Graph Theory
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Module No. # 02
Lecture No. # 12
Minors, Topological Minors and More on K-Linkedness

So, welcome to the twelfth lecture of graph theory. Today, we are starting with proof of the theorem, we were discussing in the last class. That is, if the connectivity of a graph is sufficiently large, then the graph is k-linked. We can formally state the theorem like this. So, formal statement of the theorem, I will mention like this.

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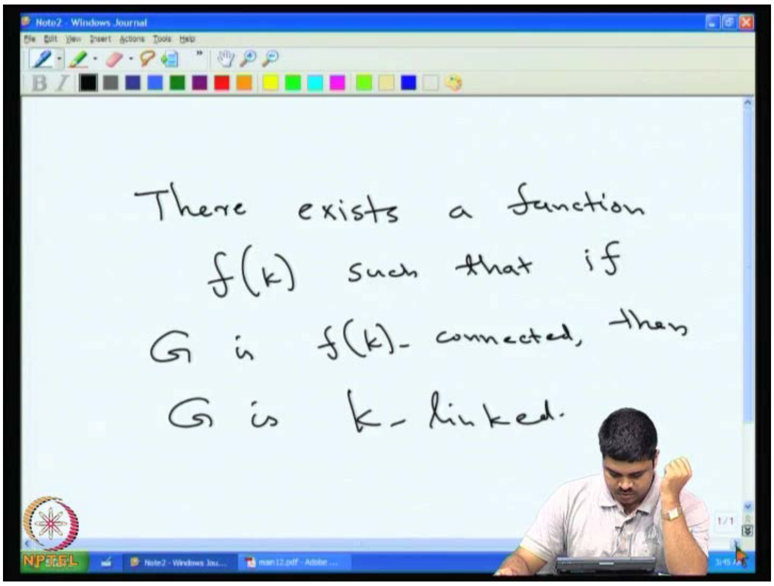

So, we will say that there exists a function f of k such that if G is f of k connected, then G is k-linked; this is what we want to show. How much should be this function - is the main question. So, we will later show that f of k equal to 2 k, I mean you can bring down to 2 k, but how we want to prove that because the proof is more complicated. We will just show that some big function of k would be enough. So, we will be showing something exponential in k, but it will give you the give a feel of what kind of proof it is, but we need some preparation for even this.

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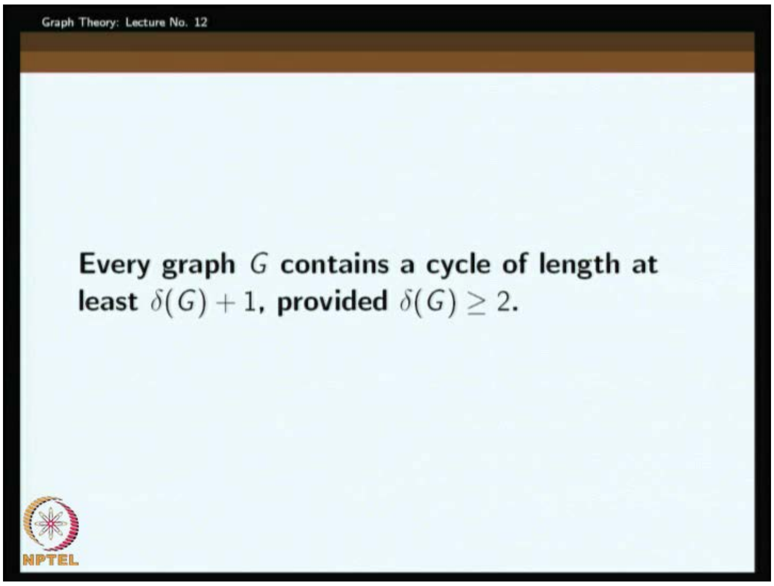

So, the first thing we need is this. Every graph G contains a cycle of length at least delta of G plus 1 provided delta of G is greater than equal to 2. What is the delta of G? Delta of G means some minimum degree. If the minimum degree of a graph is delta, then we do have a cycle of with the number of vertex cycle with the number of vertices in it as delta plus 1.

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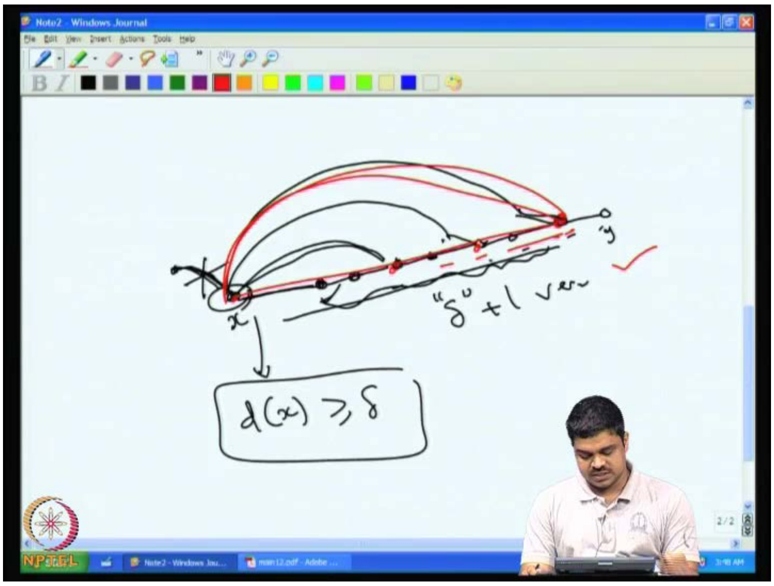

So, how do we prove this thing? This is an old theorem from Dirac. So, the proof is simple. See what we do is, we consider the longest path - longest path in the graph G.

What do you mean by longest path? You take a path, and then, if it is a longest path starting from this vertex and this vertex and if you look from this vertex, you cannot get any vertex outside it which is a neighbour of it. So, that means, if there is a neighbour like this, which is not in this set of vertices, not among this path, in this path, then we will get a longer path; so, this kind of an edge will not be there. What does it mean? But the minimum degree is delta, that is, d of x is greater than equal to delta. So, what does it mean?

So, it means that all the neighbours of x have to be here. Is not it? Somewhere, here like this. So, this is also neighbor, of course, this is. So, it is possible that there are some vertices in this thing, which are not neighbours of it, but all the neighbours of x has to be here. So, this path should contain at least delta vertices and this x included, delta plus 1 vertices, of course. What doing get the cycle now? We can always consider the neighbour which is farthest in this path and that will give a cycle. For instance, if this is the vertex, which is the farthest in the path here; one, two, three you can delta of them here including this delta plus 1. If I go like this and come back I will get a cycle. This is simple idea; very simple idea.

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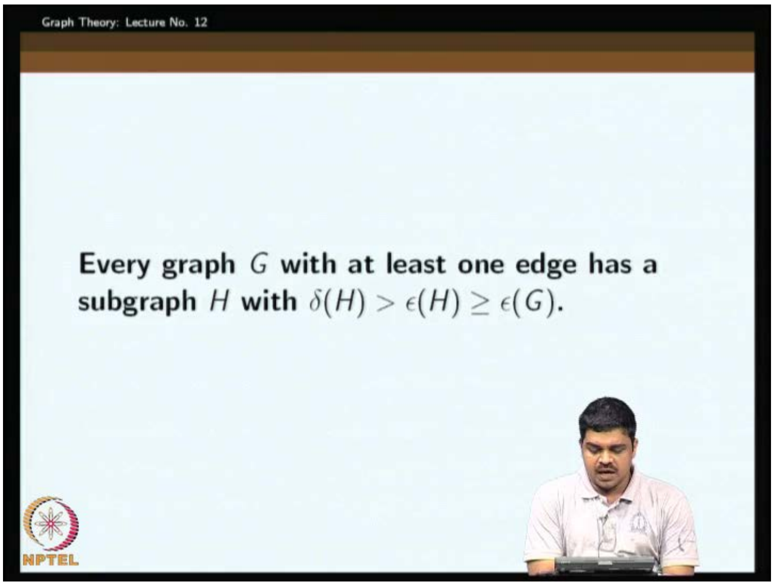

So, to repeat we are saying that if the minimum degree of the graph is delta, then we should get a cycle of length delta in this graph. The next statement, we want is this. Every graph G with at least one edge has a sub graph H with minimum degree of H greater than epsilon of H greater than equal to epsilon of G. We are using something called epsilon. What is this epsilon?

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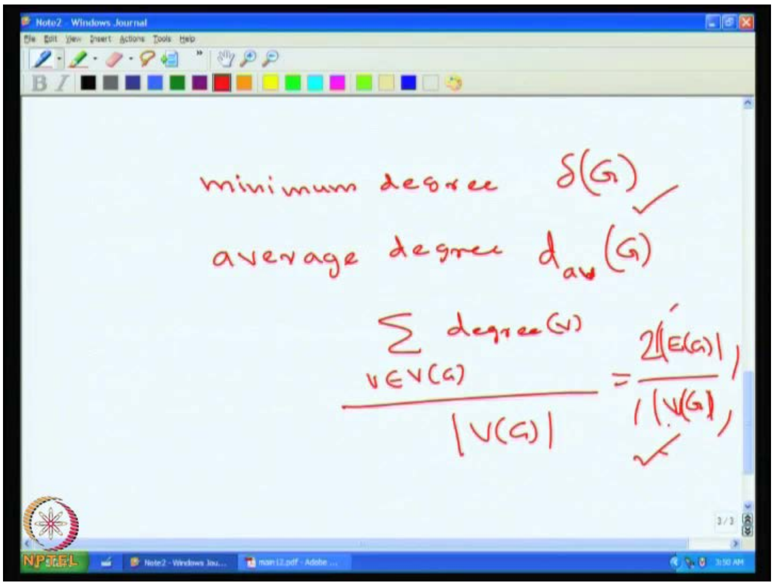

As we have seen, there are these two, three parameters. One is minimum degree, minimum degree delta of G; among all the vertices consider the degree of all the vertices; the smallest is the minimum degree. Then there is something called average degree, sometimes I write as d average or sometimes d of G. Average degree of G is essentially the sum of degrees divided by the number of vertices. So, the sum of degrees of V, V element of V of G divided by cardinality of V of G. This is essentially 2 times E of G, the number of edges in graph divided by V of G. Why two times? Because the sum of degrees has doubled the number of edges, two times the number of edges.

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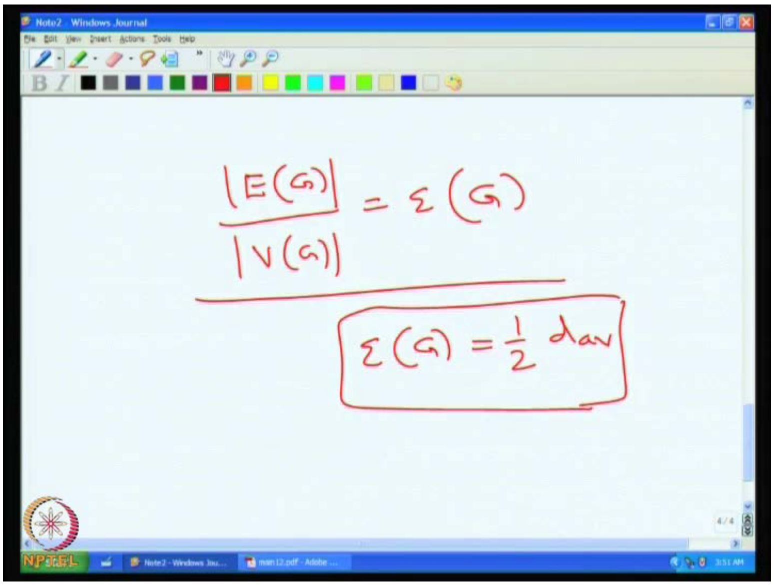

So, here is another one. Half of it suppose, if I take only this much. That means the average number of edges in the graph. No, not like that. It is essentially the edge, the number of edges to number of vertices ratio; this is called epsilon of G. How many edges are there in the graph? Of course, epsilon of G is essentially half of d average. Is not it?

So, now you know, if the average, suppose, your minimum degree is delta sorry if the number of edges in G is. So, the epsilon is given, suppose. That means number of edges by the number of vertices is given. Now, we can always find a sub graph of G such that its minimum degree itself is strictly greater than the epsilon of this.

Why is it possible? So, what we are going to do is to kind of concentrate the graph by. Of course, just because the number of edges to number of vertices ratio is high, it does not mean that all the vertices of high degree. Minimum degree need not be high. So, it is possible that several vertices are very high degree, but some vertices are low degree, very low degree. Therefore, while the average degree is high, the minimum degree can be quite small or while epsilon is very high - the number of edges to number of vertices ratio is high, when the minimum degree can be small. It is quite possible, but we can indeed throw away these bad vertices; bad vertices means the vertices which have very low degree and try to make all the vertices of degree at least epsilon; that is what we are going to do.

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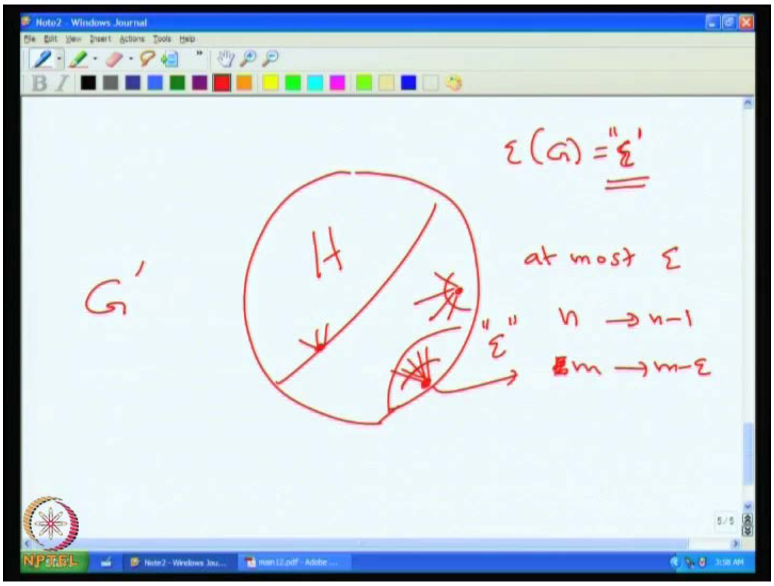

So, how will you do that? The method is like this. Here is a graph and its epsilon of G is equal to epsilon. Now, we pick up a vertex, whose number of incident edges in it is at most epsilon. Now, what can you do; you remove that vertex. So, when you remove that vertex, how many edges are removed? At most epsilon. So, one vertex is gone; that means if initially n vertices are there, it has to become n minus 1 vertices. Now, the number of edges if it was m initially, it has become m minus epsilon in the worst case. It may be a little more sometimes.

So, the reduction of one vertex, the reduction of at most epsilon edges; that means, the total epsilon of the resulting graph, say if G dash is the resulting graph, it is not reduced; it is greater than or equal to epsilon only, but on the other hand, I have thrown away a vertex of low degree; that means whose degree was at most, epsilon or less, I have thrown away. I can pick up another vertex now, in the remaining graph, if it has degree of epsilon or less, I can again do the same trick; I can through it away; I can remove it from the graph. You through away one vertex, the number of edges thrown away is at most epsilon.

So, for one vertex, you are only giving up at most epsilon edges. Originally, the edge to vertex ratio was epsilon. We are not going to reduce the ratio. Now, by doing this thing, we are kind of concentrating the graph or in other words, we are throwing away bad vertices, the vertices of low degree.

In the end, when do we stop this process? We stop this process, when we do not get any more vertices of degree epsilon or less. Of course, this will happen before the graph completely gets removed because after sometime, the number of vertices itself will go below epsilon. So, how can it even have epsilon neighbours. Total number of vertices epsilon means any vertex if you take, the degree is at most epsilon minus 1. So, it is not possible to through away the entire vertices. Every time you are throwing away one vertex at a time. Therefore, you will stop before the graph completely disappears at some point.

At that point, let us call the graph as H. Now, you know that in this graph H, every vertex has degree at least epsilon. Sorry, I just made a mistake. What I am telling is why will not it become empty because every time we are throwing away vertex, we make sure that our epsilon is same; that means epsilon is same or higher; it does not reduce. Of course, when you reach empty vertex, no edge; so, one vertex, that means no edge, one vertex. So, epsilon will go to 0.

Essentially, it has to reduce, if it becomes empty because one by one, when the vertices are removing, at some point epsilon will have to reduce. So, there will be a point when the epsilon will definitely reduce, if it becomes empty. At some point it has to stop, but then at the point we stop, it is very clear that we are stopping because every vertex has already got degree greater than equal to epsilon because if they was a vertex of degree less than equal to epsilon, we could have thrown it away without decreasing the epsilon. Now, we know that there is a point, where epsilon has to decrease.

So, therefore, we stop there; that is H and then this H has each degree greater than epsilon, not greater than equal to, strictly greater than epsilon; that is what we are saying. Their minimum degree will become greater than epsilon, strictly greater than epsilon. The epsilon of the sub graph will be at least as much as the epsilon of the original sub graph.

So, now we can prove the lemma we want. So, you remember the intension is to prove that when the connectivity is large, when the connectivity is large you know the minimum degree has to be large, minimum degree has to be large. If the minimum degree is large, the average degree is also large. So, if the connectivity is k, minimum degree has to be k because we have learnt that. kappa is less than equal to kappa dash is less than equal to minimum degree.

Now, minimum degree is also lower bound for average degree. so the average degrees So, the connectivity is high means average degree is high, then with high average degree, we can a get special structure in the graph which is called a topological minor, which is reasonably be. What do you I mean by topologically minor, topological minor?

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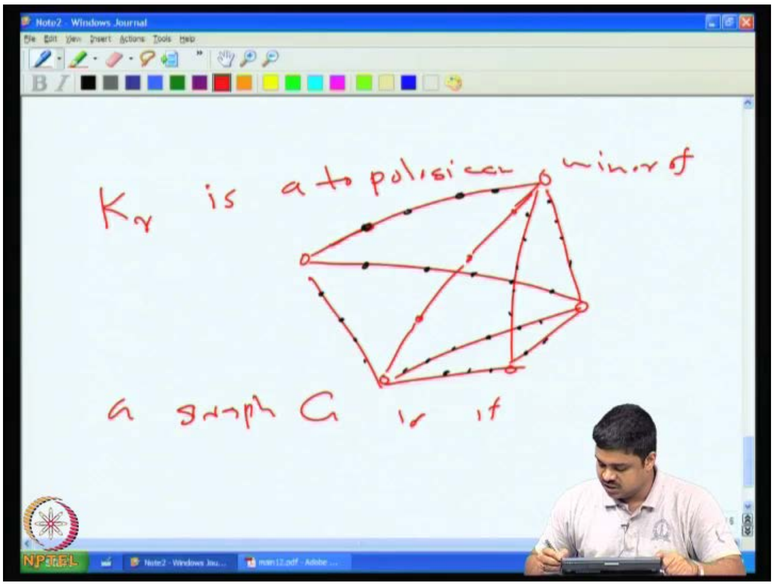

So, topological minor is some structure of this sort. For example, it is a clique, but the edges are kind of elongated. For instance, in a clique, we may see these kinds of edges, all the connections here, but in a topological minor, we may instead of edges, you may see paths. Essentially, this is called a sub divided clique, sub divided clique; that means all the edges of the clique are replaced by or it is kept as edges or it is replaced by paths.

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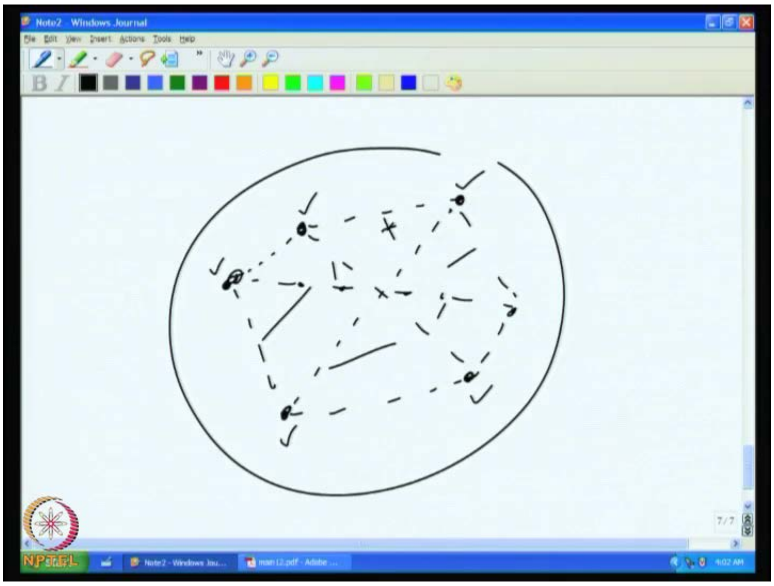

So, these vertices will not be shared by these internal vertices will not be shared by two different paths. We will say that K r is a topological minor of a graph minor of a graph G, if we can find this sub divided K r in G. What do you mean by we can find this sub divided K r in G? In other words, we should be able to locate, this is G, we should be able to locate these r vertices of K r in the graph G. There can be several other vertices, but we should be able to locate these r vertices and we should be able to trace some paths between every pair of these are r choose to paths.

So, every part of these r vertices that I should be able to trace a path such that no two of those paths intersect in any internal vertex, internally vertex disjoint path, we should get. so this is an Essentially, we should be able to identify K r - sub divided K r embedded in this graph. These vertices will be called branch vertices because these are the vertices, there can be other edges here like this, but we can as well, I mean we can think that they are all deleted and then I can just concentrate on this things. Somehow this structure is available in the graph.

So, then we say that this topological minor is available in the graph. So, that is what. If we can identify r branch vertices such that between any pair of branch vertices, we can find a path and no two of these paths share an internal vertex that means a sub divided K r can be found in graph G, then we say that K r is a topological minor of G. So, in our proof what we need is a topological minor of some large K r, for large clique, for some K r for large r; that is what we are looking for. Why do I look for this thing because you remember, K linkedness means given any collection of K vertices and collection of K source vertices and another collection of K sink vertices, we should be able to collect between any pair of them like that.

So the enemy may give us the pair as s 1, t 1, s 2, t 2, s 3, t 3 like that. Now, this connection should be possible. So, good thing about topological minor is a kind of structure available in the graph with lots of connectivity between among themselves. So, if you get into this topological minor, you can try to connect between each other; this is the point.

So, that we will come later. before so that is I just motivated that this kind of structure, finding this kind of structure will help to solve our problem. The next statement will tell us that if our average degree of the graph is sufficiently large, we do get a K r top, whichever r you want, they we can find a function h of r such that if the average degree is greater than h of r, then we can get the required K r minor. So, this is what we are going to do next.

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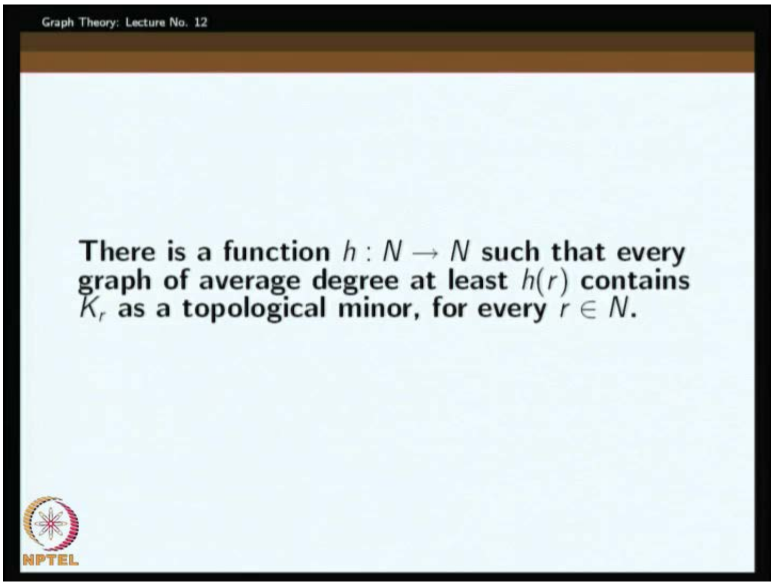

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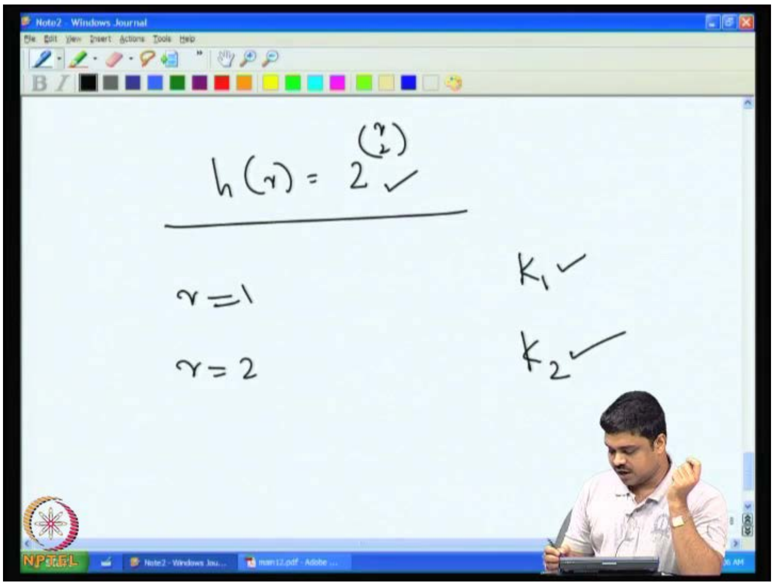

So, there is a function h such that every graph of average degree at least h of r contains K r as a topological minor, for every r element of n. So, we are going to take this function as h of r equal to 2 raise to r choose to something like that. We will do the induction, first on the number of vertices, then on the number of edges. Put r equal to 1. So, r equal to 1; that means, we are looking for a k 1, which is always available. Put r equal to 2 – so, k 2. We are looking for a k 2 as long as our minimum degree sorry average degree is not 0. So, there is one edge in the graph, then we do have a k 2. Therefore, these induction assumptions are correct. We do not have to put this much; anyway it is true.

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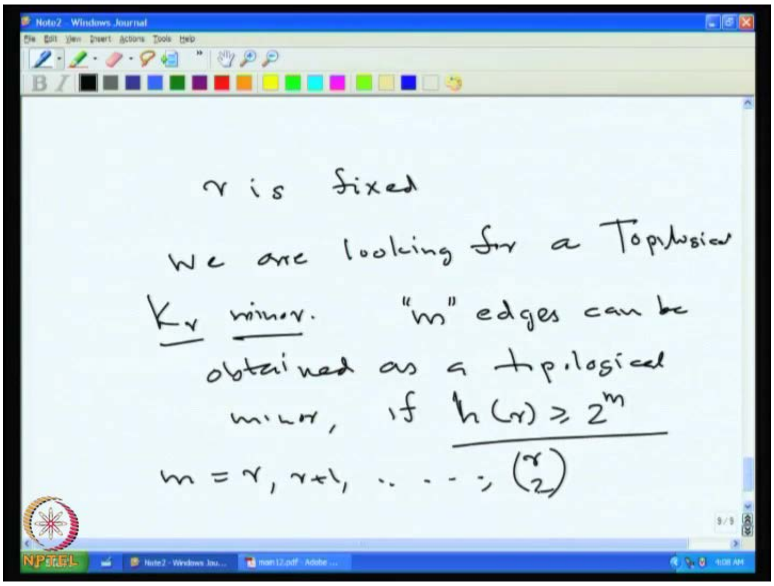

Now, let us say that for a r - given r, we will start the r is fixed now. We are looking for a topological K r minor, but instead of directly looking for K r minor, we rather would look for a minor of a graph, which is a sub graph of K r. Initially, we will say that any graph with m edges can be obtained as a topological minor, if say, if h of r is, that means that the average degree is taken to be greater than equal to 2 raise to m. This is what it means. Therefore, m equal to r, r plus 1, you will do the induction like that till r choose 2.

See, when m equal to r choose 2, when m becomes r choose 2, we can see that; that means when h of r is greater than equal to 2 raise to r choose 2. So, we will get the graph, we can get the graph with r choose 2 edges in it on r vertices of course; that is the K r minor. Of course, K r is a topological minor. So, we just have to go through this induction. We will start with a graph with r edges in it.

So, we will have a graph. We will we just have to Not any graph is not required. We just need one graph because when m equal to r choose two, you see there is only one graph with r choose 2 edges on r vertices; it is K r. So, that will automatically come.

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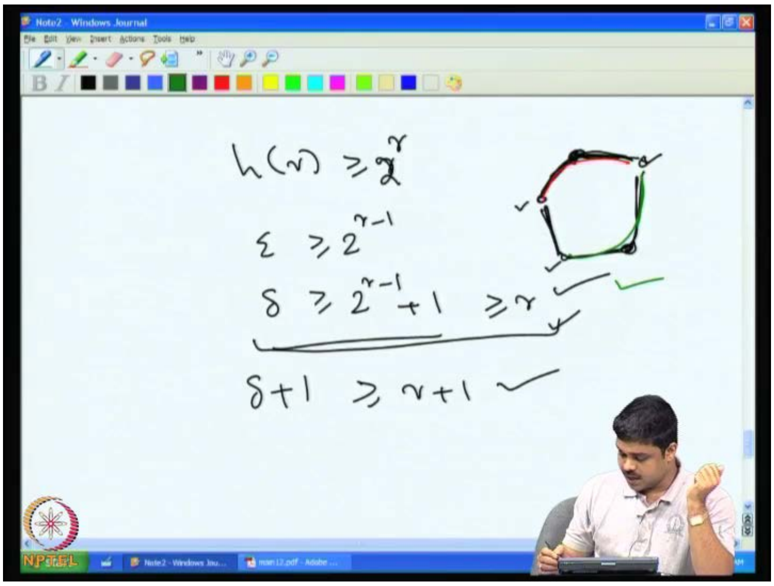

So, our idea is to start with r and we will show that there exists one graph with r edges and r vertices as a minor, if I take the average degree of the graph to be greater than equal to 2 raise to r. That is true because the average degrees, if h of r, the average degree is greater than equal to 2 raise to r. So, of course our epsilon is greater than equal to 2 raise to r minus 1 then and what about our delta? minimum degree See epsilon is greater than equal to 2 raise to r minus 1. Then you can always find out a sub graph of that, that the minimum degree is at least epsilon plus 1. So, 2 raise to r minus 1 or greater than epsilon. I mean 2 raise to r minus 1 being an integer, we can say that this is and the rest of the formula as shown in the slide. This is of course greater than equal to r.

Put r equal to 2 and onwards, 2 minus 1, 2 raise to 1 plus 1 is greater than equal to 2. So, we will get essentially minimum degree is greater than equal to r. We can assume that minimum degree is. So, we will get a cycle with number of vertices delta plus 1 equal to r plus 1. So, r plus 1 cycle has are r plus 1 edges in fact, if you delete one from that. So, it is an easy thing to see that we can start the induction with m equal to r. We can find one r vertices a graph with r plus 1 edges. So, r edges, delta is greater than, yes, we can find.

So, this is essentially r edges because the cycle itself is r plus 1. We can always Cycle is a topological minor of any smaller cycle. So, whatever number of vertices we want, we can select. For instance, this is a four cycle. So, we can consider as the topological minor of the three cycles because this edge can be considered as sub divided; this edge can be considered sub divided and these edges as such.

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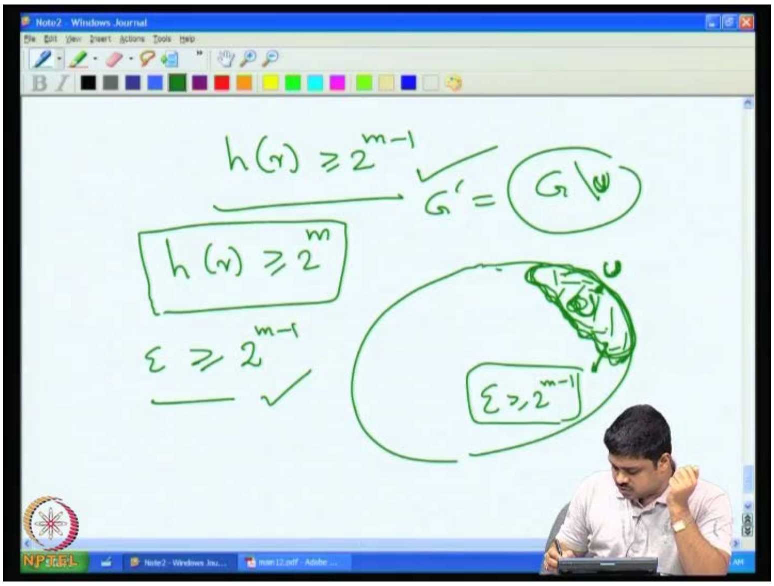

So, we just say that topological minor of C r is available. That is m equal to R v. We start of the induction easily. Now, the next thing is to consider suppose for m minus 1, when h of r is greater than or equal to 2 raise to m minus 1 or less, we know that this statement is true. That means in the graph, we do have a topological minor of some graph one r vertices with m minus 1 edges in it. Now, we will assume that h of r is greater than or equal to 2 raise to m. We want to show the existence of a topological minor with m edges in it.

So, to do this thing what we are going to do is to find a This is the graph G. So, we will find a connected sub set U here and then this a connected sub set U and you know, this h of r is greater than equal to so what is epsilon? epsilon is greater than or equal to because this is average degree so we have 2 raise to m minus 1 as the epsilon.

Now, we will find out some connected sub graph U of G and contract this thing. What we get is G bar U. So, when you contract this, the resulting graph is G bar U. So, this is G dash and it should be such that its epsilon is at least 2 raise to m minus 1 and moreover, we will try to get U such that it is maximal. That means, we cannot add any more vertex to U; that means, the biggest possible U that we have selected, it is a connected set. Two properties: one is, it is connected. When I contract it, the resulting graph has epsilon that means number edges to number of vertices ratio is greater than equal to 2 raise to m minus 1.

So, if we cannot contract anymore, its meaning is that the epsilon might be reducing by that. when you call When you try to assimilate one more vertex into U and contract, the total epsilon may be reducing it; meaning several edges may be being destroyed or removed by that contraction process. So, that is why it is happening.

But we should ask whether there is any such U at all in the graph. That is there because you can take U as a singleton vertex and then if you do this, contract G bar U is G itself. Then initially, G has epsilon greater than to 2 raise to m minus 1. There exists some U definitely, namely any singleton vertex will do.

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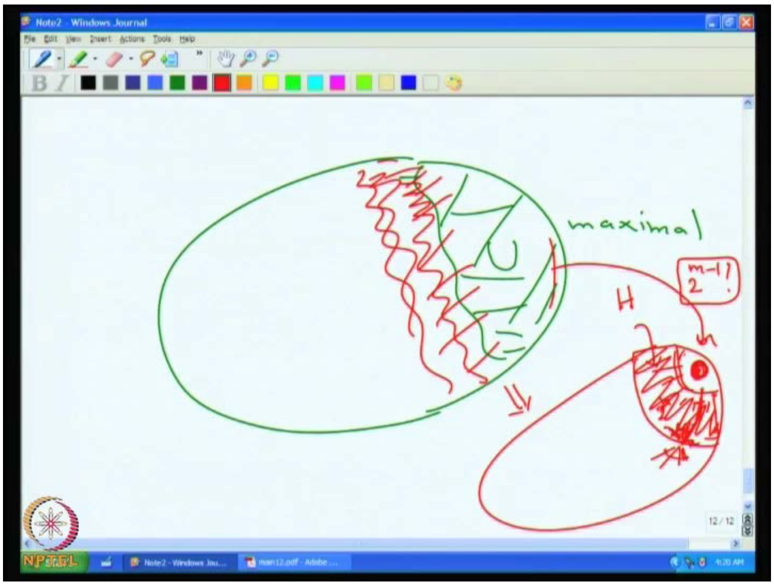

Now, you do this thing. So, this is the picture; this is G and this is our U. I am just taking this; U is connected set. Let us see and this is maximal with respect to the property that when you contract anymore vertex, the epsilon will reduce below 2 raise to m minus 1. Now, you see this has not become one single vertex because epsilon will become 0 and also the original graph is connected. Why do I assume that the original graph is connected? Because of course, original graph It is because I am only looking for a topological minor. I can take the connected component with the biggest epsilon and of course, the biggest epsilon has to be at least as much as the epsilon of the entire graph which is 2 raise to m minus 1.

So, it is unimportant to worry about whether it has several components or not. So, it is connected. Now, let us say here, is the neighbourhood of this. So, that means these vertices are the direct neighbours of u. Now, this After contracting, this has become one vertex. So, it will look like So, here we have this U, this is U, entire U and here, is its neighbourhood. So, this is the neighbourhood - direct neighbourhood. Let us call it as H. This induces sub graph on this, let us call as H; let us call, all these induced sub graph as H.

Now, what we are interested in is to analyze this graph H. In particular, we are interested to know what is the average degree of this induced sub graph H? Is it possible that the average degree of this graph or let us say minimum degree of this graph is 2 raise to m minus 1? Is it possible that minimum degree of this graph is 2 raise to m minus 1? Fine, it is possible. Why? Suppose, it is not so; there should be some vertex here, some x here such that its degree to inside is less than 2 raise to m minus 1. So, to this into this H and so, the rest is all going into this.

Now, of course, this is also connected to this. That is why it is connected. So, in that case the question is what would have happened, if we had taken this x also into U and contracted it along with U. Anyway, it is connected to U, you could have contracted. So, why did not you contract because we decided to select U as a maximal such thing, making sure that the number of edges to number of vertex ratio, for the resulting graph is still 2 raise to m minus 1. So, that is the only reason why we did not do that because if you do like that the ratio of the number of edges to vertices will go below 2 raise to m minus 1.

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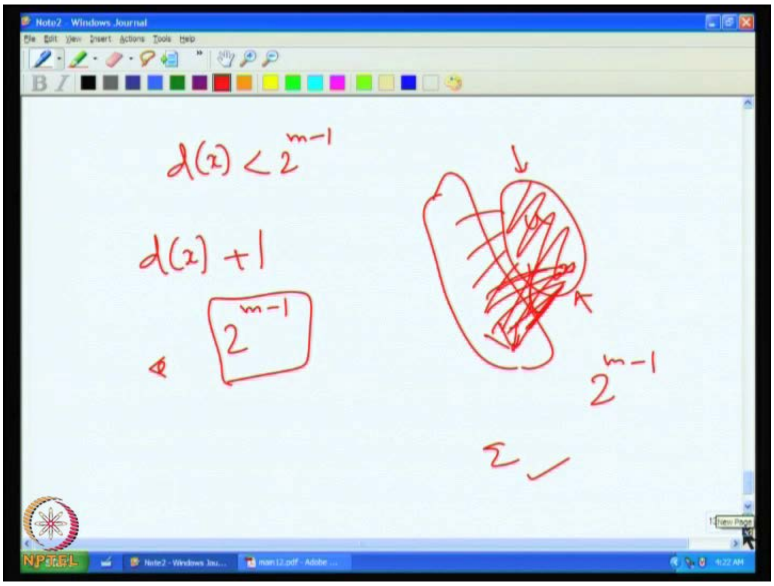

But how many When I am taking this vertex into this, one vertex is reducing of course, from this resulting graph, but how many edges are reducing. So, the resulting graph, all these outgoing edges will remain. See the problem is it is possible that when I take this vertex inside, this one edge will disappear; also, all the edges which are into this H may become multiple edges. So, I will draw this like this. For instance, this u is connected to all the vertices here already in H.

Now, this x was also x is pulled into this, but all the vertices of x, these edges will, because these contract into one vertex, will become one multiple edge. So, all these, they will not count anymore. So, we will lose all those edges, but we are losing only less than because number of edges were only equal to the degree of x here, d of x plus 1 edge into u. This much only, we will lose. If d of x was strictly less than 2 raise to m minus 1, then we will only lose 2 raise to m minus 1, but our intention, originally 2 raise to m minus 1 was the epsilon of G bar u. So, if you move x into that, we are only loose we loosing 2 raise to m minus 1 edges, but one vertex is also reduced. So, epsilon will remain same.

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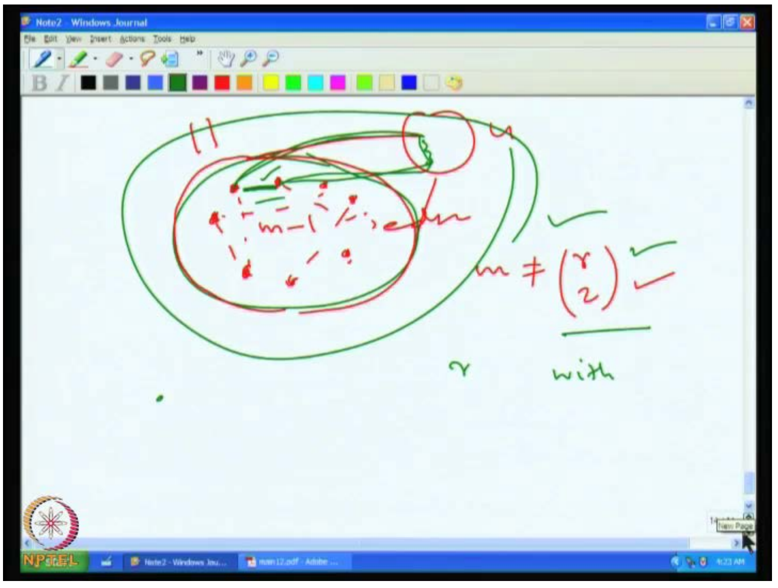

So, that will be a contradiction that u was the biggest possible V 2. You could have taken x also into that. So, that is the reason why x also can be assimilated into u. So, it follows that every vertex in H So, this is what we wrote here in H, neighborhood of u of here has to have a degree of at least 2 raise to m minus 1 in the induced sub graph, which means that the average degree of that induced sub graph is 2 raise to m minus 1. So, by induction assumption, we do have a sub graph, a topological minor on some r vertex graph with m minus 1 edges in it.

This is our u. This is entire neighborhood in H. We should be able to find out some r vertices here, branch vertices and such that some graph with m minus 1 edges is available here. Now, of course, m is still not equal to r choose two. If m is equal r choose two we are already done. So, m is not equal to r choose two, what we can do is, find out some missing edge here; some edge which is not present. Then of course, this is connected to u, this is also connected to u and there is a path here. So, this path has nothing to do with the paths inside here. Therefore, we can go here and follow this path and come back.

So, we got one more edge. That means a graph on r vertices with m edges, we got topological minor here. So, that way, the induction progresses. In the next stage, it becomes m plus 1 and slowly, we will reach r choose two and when we reach r choose two, we got a graph on r vertices with r choose two edges in it, which is the topological minor of K r. So, that is the way, we proved.

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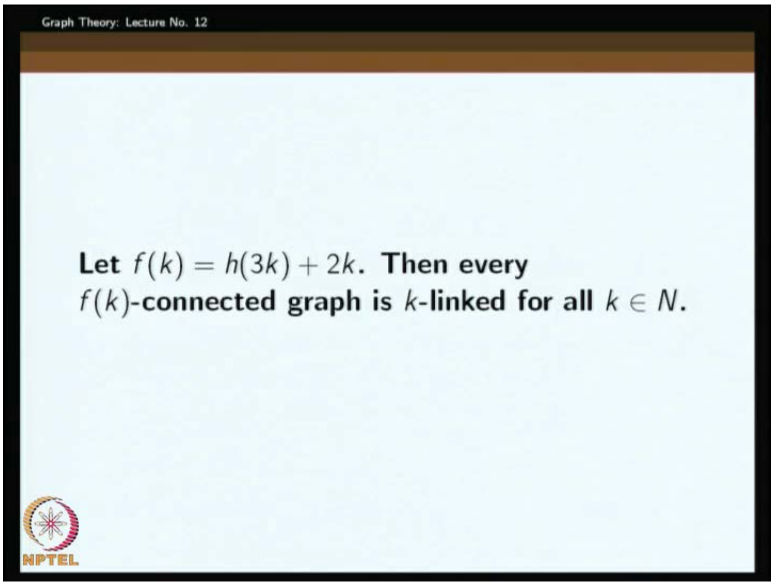

So, this proof tells us that if the average degree is greater than 2 raise to r choose two, we can just call it as h of r, then there exists a topological minor of K r in the graph. Now, we can look at the final statement. Now, we take another function f of k. So, this is what I have written. We can take this function f of k is h of 3k plus 2k. See this is to make sure two things. See the function is greater than h of 3k means, we do have a topological minor of 3k available because this connectivity would imply that same average degree at least.

So, h of 3k is the average degree means by our previously proven statement, we know that there exists a topological minor of 3 k vertices in the graph plus 2 k. If suppose, this was small, into k would assume that it is at least 2 k connected. We were making sure that the connectivity of the graph is not below 2 k, in case h of 3 k was a small number.

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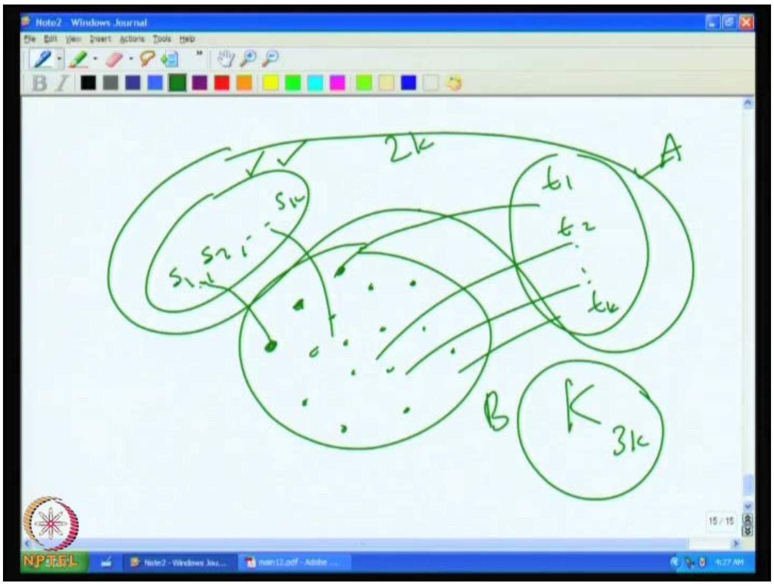

Here, it is not small number, but maybe, it is possible that somebody may come up with smaller value of h of 3k. So, suppose you put f of k equal to h of 3k plus 2k, then every f of k connected graph is k-linked for k element of n. This is what you want to show. So, how do we show this thing? So, we use this topological minor. We are going to show that k-linkedness now. So, we use this topological minor. This is the topological minor available. This 3k topological minor r equal to 3k; 3k means 3 k vertices.

Now, we only want to show k element. Let us say we have s 1, s 2 up to s k somewhere and similarly, t 1 t 2,….. t k somewhere. So, the point is that it is k element set and this is a k element set and here is a 3k element set. Now, this is together s's and t's together is 2 k element set and the graph is 2 k connected. By Menger's theorem, there are disjoint paths from this s 1, s 2,…. s k to t 1, t 2,… t k to this 3k element set. If I take this as B and this together as A, I do get 2k disjoint paths, starting from s1 and reaching somewhere here and t 1 reaching somewhere here like this.

So, of course, because it is AB path, it will start on each of this s 1, s 2,…. s k, t 1, t 2,… t k and reach here. an exactly There will not be any other vertex. For each path, there will be just exactly one vertex from these 3k vertices. So, these paths will use up 2k vertices of these 3k branch vertices; 3k branch vertices of this K 3k topological minor available.

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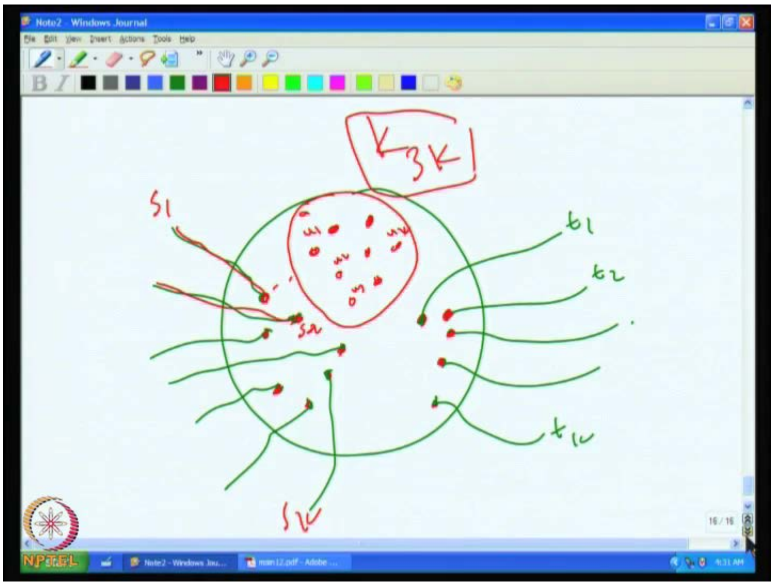

So, we still have k more branch vertices left. These k branch vertices, we are going to use for our this thing. So, this is our topological minor. These are the branch vertices say, to which s k got s k paths came and paths from s 1, s 2,… s k came and ended. So, these are the branch vertices say, the paths from t 1 to t k came and ended. See, these paths need not be long paths; it can be a singleton path also. So, these are the ones.

When we select these paths, There are also these k vertices left, which are not used by any of these paths, but when we select this path system, starting from s 1 to s k and t 1 to t k and reaching this thing, we had various possibilities. There were several possible collection of paths available. Among all the collection of paths available, we will pick up the collection of paths such that the number of edges from outside this, K 3k topological minor, this setup - this is some kind of a connected box, branch vertices and the topological these paths are there.

So, we would like to minimize the number of outside edges. Of course, we will have to use some things, may be somehow we have to reach here, but once you reach inside this system, we would you like to use as much edges from this system, rather from outside. So, among all the possible collection of paths that connected this s 1, s 2,..s k and t 1,…t k to this box - this a topological minor, we would like to take the one that minimizes the number of edges used from outside this topological minor.

If that is what we want, then we will get this picture, in fact. So, these are the end vertices of these paths coming from s’s; these are the end vertices of the paths that come from t’s and these are the vertices which are outside; these are the vertices which are outside.

Now, what am I going to do? See, I will show you how to connect s 1 to t 1, s 2 to t 2; s 1 - this path will come and here, I will show a way to reach t 1 here, without using any other paths and similarly, I will show a way to reach from this, this is the path which reaches from s 2 and reaches inside here; from here to here and then reach to t 2 and so on.

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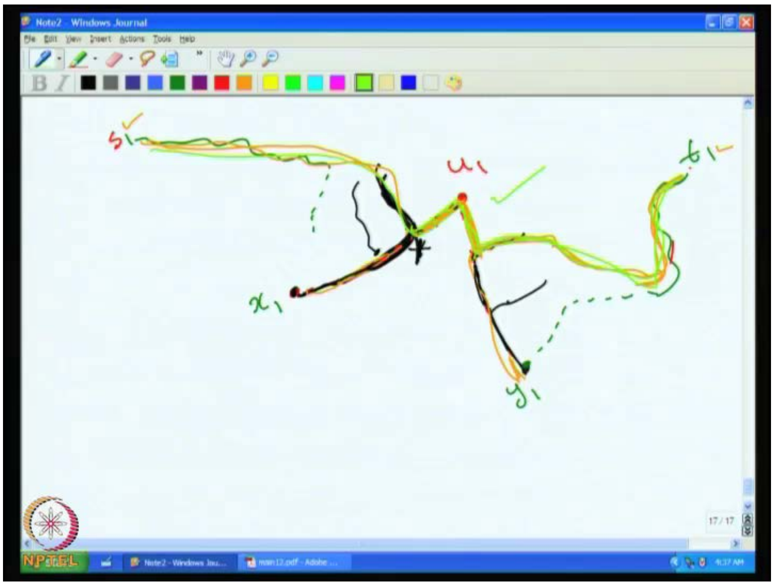

So, for that, we use this one. We can also name this as u 1, u 2, u 3,….u k. For one example, we can take u 1; this u 1 is one vertex, which is not used by the end vertices, the t paths or s paths and then we look for the edge, the end point of this path coming from s 1. It reached here; somehow, it reached here. I do not know. so it maybe how we reached on it This is the path; how it might have reached, we have no idea. It is possible that it might have travelled a long way; somehow, it reached here I know.

Similarly, I will consider this. I can call a name by say, x 1; this is x 1 to s 1. Similarly, here, I may be able to find out the vertex y 1 to which the t 1 path entered. It may have started and then somehow, entered here. I do not know how it entered, reached here. Suppose, these are the two vertices s 1 reached and now, we consider these two paths in the topological, inside the topological say, u 1 s 1 path in the topological minor; u 1 y 1 path in the topological minor; these paths are available, but it does not mean that no other path has contaminated this path. See, if it was just like that, I could have followed this and then this path and this path is disjoint. So, I take this; I take this; take this; I would have got a s 1 to x1 to u 1 to y 1 to t 1 path, but that is not necessarily pure. It is possible that when I try to use it, I may see some path using a vertex here, some other path using vertex here. I know some path may be using some portion of here and so on.

So, what do I do? I may want to examine this. I start from here and scan this portion this portion, until I see the first vertex one, which some path hit. This path need not be the path coming from s 1. Suppose, something hit here, now, the question is. So, this path, whichever path it is, it is coming from some s i or t i. Why did not it come directly to u 1? Because this entire portion was empty, I would have followed this thing. So, why should it come? You may ask, it could have gone somewhere else also.

But if it ever went out of this thing, then it will use an edge to go out. For instance, it might travel here and get out or it may get out from here itself, but whenever it gets out, it will use an outside edge, which does not belong to topological minor.

On the other hand, if it had travelled to here, then it would not have used any more outside edge. You would have reduced the number of outside edge use by that. So, it means that it never gets out, once you touched it. Once this path touched it, because it did not follow this path, it means that it has followed this path; it never went out, but rather it went to this side, but then if it went to this side, it means that it should totally reach all the way to x 1. If it reaches x 1, then it is the path which is coming from s 1 because once it uses all these things, how can s 1 reach x 1.

So, we end up concluding that the first path, when I am tracing from u 1 to x 1 moving from u 1 to x 1 and looking for a path hitting it. So, the first path which hits it, in fact is the path starting from s 1 and reaching x 1 because x 1 is reached by the path from s 1 only. So, that is it; this is the path.

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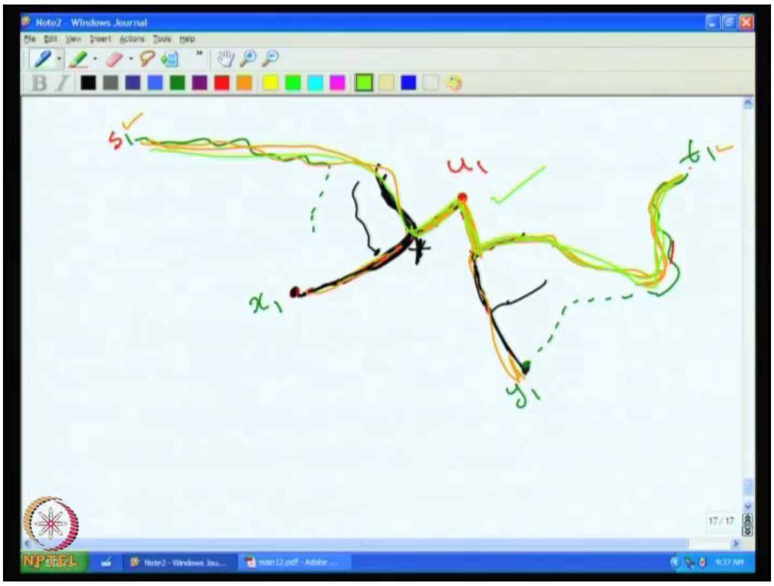

So, we can mark it with another colour maybe. So, this is a path coming from here; this is from s 1. By a similar argument, I can say that if I trace from u 1 to y 1, the first path I hit here, the first vertex I am seeing here as hitting, as an outside path hitting there, should be the path which goes all the way to y 1 because otherwise, it will get out of this thing and it will increase the number of edges used from outside. It could have come all the way to u 1 rather than why should it go out and hit somewhere else. If u 1 was empty or u 1 was not used, it could have come backward and used up u 1.

So, therefore, it should be going all the way to here and reaching y 1 and that is essentially t 1. So, we can mark that it is the path coming from t 1. This path is coming actually like this. Now, you know what to do. So, if you want reach from s 1 to t 1, you will follow this path, reach here and instead of going to this side, we would rather take this side and then come this side and then we will follow this path. This is what we will do. So, we can we can mark it with a different colour maybe. This is the way it will do.

Now, you know that this was the empty path. So, we are not using any we are not encroaching into anybody's territory. So, no other path is using these vertices. This was exclusively s 1 to x 1 path and t 1 to y 1 path and this was empty is not somebody, like that. For each s 1 t 1, s 2 t 2, you can use another u 2 to connect them together. For s 3 t 3, we can use the third vertex u 3 - the unused vertex; that is why we had 3k vertex in the topological minor, first because this k extra vertices, you wanted to use. So, we can manage to connect all of them together.

So, it follows that. So, given k source vertices, s 1 to s k and k sink vertices, t 1, t 2 to t k, then we can connect every pair s 1 t 1, s 2 t 2 with disjoint paths. The only condition, we wanted was the connectivity was to be at least f k, f k being h of 3k plus 2k; this plus 2k for connectivity, to link that the connectivity of the graph is 2k and h of 3k to tell that there exists a topological minor with 3k branch vertices in it, and other topological is subdivided 3 k clique is available in that graph and we know what is the value of h of 3k. So, the earlier theorem told us that if you put h of 3k is around 2 raise to 3k choose 2, then we will get one such thing. Big number, but we are only claiming that there exists a large value of connectivity such that once you have such kind of connectivity, we will get the k-linkedness.

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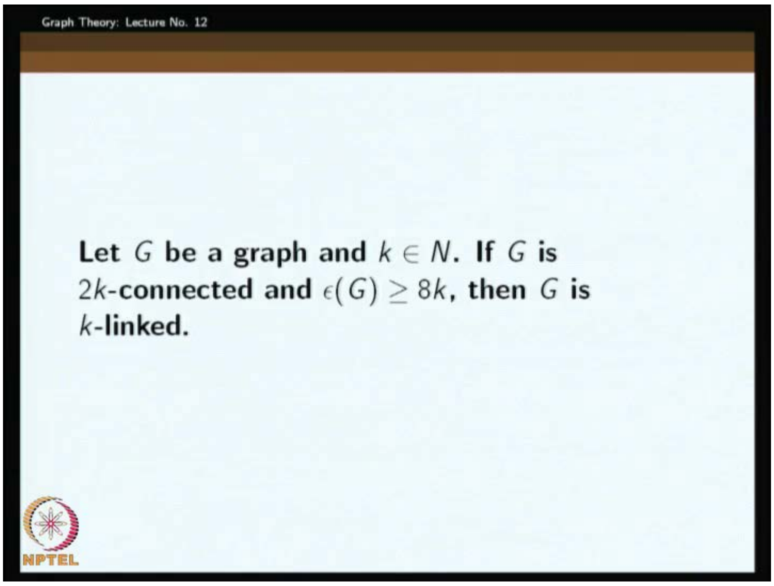

Finally, to conclude, we will mention that, we have already mentioned that this theorem is proved that G. It is much more that improved version is let G be a graph and k element of n. If G is a 2k-connected graph and average degree is greater than or equal to 8k, not average degree, the epsilon means average degree is 16k, then G is k-linked. This is by Walt Thomas. The earlier theorem was by Yung, Ju Yung and Larman and Manim; that was in some 70’s. This is a recent theorem, 2004 or 6, but then this proof is a little longer, but, he still has the proof in his text book. So, students can read it from there. So, that is all for today. In the next class, we will start with colouring.

Thank you.