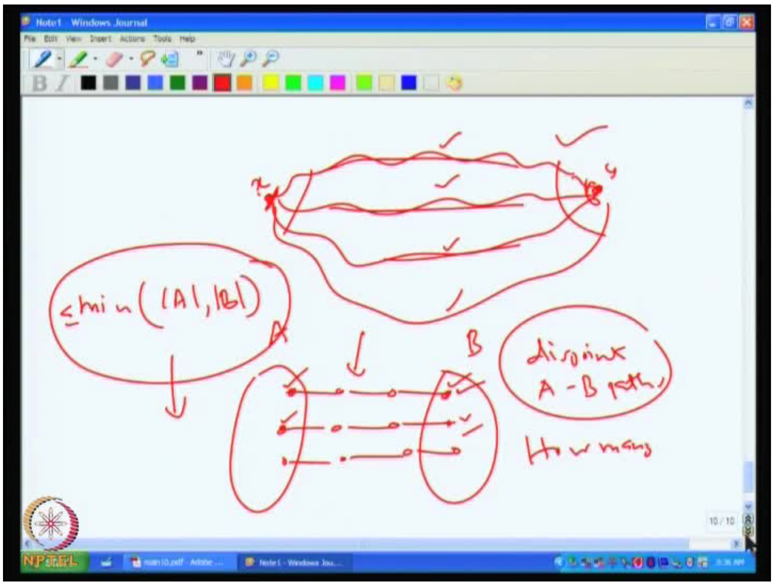

So, in other words, when I say A-B path, I am not assuming that A intersection B not equal to 5 or something, so there is there can be common vertices A and B, in that case, that single vertex, each vertex in the A intersection B, in a section B, is an A-B path. Now, also it is possible to have direct edge from a to the set A to B, all such are possible, allowed. For instance, so the is for instance it is possible that A is a subset of B, in which case, the vertices of A, each of them will define one A-B path of just one, having one vertex each; so, that is all the A-B paths. So, whenever say we say A-B paths, like in the manger's theorem, we were interested in the number of internally vertex, this shows in path between 2 vertices A and B, here we have converted the notion to just, so in that case. There were 2 vertex, is x and y, we were interested in the number of internally - maximum number of internally vertex disjoint paths between x and y in the graph. So, how many are possible? So, the only conditional was it should be internally vertex disjoint, the only common vertex is x and y. Here, we have generalized it to some sets, A inside of one vertex, we take A and B, and this internally disjoint properties kept in some sense, because we want disjoint A-B paths, we do not want not internally disjoint, we say disjoined A-B paths are required, we can always like here. We can ask here also how many such A-B paths can be obtained. When I say A-B path, they have to be disjoint. Here, I only wanted internally vertex disjoint things, here I want disjoint, that means 1 A-B path, the even the end point should not be shared.

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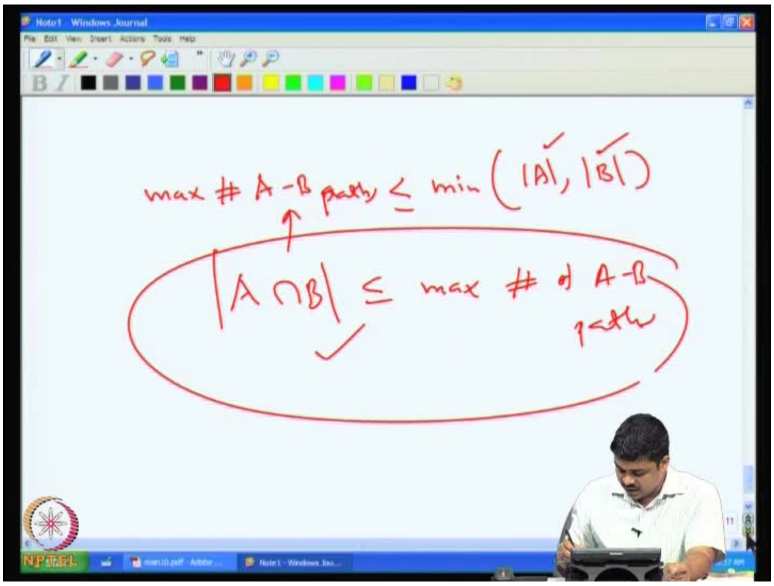

So, in other words, how many such A-B paths can be there? So, definitely we cannot have more than minimum of cardinality of A, B, so that is an upper bound for the number of possible disjoint A-B paths. So, this is the parameter we will be studying. So, as I told, so minimum of cardinality of A, B is an upper bound for the maximum number of A-B paths, disjoint the A-B paths. So, because each A-B path will take away 1 vertex from A, and 1 vertex from B, so when one of them finishes, then we would not get any more.

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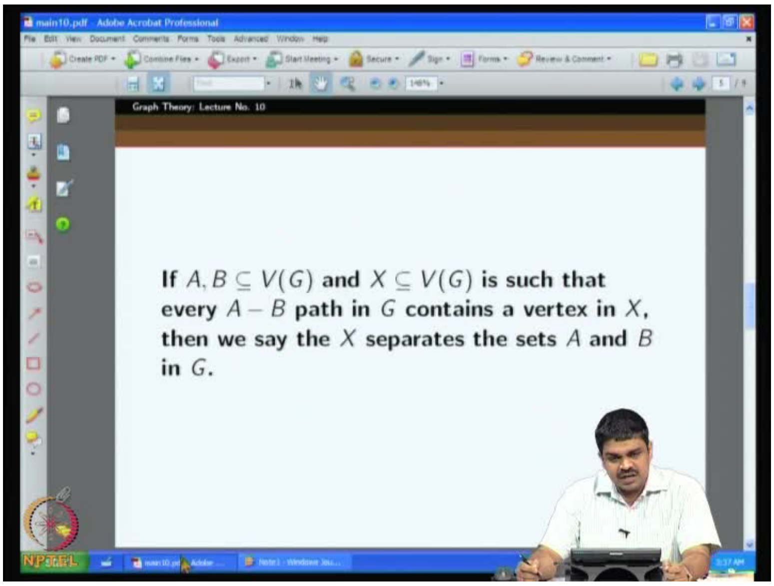

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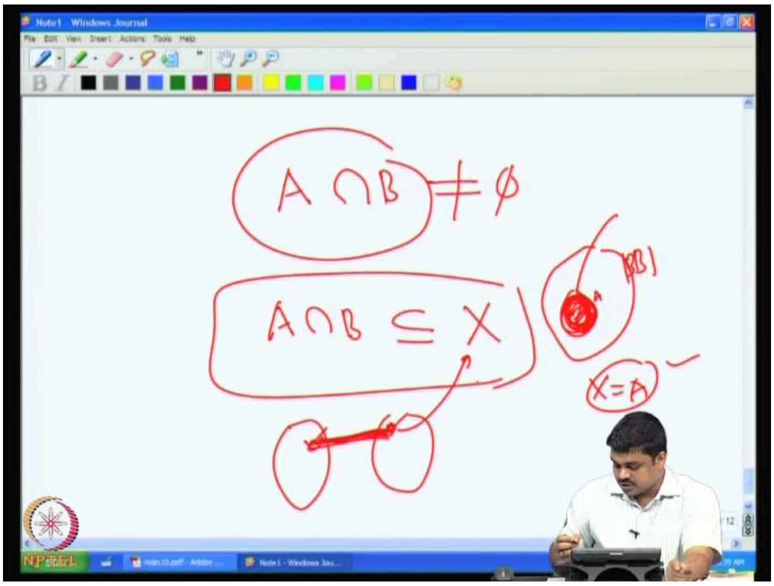

Similarly, if you considered this quantity A intersection B, so each vertex an A intersection B we will define an A-B path, therefore this maximum number of A-B paths has to be at least as much as this, so this is some trivial lower and upper bounds for the number of A-B paths. So, we will come back to the number of A-B paths a little later, we will first look at another course concept here. So, here is another concept, given A-B, like earlier I am not assuming that A and B are disjoint or anything, so A and B. And then, we say that a collection of vertex X is such that, if it is such that every A-B path in G contains a vertex in X, then we say that the X, this set x separates the sets A from B in

G. So, in other words, given A and B, we are looking for another subset X, such that if I remove X from the graph they would not be any more A-B path in the graph, or every A B path should pass through this X, so then we say that X separates A-B. So, here again you can see that… Like earlier, because we do not have any condition that A and B has to be separated, so we can have A intersection B not equal to phi, right. In that case, you can see, if this A intersection B has to be a subset of the X, if X separates A from B, because any vertex in A intersection B remains in the graph, then there is an A-B paths, namely that trivial single ton vertex paths, single vertex path, right; A to B we can reach.

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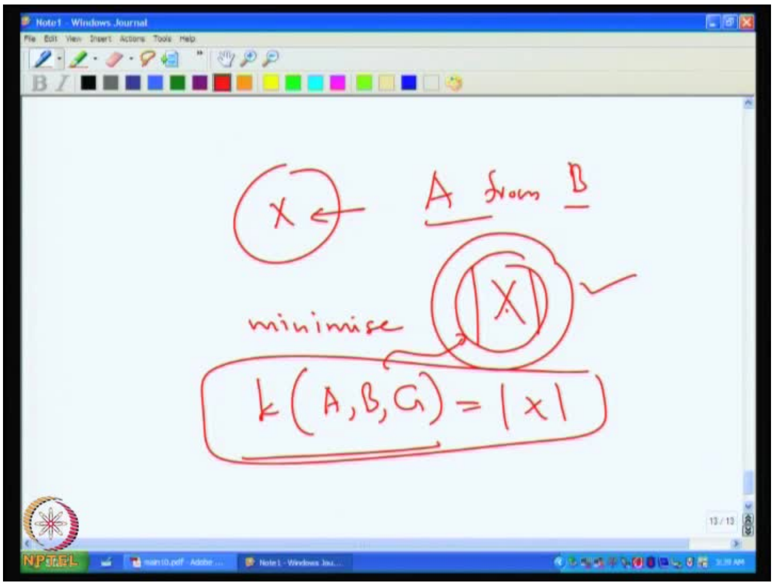

Similarly, if you can see that if A is this, and B is this, there is an edge between them. So, at least one of this vertex is should belong to x, otherwise this path will remain, right, this is one edge path. so right a so if for instance, if A is a subset of B, then all the vertex is of A are the only possible collection of A-B paths… Sorry, the vertex is to separate A from B, you have to remove, take X is equal to A itself, otherwise you will not able to be separate. So, essentially you can see that X, when you say X, the sets that separate A from B, I am interested in minimizing the cardinality of X, I want to give an A B, I will be interested in getting this parameter k of A,B,G, that is the minimum number of vertices in any set X, so that X separates A from B. k A B is equal to cardinality of X, which minimizes the sets A and B, so the minimizes the set X, the cardinality of X, which can separate A from B.

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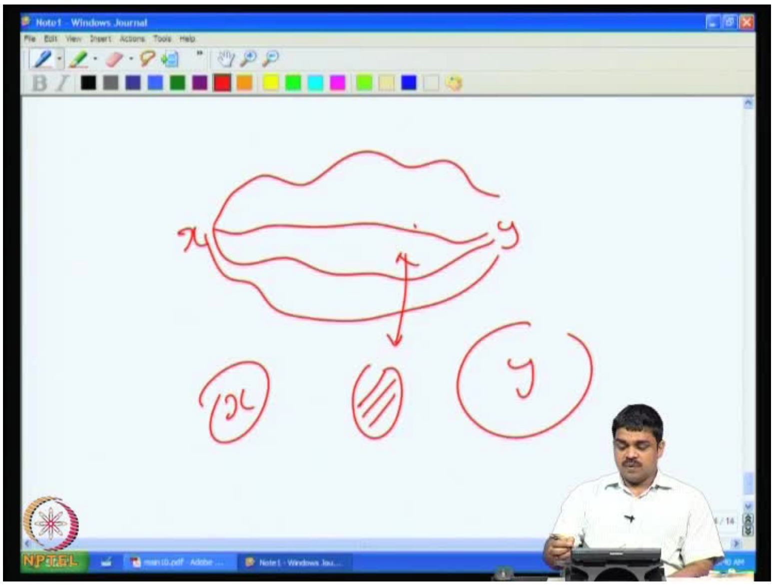

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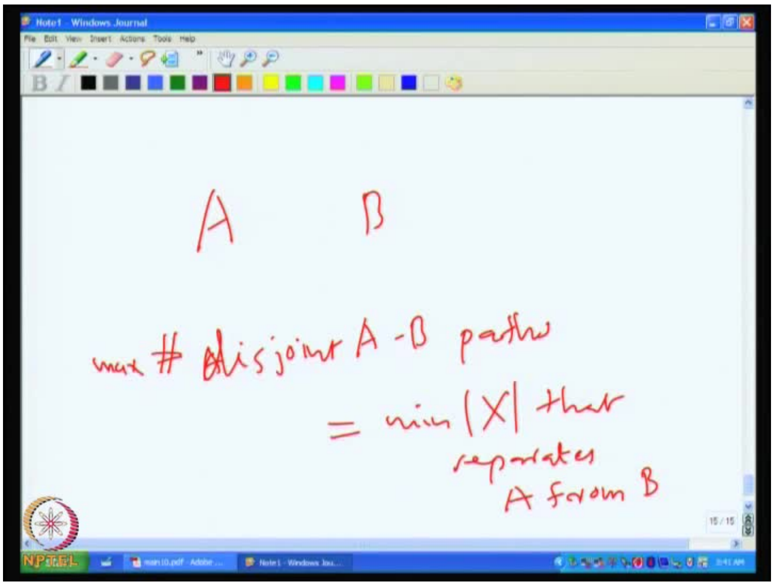

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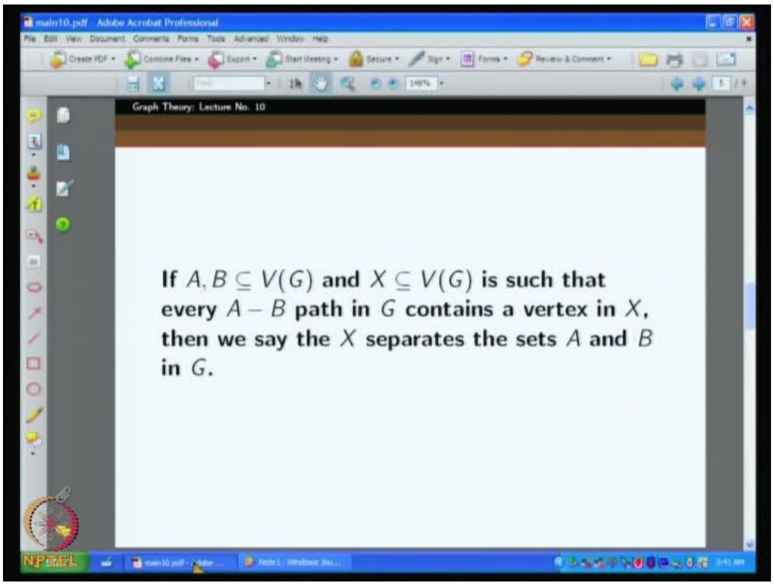

So, earlier, so of case, these two parameters, the cardinality of the path - A-B paths, maximum cardinality of the number of A-B paths, the minimum cardinality of a set of vertices - subset of vertices X, which can separate A from B, these two parameters somehow correspond to the two parameters. We earlier discussed, namely given two vertices, the local connectivity was defined to be the number of - maximum number of internally vertices disjoined paths, and the I also we told that the minimum number of vertices to separate x from y was another parameters. We our intension was to prove that these two are equal, right. Instead of proving this thing, we will prove this general statement that give me any two subsets A B, then will show that the maximum number of disjoint A-B paths equal to the minimum X - cardinality of X that separates A from B. So, minimum cardinality of X that separates A from B, so formally more formally I can show that the A B subset of V of G.

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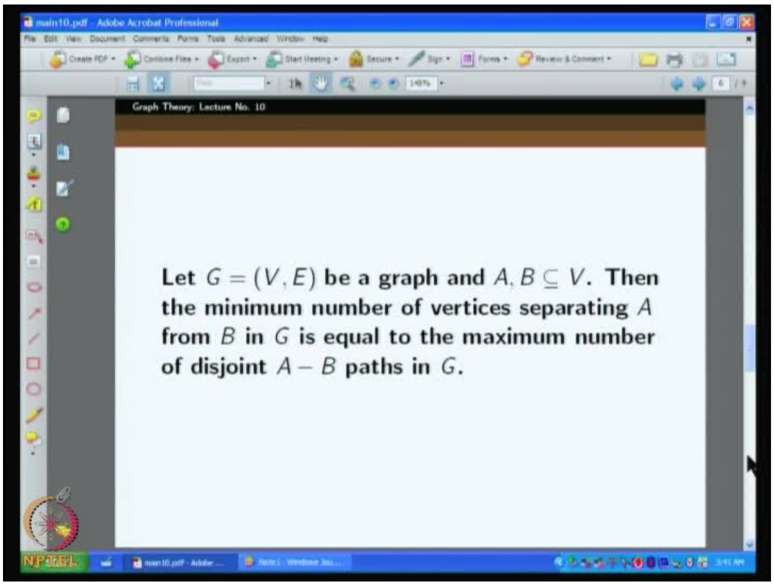

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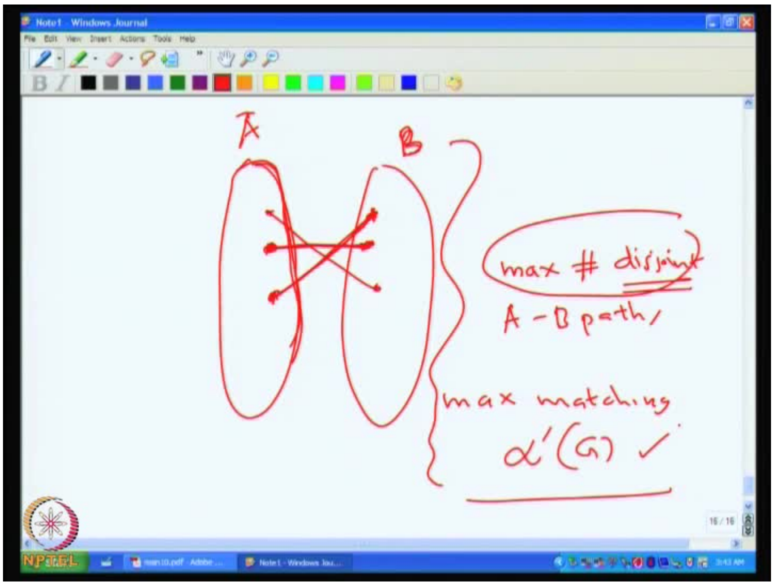

So, now, here, A graph and A B subset of V, and the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A-B paths and G, right, this is what we want to prove. See, before going to we before we get into the proof of this thing, let us look at some interesting features, like a you can take a bipartite graph, so A B, you can take a bipartite graph, and then let this side of the bipartite graph itself is the set A and this is B. Now, what are the if I ask for the maximum number of disjoint A-B paths in this graph, what is that? So, once you can say that if there are no single ton paths, we have to because of the disjoint right, that is no intersection here, so you can, so you have to starts from here and reach here. So, you can, you can take an edge, but then once you take an edge ,you cannot come back, because you will already touched B. So, the only possible A-B paths are some the edges, any edges, an A-B path, you can start from here and you reach here. You start from here… Why you cannot take more than an edge? Because, if you start, if you take an edge, already you are in B, and A-B path does not allow you to go further once you reach B.

So, therefore, the only A-B paths you can take are the edges, but then they have to be disjoint, independent. So, what are they in fact? The maximum number, that is an essentially the maximum matching, right, the maximum matching. So, you can say this is the alpha dash of G, right, so this is may come as, come out as an interesting observation, that is maximum cardinality of the A-B paths in this graph. This bipartite, in any bipartite graph, when I take A to B in one side of the bipartite graph, and B is other side, and then it is indeed the maximum matching that we are looking, talking about.

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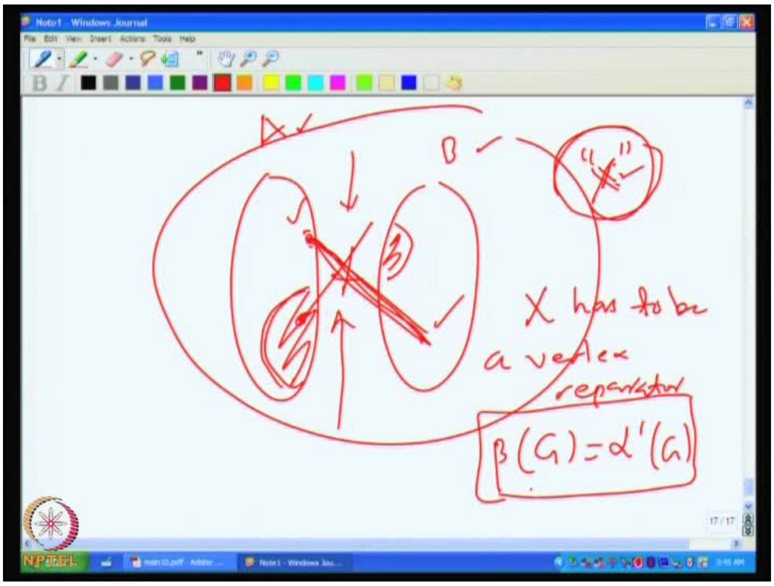

Now, again for this bipartite graph case, when you take this A and B, and you can ask what will be the minimum cardinality of X, any separator of A and B, right. But, according to the definition, now, you see the separator should be such that once you remove the vertex, suppose here is the separator, so once you remove the vertices of separator, we will not get even one edge in this across. Why because, even if one edge remains, then A is not separated from B, because there will be… this edge is an A-B path and then there will be an A-B path, a path from A to B, so all the edges should get removed when we remove the vertices of X from the graph G, if it is an A B separator. In other words, every edge should have at least end point in this X, which means X is a vertex separator. X is a X has to be a vertex separator, if it is an A-B separator here.

In other words, the minimum cardinality of the X, the sets X that separates A from B, is essentially the cardinality of the minimum vertex cover beta of G. So, if we manage to prove manger's theorem, that is the maximum number of A-B paths, a disjoint A B paths is equal to the minimum possible cardinality of a set X that can separate A from B. It is equivalent to proving here, in this case, at least the in this case, that alpha dash of G is equal to beta dash of G in the bipartite case, when we take the A and B to be the two sides of the bipartite graph.

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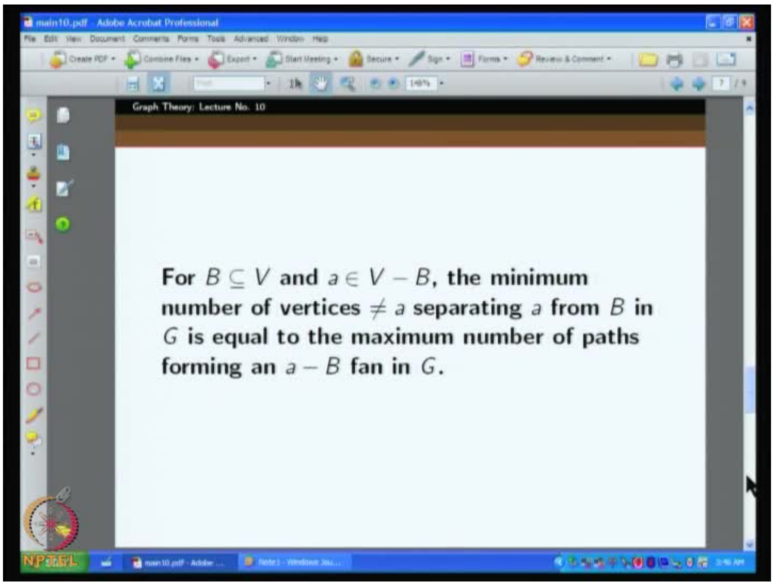

So, therefore, so it is so this generalization is not that bad, so it is able to capture some things, which at one look we may not notices. So, from the original statement, it is indeed a more powerful generalization.

Now, let us also see the earlier statement that we wanted, some of the more intuitive statements that we wanted, like if two vertices are there, which is non-adjacent, then the number of vertex disjoint paths between them, internally vertex disjoint paths between them, the maximum number of internally vertex disjoint paths between this two vertices A and B, will be equal to the minimum number of vertices that I have to remove from the graph, so that A and B gets separated. The minimum number of vertices other than A and B that I have to remove from the graph, so that vertex A is separated from B in the remaining graph, disconnected from in the remaining graph. These two numbers are equal; see this kind of statements will follow from what we have proved that, we will first look at how it follows from this already proved statement.

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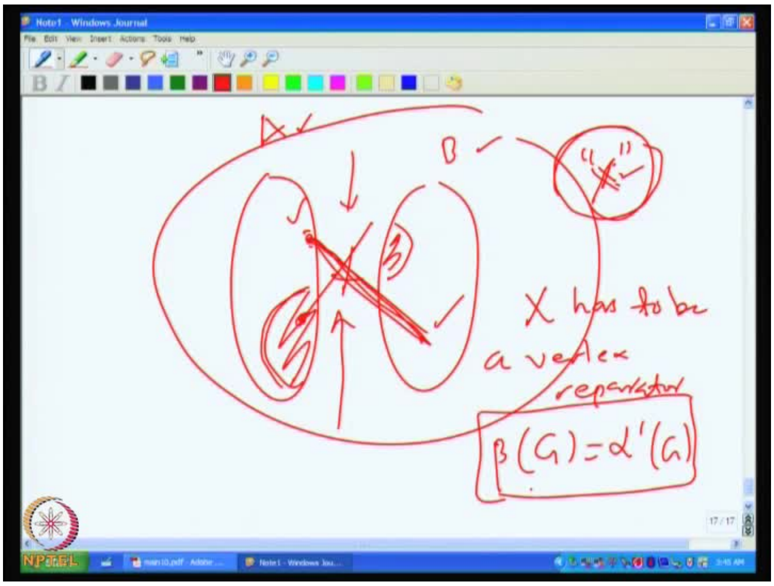

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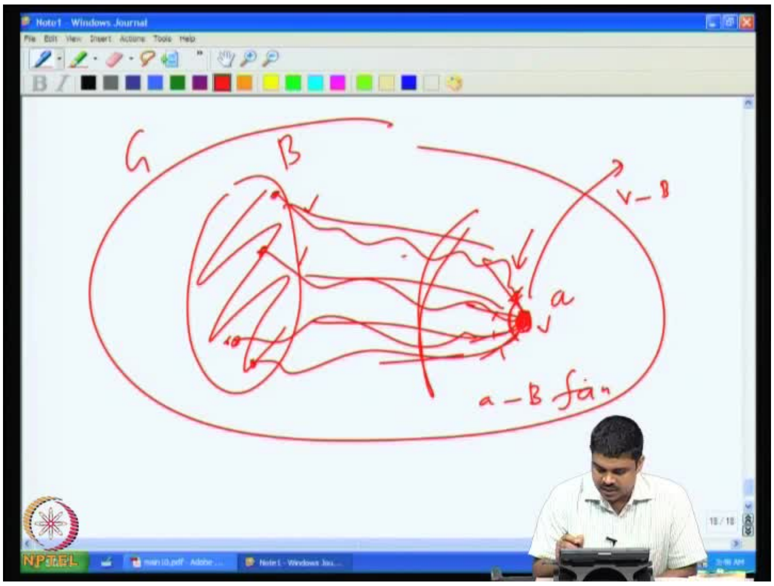

So, before the first statement I would look, because I want to look at this thing first, because there are once you do this thing, the other proofs will be much easier. So, here, there is a concept called a-B fan, what do you mean by a-B fan? So, here is a set B, in this is the graph G, and then here is a set B, and then we take a vertex a outside B, that means this a belongs to V minus B.

Now, we want to find some disjoint path starting from a to B, disjoint means disjoint paths except for this vertex a, means other than this vertex a, there should be internally vertex disjoint. And that means here vertex disjoint, that means nothing else should be shade, they should be reaching in different points, and then this should all be disjoint, right, except this vertex, they starts here. This kind of a structure is called an a-B fan, so you say a-B fan, it is look like a fan probably, so the fan structure.

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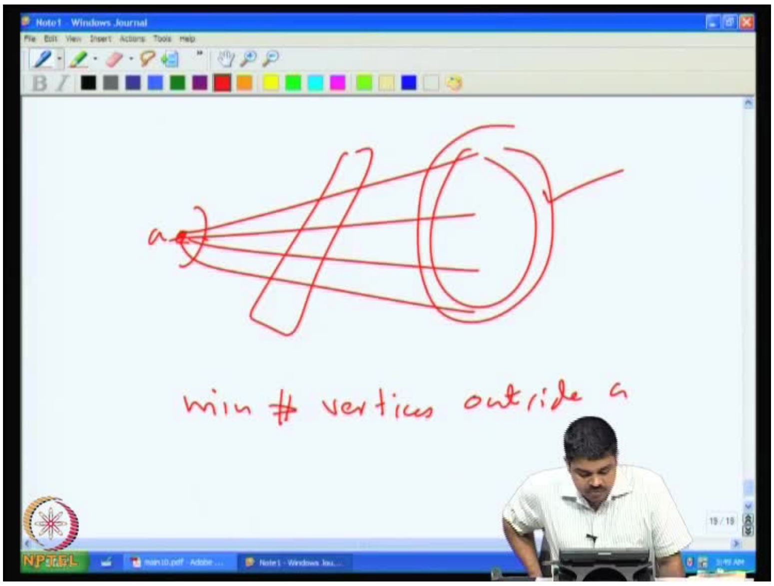

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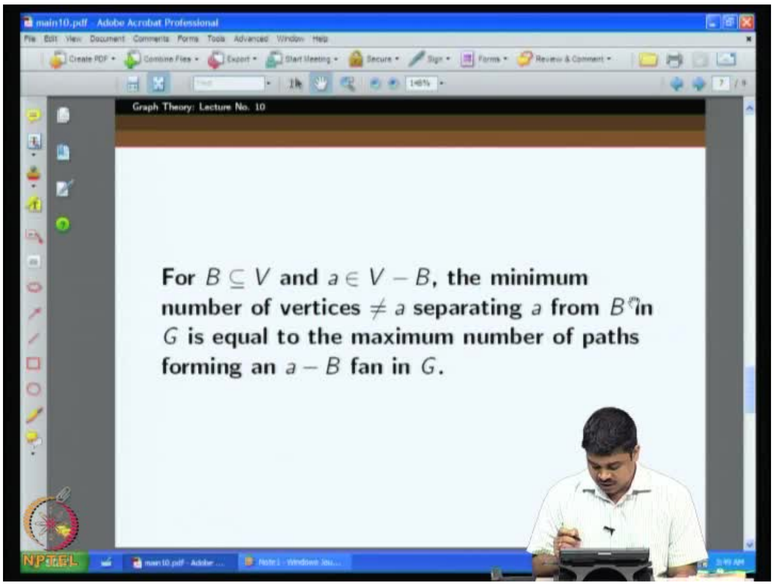

So, this is called a - B fan, the cardinality of a - B fan is how many such paths are there in from a to B, that is the cardinality of, the maximum cardinality of a - B fan. So, we will show that the essentially the maximum cardinality of the a - B fan will be equal to this is a, this is B, so this maximum cardinality of the a - B fan will be equal to the minimum number of vertices outside a, other than a, that I have to remove from the graph, so that this get separated from a. There is no more path from a to B, that numbers are same, so you can look at it once again. B subset of V and a element of V minus B, the minimum number of vertices is not equal to a separating a from B in G, is equal to the maximum number of paths forming an a - B fan in G.

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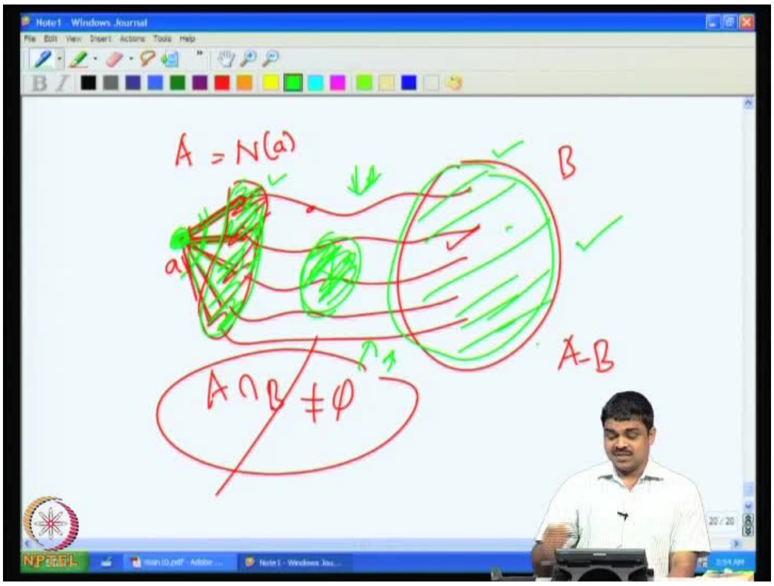

So, how do I prove this thing? So, this proof is simple, so we will this is our a, we rather considered the neighbors of a, this is called N of a. And let us defined this is capital A, and so this is our B, so it is possible that some vertices here A intersection B need not be phi, there no is nothing like that. It is possible that there are some vertices and this neighbors may be in this B.

Now, it is easy to see that you consider the maximum number of paths starting from A and reaching B, this number essentially will be equal to the number of, maximum number of A-B; this capital A-B disjoint paths. Because, if you considered this A-B disjoint paths, they are not going to contain any, none of this thing is going to contain A, because if you go to A, you have to come back to this, right, because B was totally outside it. Because, the only way once you go, you the it is only ways to come back to its neighborhood, so because any A-B disjoint A-B paths is going to contain a vertex of A, once so it will never contain any of these things.

So, what you can do is, all these paths can be extended to a, and then you will get the extended to A, and you will get the an a - B fan of the same cardinality. On the other hand, if you have an a - B fan, you can discard a and it is connection, and the remaining is going to be an A-B, the collection of disjoint A-B paths, right. You can immediately see that… Now, it is already disjoint A-B paths, therefore the this parameter, the number of disjoint A-B paths is essentially equal to the maximum cardinality of an A-B path here, once you take this capital A to B N of a, that is the only trick.

Now also we know that this number of A-B paths - disjoint A-B path, if the manger's theorem is proved in the general setting, we know that it is equal to the cardinality of a minimum X that will separate this portion from this portion A from B. And then we just have to prove that this number essentially is equal to the number of vertices, is required to be removed outside A, to separate this vertices a from this. You know that this cannot contain a, because if you contain a, right, even without a, this could have go out separated from this portion, it is very easy to verify. Therefore, by the same kind of argument, we can say that the minimum number of vertices that separate rate, this thing, will also separate a from this small vertex a from B, and of case that set is without a in it. And of case, the other way, right if any collection of vertices that can separate a from this thing, will necessarily have to separate this neighborhood from this, otherwise we still have a path, just matter of carefully looking at it, it is very easy.

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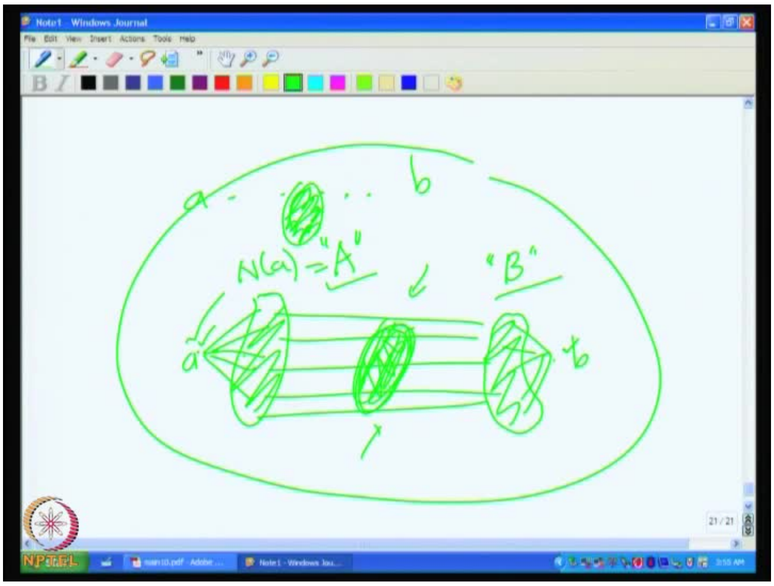

So, therefore, this both the parameters are equal, and by manger's theorem we have already proved that the number of disjoint A-B paths is equal to the minimum cardinality of a separator between A and B, so that is it, so that will prove this thing. See, the key point is only to consider the neighborhood, instead of the vertex itself, and then take it as capital A, and then B is original. Now, in a similar way, we can show that suppose a and b are two disjoint vertices on the graph, then the number of internally disjoint vertex - disjoint paths, maximum number of internally vertex disjoint paths between a and b is equal to the minimum cardinality of a set of vertices required to be removed, so that a gets separated from b.

For this thing also we do the same thing, we considered the neighborhood of a, this N of a as the capital A, and the neighborhood of b as the set B. The neighborhood of a as the set A, and neighborhood of b as the set B, and now we consider these two parameters. The mangers theorem would say that the number of disjoint, this capital A to capital B paths will be equal to the minimum cardinality of a separator to separate a from b. We should prove that this number of disjoint A-B paths, the maximum number of disjoint A-B paths is essentially equal to the number of internally disjoint vertex, so vertex disjoint paths between the small a and small b the vertices. And similarly, we have to prove that the number of vertices in the minimum separator for a and b will be equal to the number of vertices to b, removed to separate a from b.

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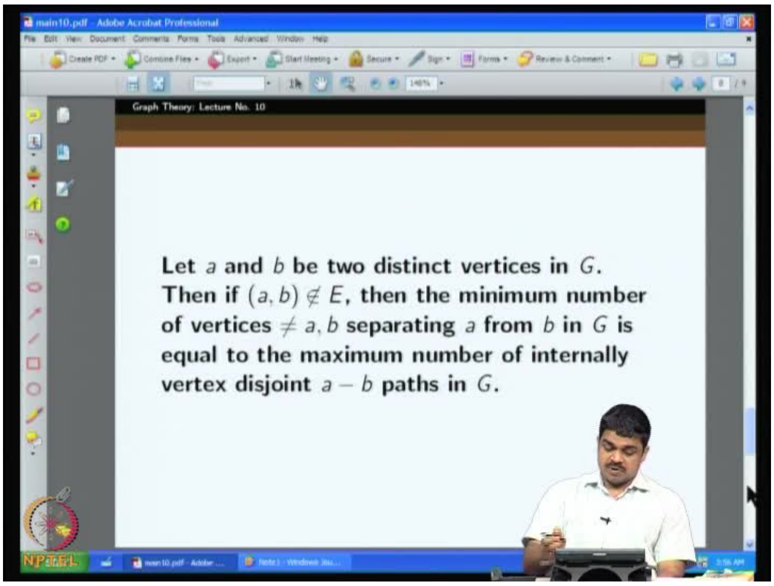

So, once you get these two things, then it will the proof is automatic. While, I leave it to the reader to, the student to, verify that. So, it is same as the earlier one, while it is not very interesting to teach. Now, finally, this would also allow us to make this statement, so can so this statement, namely, so this is, so last one we proved that a and b be two distinct vertices in g, then if A-B is not element of E, then the minimum number of vertices is not equal to a, b. I mean any minimum number of vertices other than a and b separating a from b in G, is equal to the maximum number of internally vertex disjoint A-B paths in G.

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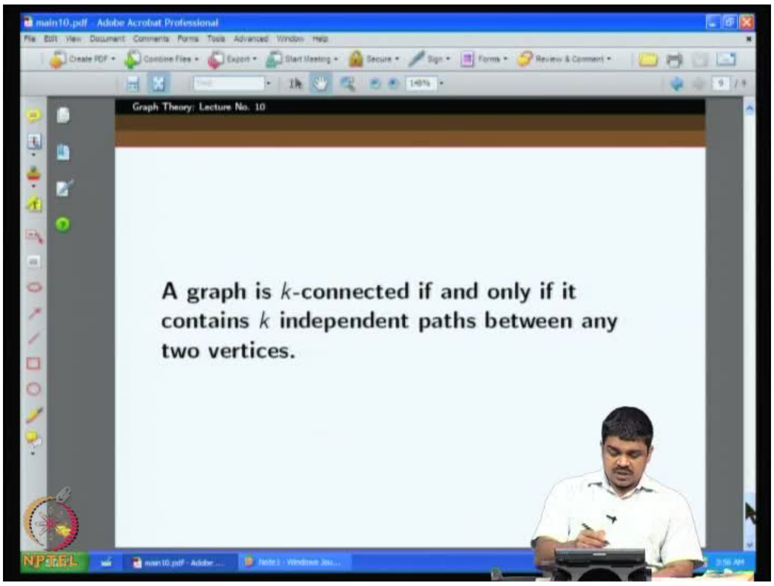

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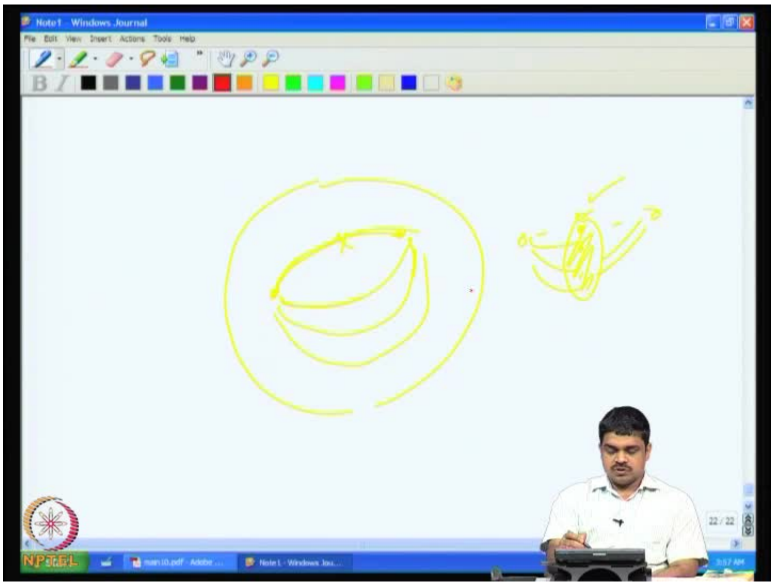

Now, a graph is k connected if and only if it contains k independent paths between any two vertices, so that is what, so k independent path, k disjoint paths between any two vertices. How do I show this thing? So, this is also, you can see that suppose between any two vertices, suppose there are k disjoint paths, then by definition, we know that it is k connected, because you have to remove… See if it is adjacent by removing some vertices, you cannot disconnect them, if it is non-adjacent, so of case, you can remove, we have to remove at least one vertex from each path, then only we will be able to disconnect it, so it is k connected.

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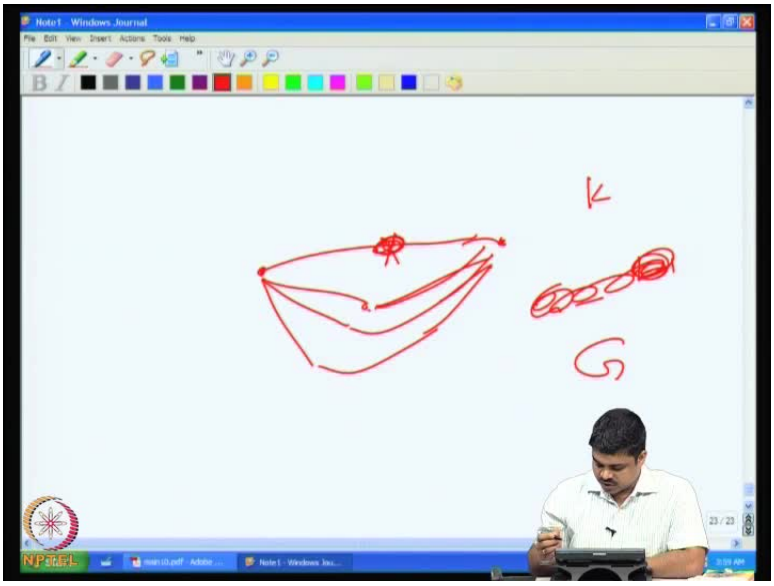

So, for any pair, if it is if there is k internally vertex disjoint, or sorry in disjoint a - b paths, then its k connected. Now, is also see that suppose it is k connected, right, that means, you have to remove k vertices to disconnect the graph. That means, for every pair, we want to show that, then in that case, for every pair, there will be at least k internally disjoint vertex paths. So, how do you show that? So, because see for two vertices, which are non-adjacent, definitely we have already shown that, because that is just previously we show that, if it is adjacent, one path already is here, now you can see that… So, if there are no k internally disjoint joint path between them other than this thing, they will only be at most k minus 2 disjoint path in G minus this edge. If I remove this edge, only k minus 2 disjoint path will be there, so that will essentially tell me that so it is k minus 2 plus 1 is equal to k minus 1 connected, which is, so that I mean you can, you can separate the graph G by removing just k minus 1 vertices that will be contradiction. How do I do that?

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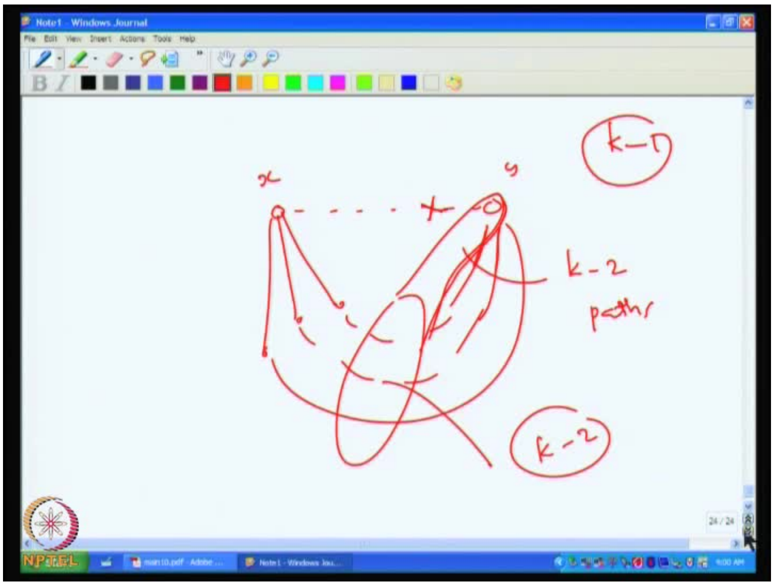

So, suppose, so, this is the x y, so what I do is, I remove this edge from this thing. Now, you know, by assumption we have only k minus 1 disjoint path between x and y, so one is gone now, so how many more or left? Only at most here, like this, so at most k minus 2 paths are left, 2 paths a left, but if there are 2, k minus 2 vertex disjoint path in between x and y, then we can remove a set of cardinality k minus 2 and separate x from y, right. Separate x from y, now you can see that so it is not to that, so there should be one more vertex, which if you add x, this y also, see one of this vertex can be added, and you can separate, disconnect the remaining graph, because this edge will go away with this thing and then we will get a k minus 1 separator.

So, this is y, we will end up with k minus 1 separator, if there are no k disjoint paths between any 2 vertices in the this thing, so in other words, the k connectivity of a graph, namely the number of, minimum number of vertices is to separate graph, so disconnected graph is equal to the minimum maximum number of… right. Sorry, over all pair, if you if you take any pair, right, you have to minimize over that over all pairs, so then the we consider this maximum number of disjoint paths between those pairs, so that is what manger's says.

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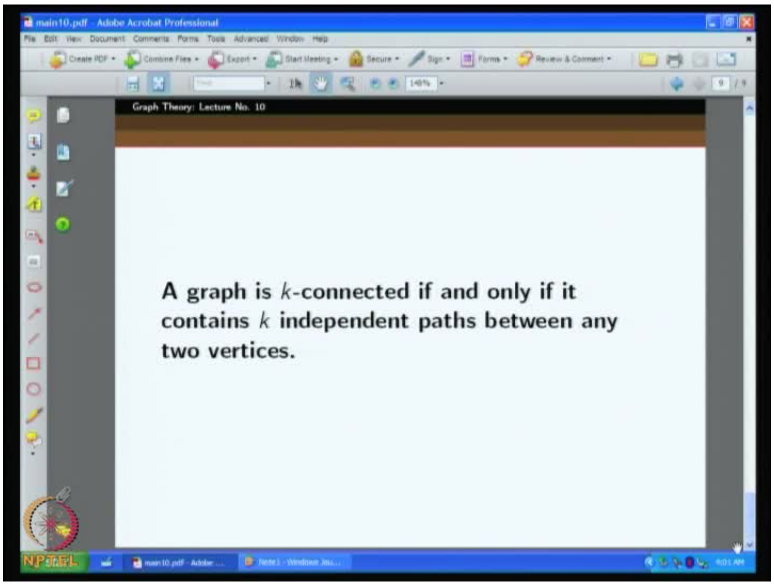

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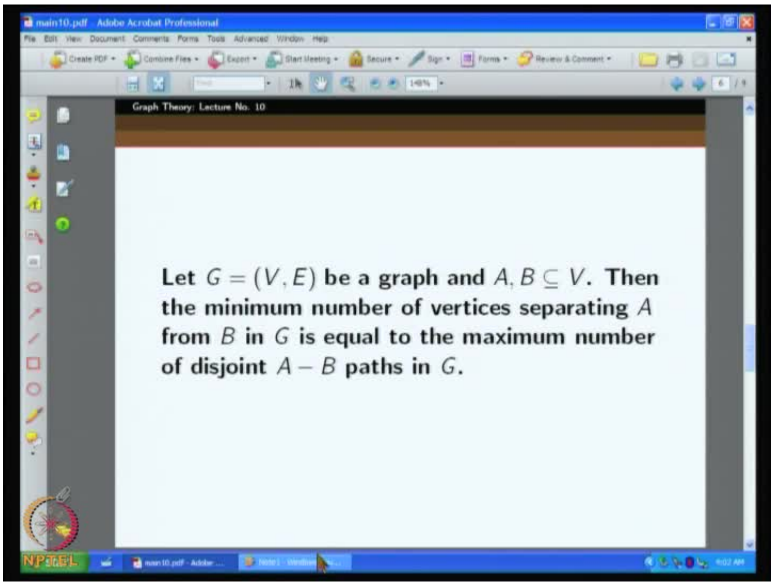

So, now, we will go back to our proof. So, this essentially is what all these consequences are more or less immediate, though when we are explaining it, it looks a little dull and then they not very difficult to prove. if you Though, even if you have missed some of the points here, you can easily figure it out yourself, so we will go to the more non trivial part, namely the proof of this statement, that is this statement. So, let G equal to V comma E be a graph and A-B subset of V, then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A-B paths in G, so prove this thing.

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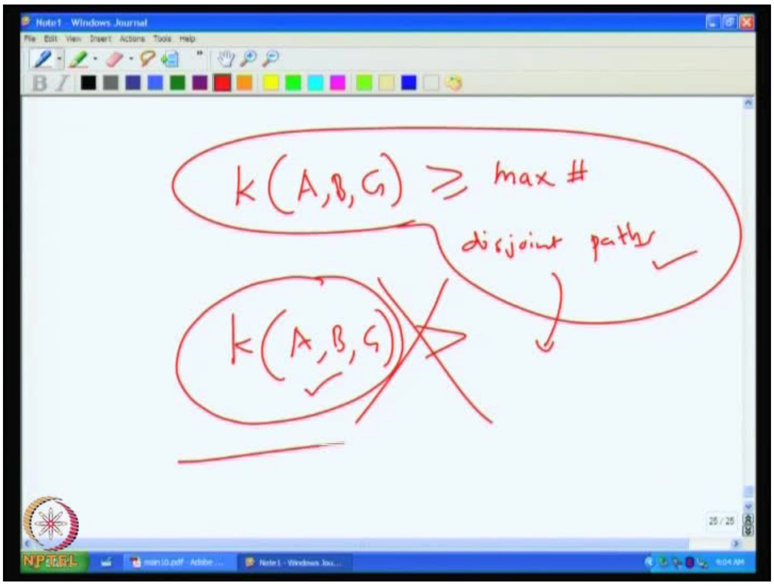

So, we first observe that so suppose this k A B of G, the minimal number of what is to be separated has to be always greater than or equal to the maximum number of disjoint A-B paths. Why because, you have to remove at least one vertex from the disjoint paths, at least one vertex from each path, otherwise at least one of the path will remain as such. So, how can we disconnect A from B without removing at least one vertex from each path? So, this is true, the only thing we have to make sure is this will not happen, right, A B of G is strictly greater than this quantity, is not possible, is what we want to show. In other words, so the maximum number of disjoint, this is what we want to show. So, suppose this is the cardinality of the minimal separator of A and B, then we want to show that there exists these many disjoint paths between A and B, that we will do in the next class.

Thank you