Graph Theory

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Module No. # 03
Lecture No. # 15
Edge Coloring: Vizing's Theorem

Welcome to the fifteenth lecture of graph theory. In the last class, we were looking at the class of k-constructible graphs. So, to remind you what is a k-constructible graph, it is a graph which can be constructed using the following rules: the first rule is the complete graph of k vertices is a k-constructible graph. Now, if G is a k-constructible graph and x and y are two vertices which are non-adjacent in G, then you add that edge xy and contract, after that the resulting graph will also be k-constructible.

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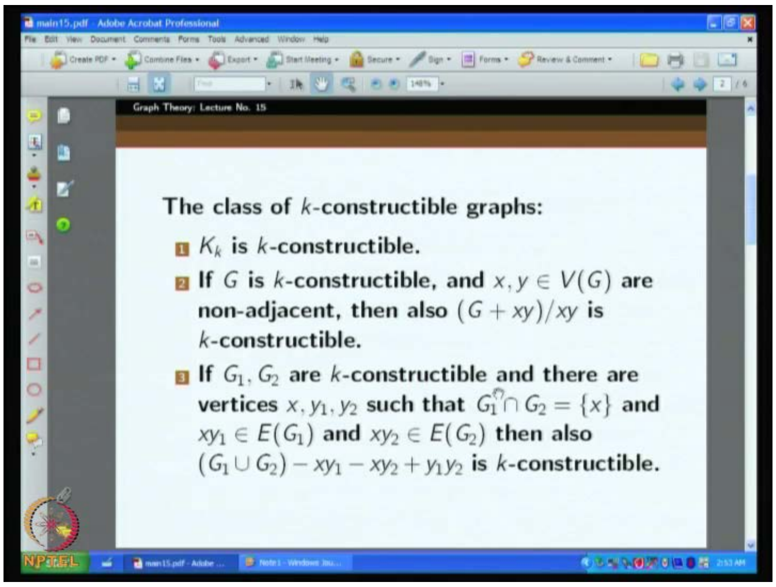
Graph Theory: Lecture No. 15

The class of k-constructible graphs:

1. K_k is k-constructible.
2. If G is k-constructible, and $x, y \in V(G)$ are non-adjacent, then also $(G + xy)/xy$ is k-constructible.
3. If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$ then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible.

The third rule is if G 1 and G 2 are two k-constructible graphs and there are vertices x, y 1 and y 2 such that the only common vertex of G 1 and G 2 is x; and x y one is an edge of G 1 and x y two is an edge of G 2, then the union of G 1 and G 2 minus x y one minus x y 2 plus y one y two is k-constructible, which means if we take these two things together and then delete those edges x y one and x y two, and then add the new edge y one y two, this will also be k-constructible.

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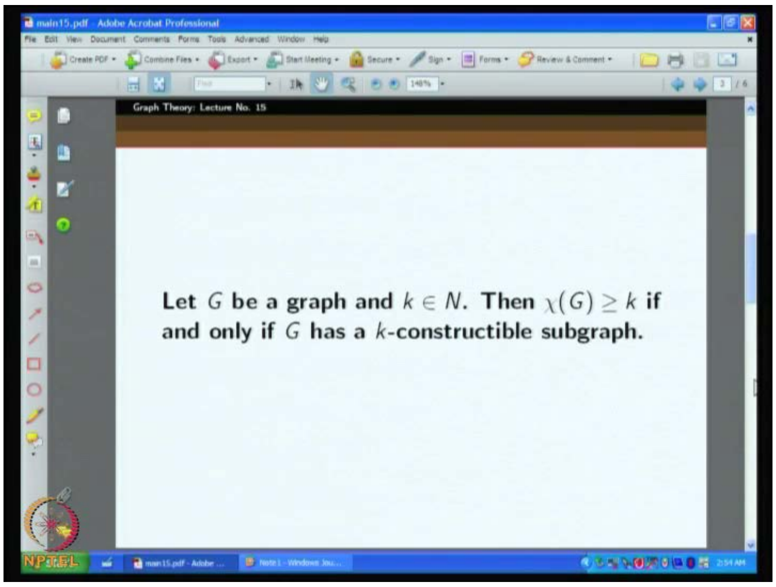
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Let G be a graph and $k \in N$. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.

All graphs which can be constructed in this way is k-constructible. This is what yesterday we talked, and also we argued that if a graph is k-constructible, then it is, it needs k colors to color. Its chromatic number is k, this is what we argued, what was the argument? So, for instance the first rule there is very clear, that means if it is a complete graph on k vertices, then it needs k colors to color, its chromatic number is k.

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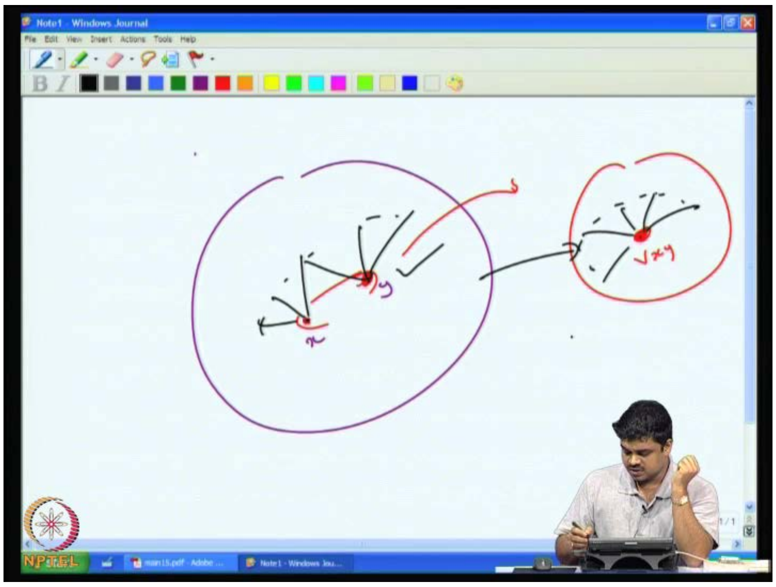

The second rule was also clear, because if the resulting graph after putting the edge and contracting it, x y edge x y were two new non-adjacent vertices and we added that edge, so this is the graph, so x y were two non-adjacent vertices, we added that edge like this and then contracted it.

So, if the resulting graph requires only less than k colors of case, it is easy to see that the original graph also requires the same number of colors, because the same color can be retained here . For instance, after contracting, suppose the graph is like this (Refer Slide Time: 02:52), so this is the V x y after contracting. Whatever color you give, say red color, then the same red color can be given to both the vertices. Why? Because you know, the original of these edges is not there. So whichever neighbors were there, they were here also.

So, the colors the neighbors get here will still be here. Now, this color can be given to both these vertices because they will be different from the colors on the neighbors. Therefore, if after putting this edge in contracting, the graph requires only less than k colors, then the original graph also can be colored with less than k colors. That will be a contradiction to the assumption that the original graph required more than or equal to k colors. So the assumption that the original graph is k-constructible.

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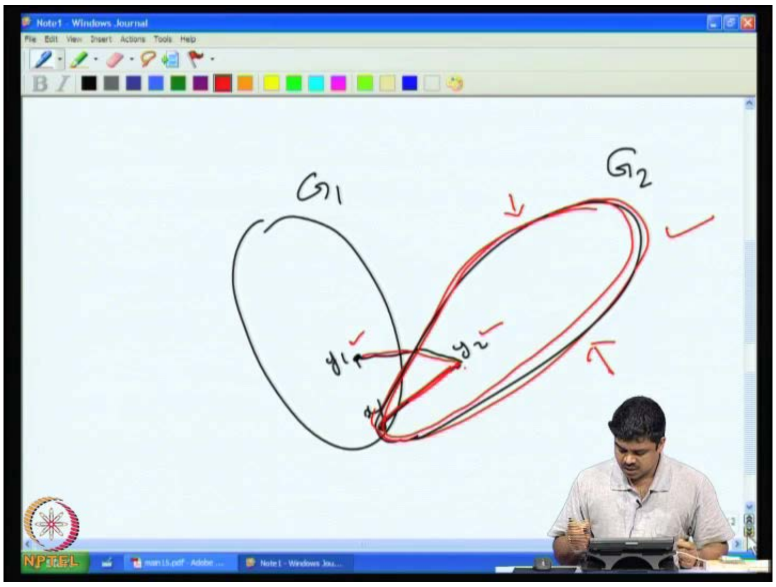

Now, the third rule also was easy, because you know that if the resulting graph we had this kind of structure, this was the common vertex x, so this was G one which is assumed to be k-constructible, and therefore, assumed to require k colors to vertex color. Now what we did, we identified these two edges and then we decided to delete these edges on the graph, and then add this new edge. This is what we did, add this new edge, so we can say this is the new edge - red colored.

In case, if after doing this thing, if you could color the graph using less than k colors, what can we say? We can say that in that case, we can definitely look at the color of y one and y two. It is because of this edge, at least one of them have a color which is different from that of x. Then, say suppose y two has different color from x, then naturally you can consider this part and then if you put back this edge, it will not violate any condition because this and this have different colors.

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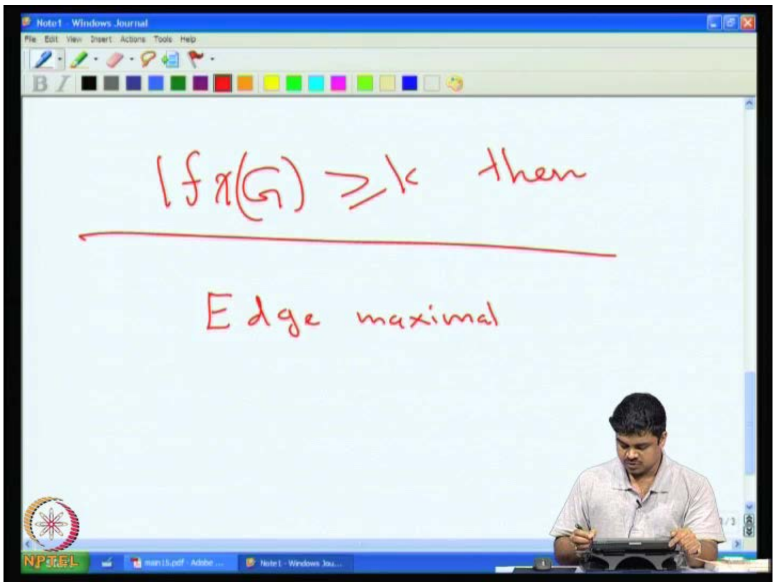
If $\chi(G) \geq k$ then

Edge maximal

So the same k came less than k colors will suffice for this G two also, that will be a contradiction because we assume that this is k-constructible and requires more than or equal to k colors. This was the argument we did. Now, the other part of the statement says that if G, the G such that the chromatic number of G is greater than or equal to k, then there exists sub graph in G, such that it is k-constructible, this is what.

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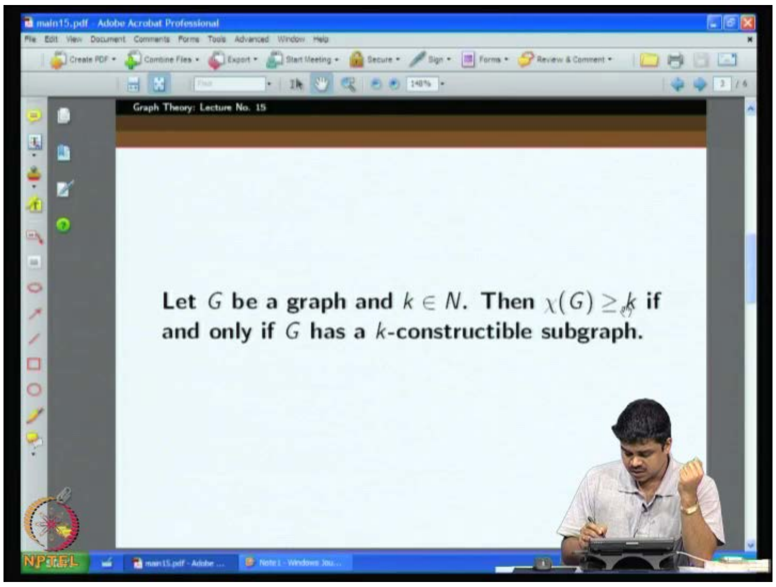
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Let G be a graph and $k \in N$. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.

So let G be a graph and k element of N. Then, if chromatic number of the graph is greater than or equal to k if and only if G has a k-constructible sub graph in it, how do you show this? To show this thing, we will prove by contradiction. Suppose there is such a graph, that means, this G is a k chromatic graph, that means it requires at least k colors to vertex color; now suppose none of its sub graphs is k-constructible, now what we can do is we can add edges to the graph. So when you add edges, the chromatic number will only go up. But, we will keep adding edges as long as there is no k-constructible sub graph in it, results in it. See for instance, if you add one more edge, it is possible that many new sub graphs will come and then one of the sub graphs may be k-constructible, but if such a k-constructible sub graph does not result by adding a new edge, you can add that edge.

So, in that way we make the graph edge maximal. We are familiar with this word edge maximal with this property that none of its sub graphs are k-constructible. We will come to a contradiction to this. This is not possible because see, you will say that if suppose there is a graph with chromatic number greater than or equal to k and with none of its sub graphs not k-constructible, then we can indeed make it edge maximal with that property, namely keep on adding edges until it retains the property as long as it retains the property that there is no k-constructible sub graph of it.

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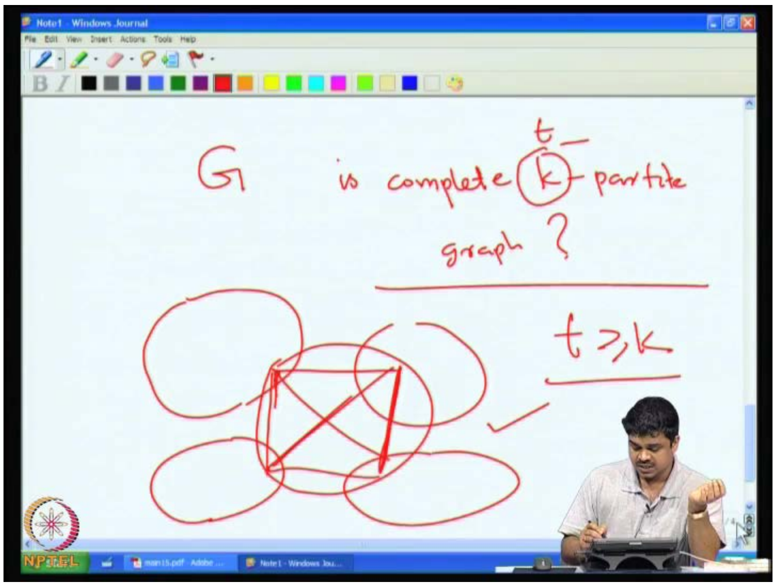

At some point of time, any new edge you add, you will end up with a k-constructible sub graph in it. In this stage, we will show a contradiction. What do we do? We first ask this question. Is it possible that this graph G, this edge graph, this maximal graph G is a complete k-partite graph? Is it a complete k-partite graph? Is it possible?

So we claim that it is not a complete k-partite graph, because if it is a complete k-partite graph, then which for some k, I mean k-partite graph in the sense of some k, but then you know it is a complete k-partite graph means that k has to be at least the chromatic number. So, it should be at least k. Some places you must say complete t-partite graph or something, definitely t greater than or equal to k, because otherwise you can color it with t colors if t is less than k. So the assumption that the G is chromatic number is k will be valid.

But, then if this is a complete k t-partite graph with t greater than or equal to k then definitely you can consider the one vertex from each of this part and then what will you get? You will get a complete graph on t vertices that means complete graph on at least k vertices in the graph and that is indeed k constructible k k right it is a complete graph on k vertices and indeed k constructible sub graph of G.

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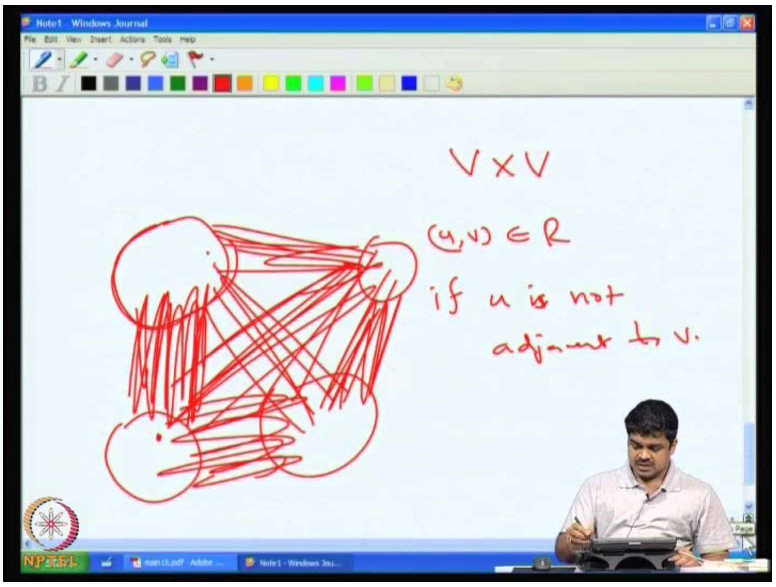

So we can assume that it is not a complete k partite graph. Then what can we tell? If it is not a complete k partite graph, then we can immediately say that we look at the relation induced by the non-adjacency of vertices. See the relation defined on V cross V. By this rule that u and v belong to the relation if u is not adjacent to v, this relation is a (( )), if it is an equivalent relation, what will happen? It will give you partitions.

So like this and then such that none of them. So this will essentially form a complete graph. When the underlying graph is not a complete k partite graph, this relation of non-adjacency cannot be an equivalence relation. Why? Because an equivalence relation will give you equivalence classes that means, if you take a class that none of them will be non adjacent with each other so, none of them will be adjacent with each other, that means any pair among them will be non adjacent.

So that is a class and if you take any other vertex outside, you will not see that the pair is non adjacent to this. That is why this complete connection comes. Because we know that this is not that equation, it cannot be an equivalence relation, why is it not an equivalence relation?

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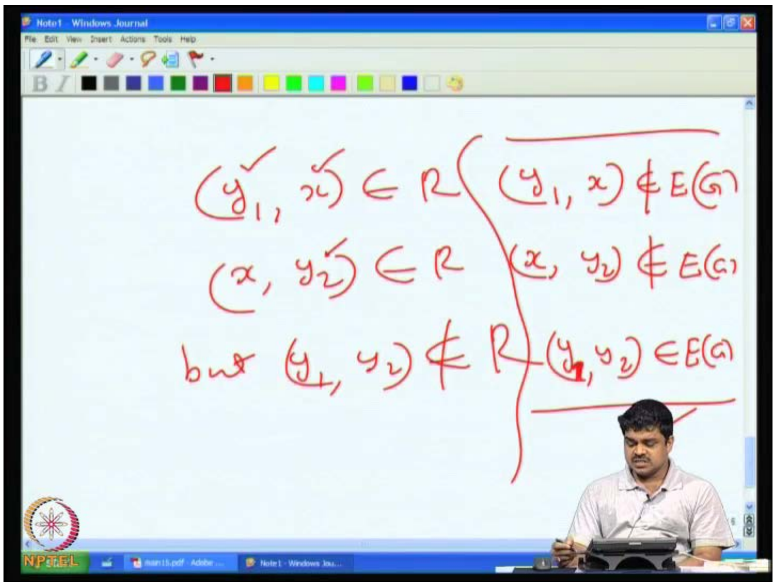

$(y_1, x) \in R$ $(y_1, x) \notin E(G)$

$(x, y_2) \in R$ $(x, y_2) \notin E(G)$

but $(y_1, y_2) \notin R$ $(y_1, y_2) \in E(G)$

If you remember the properties for graph to be, for a relation to be an equivalence relation, there are these three things - reflexive where we can always assume a is there. Write it here also, mini graph we can simply assume that. Then the second is the symmetry - if a b is there b a is also there; non adjacency such a relation definitely it will be true. So what would be violated then if it is not an equivalent? The third condition namely the transitivity will be violated. Why is transitivity violated? See, it should be because there are some vertices in the graph some (y one, x) element of the relation, some (x, y two) element of the relation but, some (y one, y two) is not element of the relation. This is why the transitivity is violated. For there should be some pair (y one, x) and (x, y two) - an element of the relation but, (y one, y two) is not an element of the relation. So if it is transitive, (y one, y two) would have been in the relation.

If for every possible (y one, x), (x, y two) pair, if this was (y one, y two) also as element, then we would have told that it is transitive relation along with reflexivity and symmetry. We would have told that it is an equivalence relation. Since we know that it is not an equivalence relation and that transitive has, transitivity is not true, we should be able to find some three vertices of this type.

So now, we have (y one, x) there, (x, y two) there but, (y one, y two) is not there, which means that the (y one, x) is not in the edge set of the graph. (x, y two) is not in the edge set of the graph and but, (y one, y two) is in the edge set of the graph. This is what it says because the relation was non adjacency. So these two are y one and x are non adjacent, x and y two are non adjacent, so y one and x are and (()) y one and y two are adjacent. This is what we see. Such three vertices should be available.

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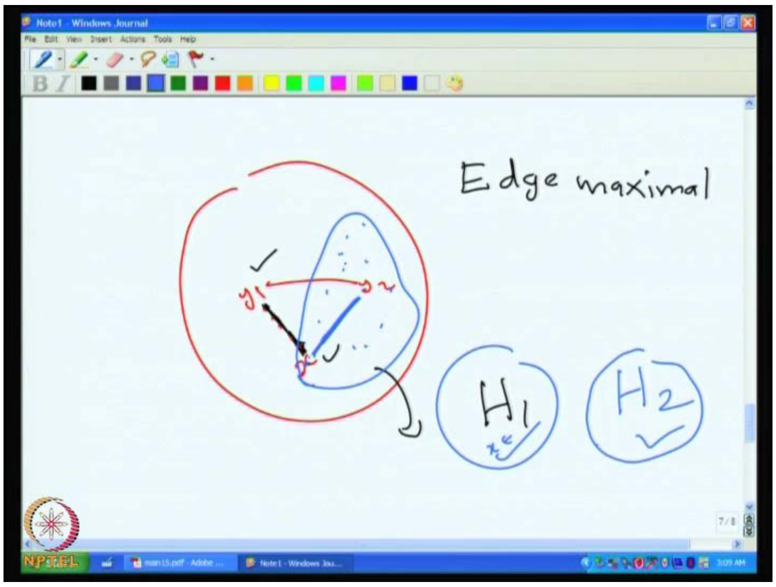

So now, if such three vertices are available, let us put it like this. Suppose this is my x. We have (x, y one) - this is non adjacent. This is non adjacent (x, y two) and this is adjacent (y one, y two). This is the way the graph is. See the question is what will happen if I add this edge here ? Because of the edge, (assumption of edge maximality) because our graph was edge maximal with the property that there is no k-constructible sub graph in it, if you add one more edge, a k-constructible sub graph will come. For instance, if I add this edge, a k-constructible sub graph will immediately result in it. Let us say that k-constructible sub graph is H1. It contains (x, y one) in it because the newly added edge should be there. Otherwise, earlier also it was there. Similarly, suppose if I add this edge, you will get another sub graph H2, which contains x and y two and which is a k-constructible sub graph. So both this and this have to be k-constructible sub graph. They may share something.

What we are going to do is to create a copy of. We will take H1. x is there in it. While H2, we take only a copy. We do not use H2 as such from G. So, what we are going to do is we will take a copy of H2, but retain the original vertex x in it. So it is a copy but, see for instance, H can be somewhere here like this. We will take x, but then we will, for all other vertices in it, we will make a copy. Rather than taking the original vertices, we will just make a copy.

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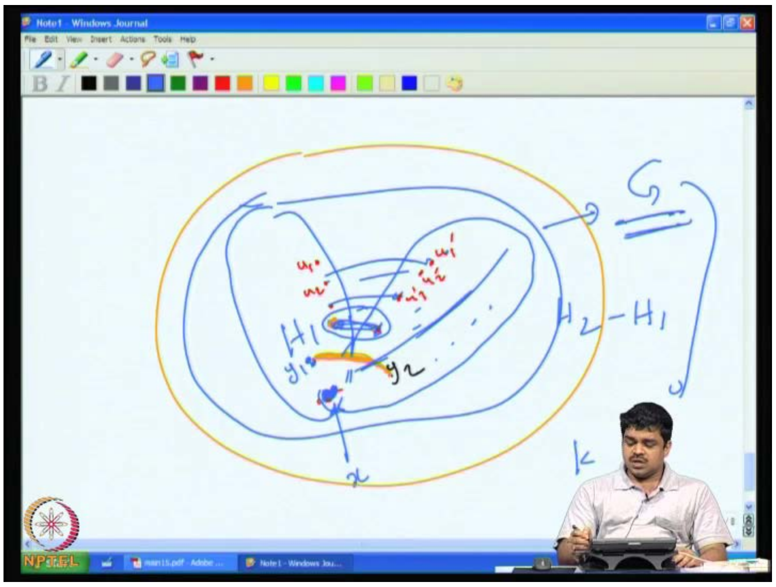

So this is the thing. We will have H1. H1 is as taken from the original graph G and then here, I retain x. We will make H2. For instance, the vertices which were in H2 minus H1, I can take from G. As usual, the original ones I can take. But those vertices which were shared, we have to only make the copy of these things because we want only x to be common to this. So what will happen is, here is y one. Now there is edge. So we have this edge we have added.

Now similarly, this y two is here. Here we have our y two. Similarly, this edge we have added, remember? Now what you see is two graphs; H1, which is k-constructible, another graph H2, which is k-constructible. so like we have It is possible that in the original graph, some of these vertices are common but, for instance if this was u1, u2 etcetera we will take a copy of that u1 dash, u2 dash, u3 dash, so that this will not share anything with this other than x. That is the intention. So, we want only x to be common to both of them.

Now we know this is like our original construction. What we do is, we just delete these two edges - third rule of the k-constructability. We will add this one instead. We will add this edge like the third one. After deleting this xy one and xy two, we will add these two.

You remember this yy two edge was there on the original graph. Because we have copies here therefore, we can add this one. It will be an edge of the original graph only.

Now what you do, whichever was common vertices here we should. You see that this graph, because it is constructed from two k-constructible graphs by the third rule, it is again k-constructible. Now we can identify two vertices which are as such non adjacent here but, actually copies of the original thing.

So now what you can do is, you can try to identify these vertices - corresponding vertices here - the same vertices and then if you identify here (Refer Slide Time: 16:55) and then here, then here like that, it is like merging them together and you will definitely get the original graph G. Each of these merging correspond to the second rule, because whenever two non adjacent vertices are identified and then you put the edge and contract them - that means they identify those two vertices- two non adjacent vertices with the same neighbors.

So because the neighbors - Slowly all the neighbors will get merged. Therefore, in the graph so it will this if only the original neighbors which were there in the G minus, because we are only taking a sub graph of it, some vertices will not come. But, we will get a sub graph of G by this merging process.

Because we applied the second rule to an already k-constructible graph, we end up getting a k-constructible graph. What have we proved now? We have shown that any k-chromatic graph, if the chromatic number of the graph is at least k, then it contains k-constructible sub graph in it. This is what we have shown. This is the result of Hajose.

So the key point is, to identify some structure in a k-chromatic graph, if the coloring, some sub graph which causes the chromatic number to be high. This is the intention. But, if you ask a novice or a beginner this question – “Why is the chromatic number high?”, it is very likely that he may be misled to tell that it is because of some large cliques in the graph.

So it is the initial intuition probably, to believe that when there are large cliques in the graph, the chromatic number is high. One may even, in the beginning think that it is because of the large cliques that the chromatic number is high. It is not true because even in graphs where the clique number that means the maximum clique size is as small as two, that means not even a triangle is available, not even a k three is available, then also your chromatic number can be arbitrarily large. It can be as large as possible.

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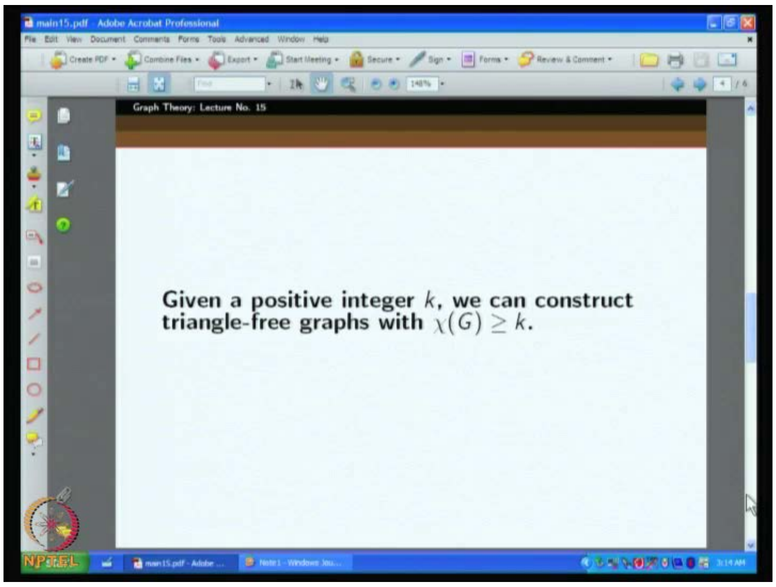
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Given a positive integer k, we can construct triangle-free graphs with $\chi(G) \geq k$.

So, for instance, we can make this statement; Given a positive integer k, we can construct a triangle-free graph with chromatic number greater than or equal to k.

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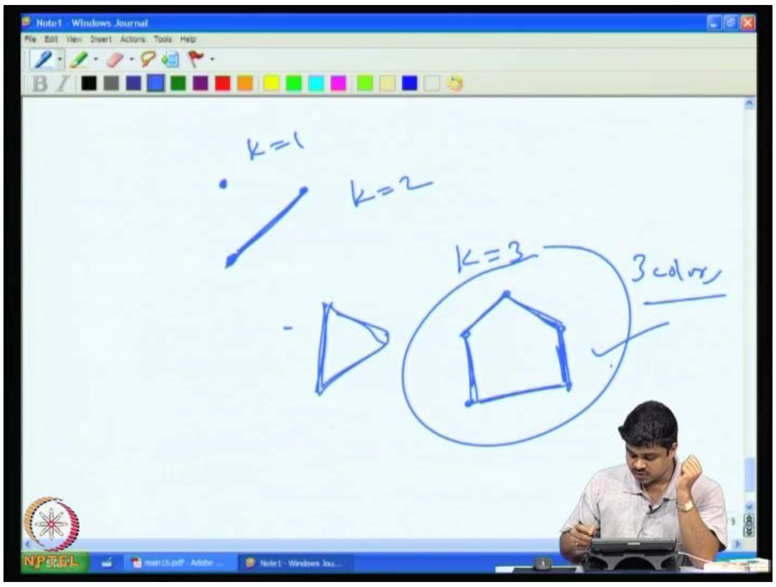

Initially it may look like a little contradictory. For instance, if it is one vertex, it is very trivial. Also if you have two vertices, you can just take an edge. There is no triangle in it. Its chromatic number is two. k equal to two. k equal to one - case. k equal to two - case.

k equal to three - case. You cannot take this. This is a triangle. This is not the one we are looking for. This one will do - this is a pentagon - a cycle on five vertices. This will allow us to see this is essentially a triangle-free graph which requires three colors to color it . So chromatic number is equal to three but, there is no triangle in it.

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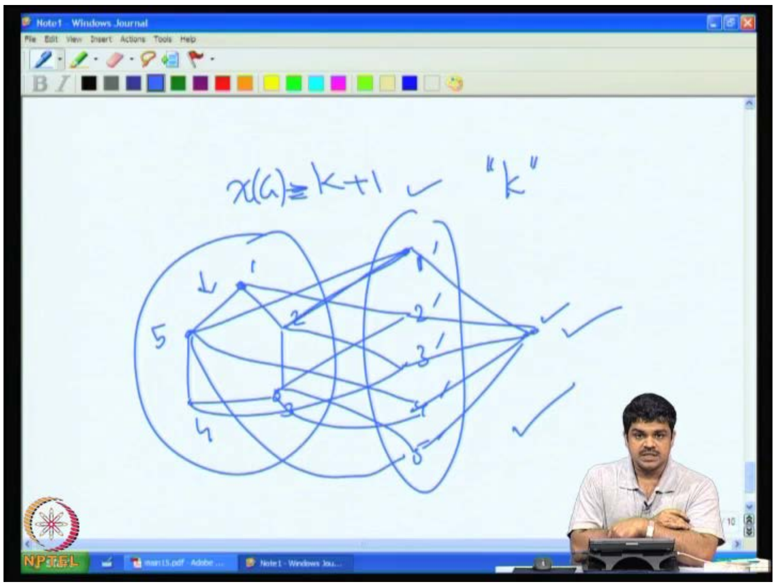

Now, one way to construct a graph without triangle and having a k plus one chromatic number i.e. chromatic number is greater than or equal to k plus one, is to take one graph. For instance, if you take a pentagon, the chromatic number is three or in general a chromatic number equal to" k" graph. Then, you just , this is one, two, three, four, five - the vertices. You just make copies one dash, two dash, three dash, four dash and five dash and then we have to connect to the neighbors of it like this. This construction will give us a graph for one dash, two for instance you can put to one and three, three you can put to two and four and four you can put to five and three. So for five dash you can put its neighbors. This is what we.

So if you construct and then, if you put one vertex here and this. In general, what we do is you can make k vertices here and copies of k vertices here and then one more vertex if you add, then you can create a graph without. It is very easy to check that there is no triangle in it and then you can also argue that. So you do not have your chromatic number will be more than k.

So you start with a k chromatic graph without a triangle here. Then suppose there are n vertices, make copies of n vertices here. Each vertex will be made adjacent to the neighbors of its corresponding vertex in the original graph and then finally, you will add one vertex in the last. Then it is connected to all of them. You can argue that this requires k plus one colors because the original one required k colors and also it is very easy to show that it is triangle free. So this is the way to show that in fact we can construct triangle free graphs with chromatic number at least k.

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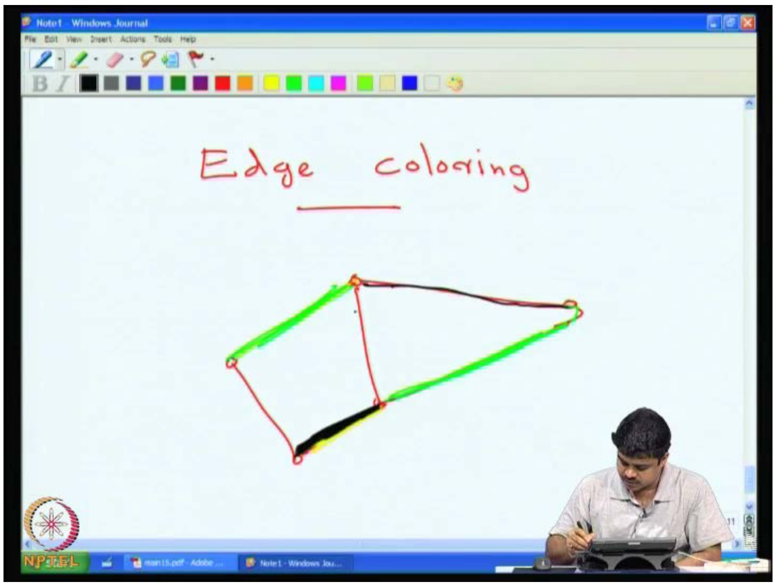

I will move on to a new topic namely edge coloring. The edge coloring, like vertex coloring is the problem of coloring the edges of a graph in such a way that two adjacent edges get different colors.

For instance, if this is the graph, (Refer Slide Time: 26:24) how do you edge color it? You may want to use green color for this, red color for this and may be a black color for this. These three are red, black and yellow. Here it is edge colored. What should I do? I can use, for instance, yellow color here and then can I use the same black color here? We may have to use different colors. We use three colors.

So one possibility is, if I want to reduce the number of colors, what I should do is, I should use green color for this thing, I should use black color for this thing. Then I end up coloring this way. This is green color and black color.

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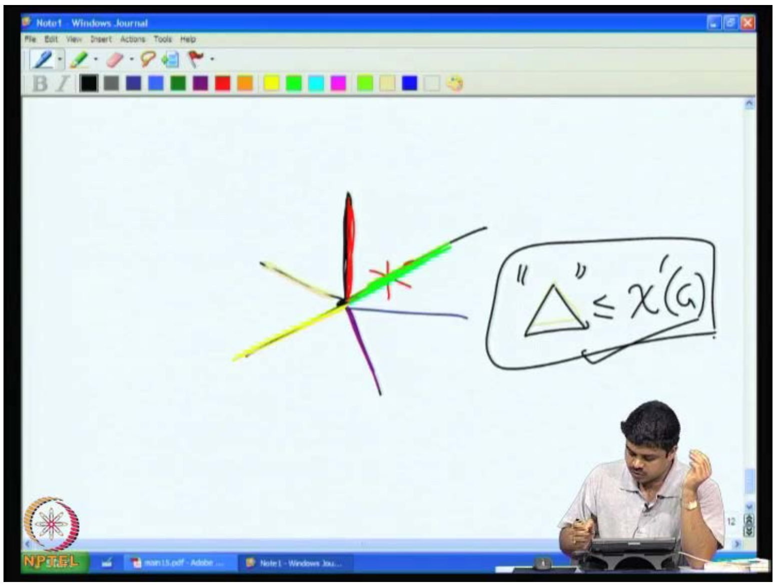

Now let us see some examples, for instance, what will be a very natural and easy lower bound for the edge chromatic index. What do I mean by edge chromatic index? It is the minimum number of colors required to edge color the graph properly. ‘Properly’ means any two edges which are adjacent, which are incident on the same vertex like this. We should get different colors, that is the only condition.

Consider most natural upper bound for edge coloring, if you think for some time is this. In the sense, if there are so many edges incident on the same vertex, they cannot share colors. They should get all different colors, so it is not possible to give a red here and again a red here. That will be a violation. If I can only give different colors to these things, you cannot repeat colors.

So this should be the way it is colored. Then the question is, what will be the biggest upper bound I can get by this argument? Naturally, it is the maximum degree delta. For instance, if I consider this parameter delta the maximum degree - the biggest of the degrees of the vertices so that will be the natural lower bound for the chromatic index of G.

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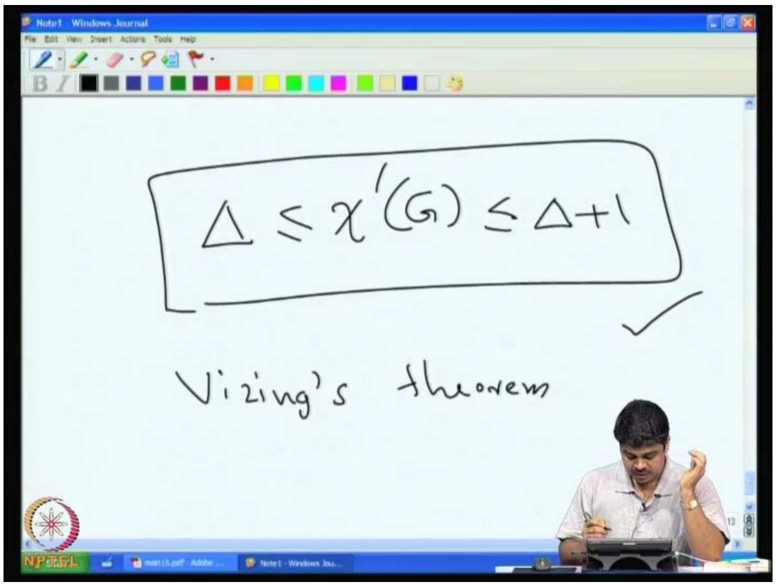

$$\boxed{\Delta \leq \chi'(G) \leq \Delta+1}$$ ✓

Vizing's theorem

Now the most interesting result for here, about the edge coloring would be this. If you consider the chromatic index of a graph, that means the biggest number of colors required to edge color the graph, it is at least delta. It is also known that it is utmost delta plus one. This is called Vizing's theorem. This is about proper edge coloring. I have told the definition of the proper edge coloring. Then we will prove Vizing's theorem in the next, after proving one special case of it namely the bipartite case.

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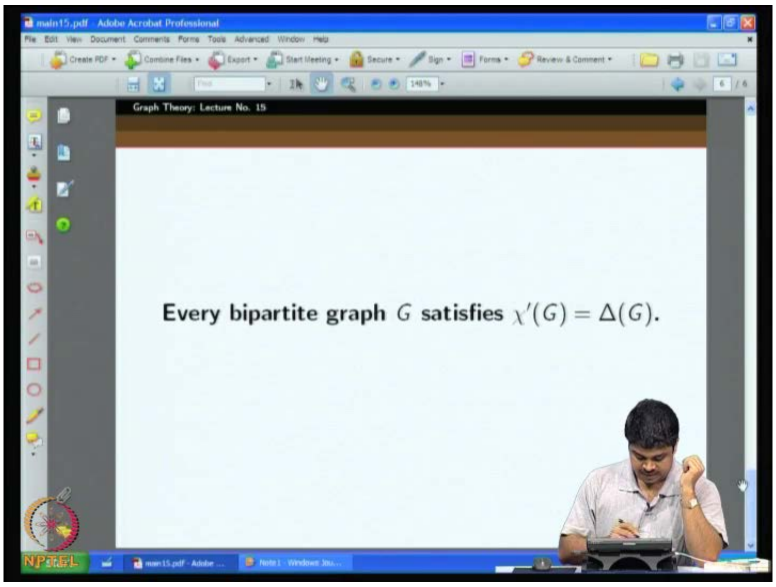

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Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

The bipartite case says 'If it is a bipartite graph, this lower bound - this obvious lower bound of delta is exactly is the actual value of the chromatic index of the graph'. Essentially it requires only delta colors. That is what happens for a bipartite graph.

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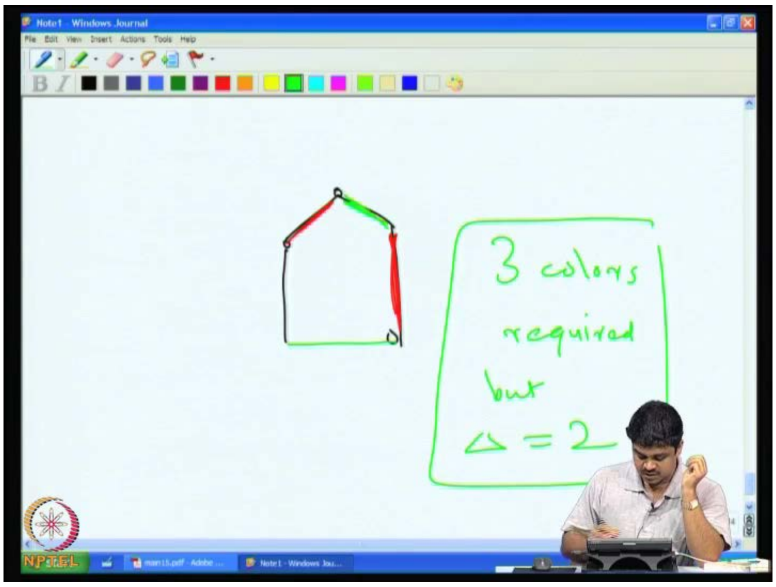

But in general, some graphs may require delta plus one. Can you see some examples for instance, some simple cases when we consider the odd cycles. For instance, our pentagon, it requires three colors, isn't it? One, two, if I start coloring like this, here I will need new color – one, two, three will be required - three colors. But, delta equal to two only, right?

So, there are some examples where you need one more color, but no example will be there which requires more than one color. This is what Vizing is saying, if it is a simple graph. But, before moving to that, because it requires a little more sophisticated argument, we will take a very simple case of bipartite graph and we will show that in fact in the case of bipartite graphs, only delta colors are required. But, how do you show this thing?

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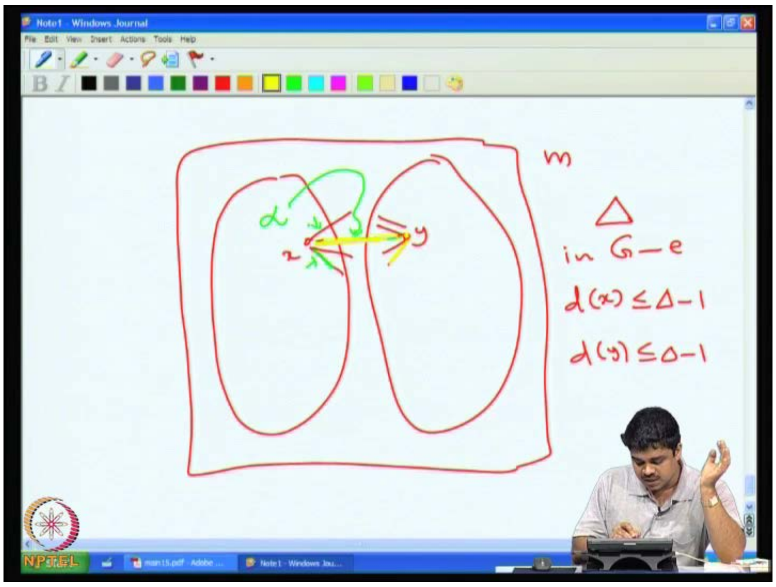

Suppose you consider a delta coloring. This is the bipartite graph. We will show by induction on the number of edges, as usual, if there are no edges, there is nothing to prove - delta equal to zero. We do not need any colors. So, delta colors are enough. Suppose for less than m edges, we have proved.

Now, I consider a graph with m edges. The maximum degree is delta here. You can pick up a vertex and delete an edge. This is xy. You know the original degree has utmost delta here x and y. What you do is you delete an edge. How many edges are present here in both cases? d of x now in G minus e. So, d of x is less than or equal to delta minus one and d of y is also less than or equal to delta minus one. One less - so it may be just exactly delta minus one.

Now, by induction we know that it can be colored using delta colors. If it so happens that the delta itself is reduced, then we are done, because any way we are allowed to use one more color. We can use that new color. We can assume the delta is not reduced. This is colored using the delta colors. Delta colors are used.

The point here is to consider whether I can give a new color to this edge - this removed edge without any conflict. Where does the conflict arise? For instance, when I try to give a green color to this, what will be preventing me from giving green color to this? This is if some green color is already here, with respect to our coloring we got by induction. So, this may be possible.

We know that only delta minus one colors are used here, because only delta minus one edges are edges after this removal. One extra color is available. Let us say this is alpha. Alpha is available. I can try to give this alpha color to color alpha. Maybe let me give the yellow color. I can try to give yellow color to this but, then the issue is that the yellow color may not be the color which is away - which is missing here. Yellow color may be here. Then, it will conflict on the other side; that means x allows yellow color to be given to this edge but, y may not allow.

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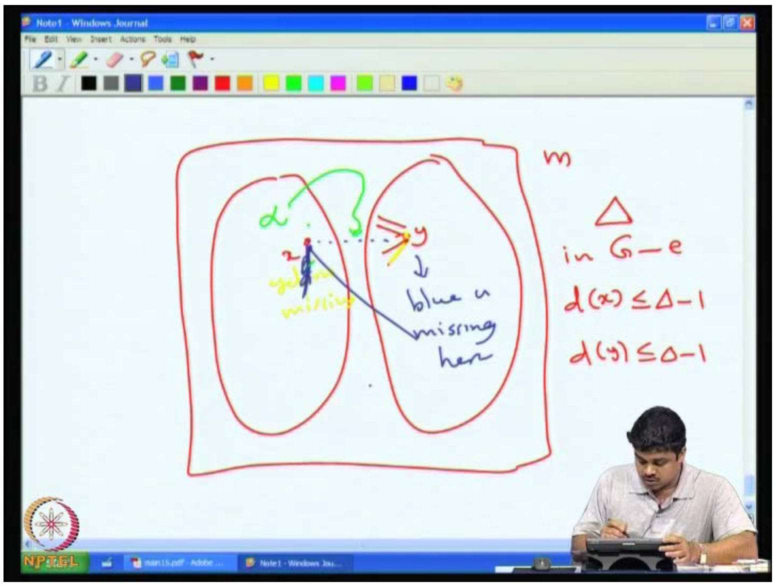

So, y may say that the only color which is missing in its neighbors - on its edge set – incident edges is say, blue color. It is possible that it may allow only blue color. Maybe, I can take blue color - the missing color so blue is missing here and then yellow is missing here.

So what we can try to do now is somehow make both the missing colors at x and y the same. If the missing colors at x and y are same, then we can use that color to color xy, because there would not be any conflict - x also will be happy, y also will we happy.

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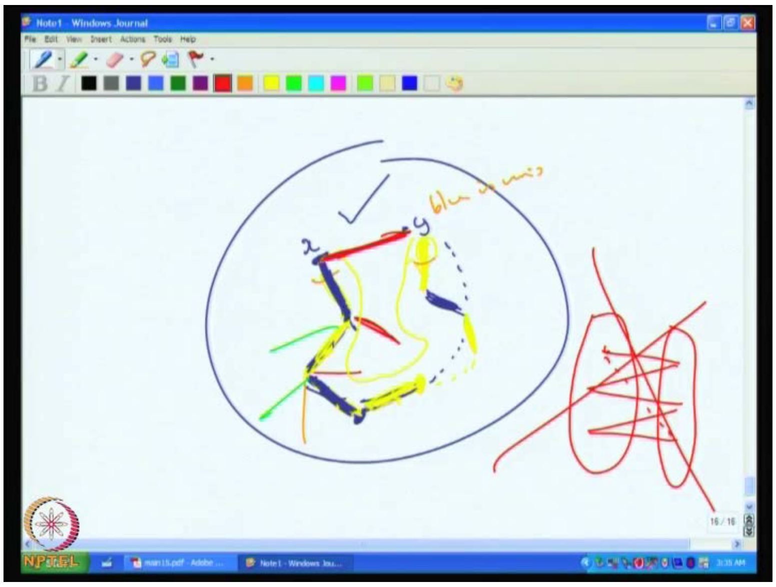

But, how can I do that? One strategy is to try out. Because you know, this blue is available here. If blue was missing here, I could have colored with blue. This xy edge could have been colored with blue. Blue may be here. I will do like this. This is the big graph. x is here. Remember this a bipartite graph. I know that blue is here. Yellow is available here.

Now, what I do is I trace the blue yellow track here. For instance, when I start with a blue edge, then I can at this vertex, look for a yellow edge. How many yellow edges? Only one yellow edge can be there, if at all. Otherwise it will stop. So, yellow edge. So, then it will take. Then in this point I can again start looking for a blue edge. Is it possible that it will come back to some of the already visited vertices? No, it is not possible, because every vertex will have one yellow edge and one blue edge and here x does not have a yellow edge. It is not possible to come back.

So, it will keep going. At some point, where will it stop? Suppose it stopped here, somewhere. What we can do is, we can do an exchange of colors. So, for instance, all the in this path, I can make these blue edges yellow and yellow edges blue. Let us do that. For instance, I will make this yellow and then of course, there is a conflict here. The only conflict is here. Then, I will make this blue, then of course, I can make this yellow and then I can make this blue. So, there will not be any conflict after this thing. I just switched colors in this path, no other, because if there are some other edges here, it will be red or green or whatever other color. But, they will not have any problem whether it is yellow or blue. As long as there are different colors, it is okay for them. Only when I change from yellow to blue, the only edge that is likely to complain is the blue edge, because blue-blue will come but, then I am changing the blue-blue to yellow also. Therefore, there will not be any complaint.

But, the effect is that, yellow is there and yellow is there, but blue is missing here. Earlier blue was there but, it is missing now. Blue is missing here also. Now, I can use the blue edge to color this. So, that will settle. Without using any new color, I can give a color to that xy edge, isn't it?

So, what do I conclude? It means that if I trace the blue yellow path, (Refer Slide Time: 36:52) -this is the blue-yellow path, right? - the original blue yellow path here, I will not abruptly stop anywhere. Because, if I stop somewhere other than y, then I can always do this exchanging yellow to blue and blue to yellow in that path. Then, I will free blue from here and then, this will be. I can color this xy edge.

So, what can happen is, it can go all the way to y. The only thing that can happen is it will go all the way to y, but how can it enter y? There is no blue edge at y. So, it has to allow y through yellow edge. It has to enter through an yellow edge. This is the yellow edge. Then there should be this blue edge and then there should be this yellow edge.

Like that, it will somehow enter there. What is the issue here? If suppose, this happens, then our earlier strategy will not work. For instance, if you try to interchange yellow and blue here, of course yellow will come and blue will be freed from x side but, then earlier y had a blue missing edge but, here instead of yellow, we will have a blue now and then y is missing and will become yellow rather than blue. So that will not help us, because what was missing at x is now missing at y. That is all. So, just interchanging the role between x and y - the colors, right?

That will not help us to resolve the issue. If this happens, we are stuck. We cannot do the earlier thing but, then if you carefully look at it, this situation will never occur because of the following reason. Suppose, if there is a path like this, start with a blue-yellow-blue-yellow-blue-yellow and then, when I am entering y, it is a yellow edge.

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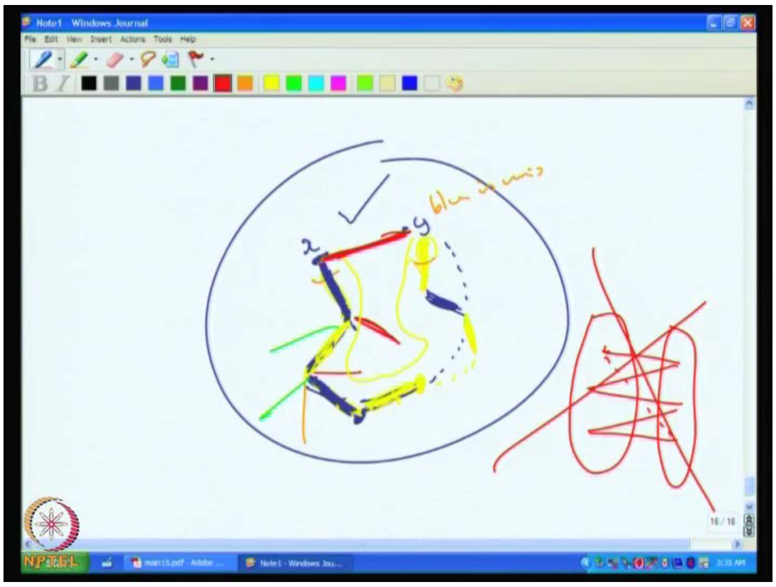

So, how many edges have I used? Because I started with the blue, second is a yellow edge, third is a blue and fourth is a yellow edge. Therefore, any yellow edge has to be an even number at the second, fourth, sixth, like that. Therefore, there will be an even number of edges in this path, when I start from x and reach y.

In the end, if I put this edge, if I try to add this edge, then what you get is an odd cycle - an even number of edges in the path and one, this xy edge - together is an odd cycle. But, then a bipartite graph cannot have an odd cycle, because that is very easy to see. For instance, if this is the bipartite graph, any cycle has to go once here, then come back here, then here, then here, then here. So, if it is reaching back, it should be an even number of edges. It can never have an odd number of edges in it, in its cycle, in a bipartite graph, because it should zigzag from two sides, between two sides and it should come back to the original side where it starts at.

Therefore, we can immediately assume that this will never happen. Then what can happen? It will, if at all I try the blue- yellow edge, it should stop at some other vertex. It cannot keep on going infinitely. It should stop because it is a finite graph. We will stop at some point and we will do with exchange of blue and yellow and then by blue will be released at the x vertex. Blue was also missing at y and then that can be used to color the x y edge. blue can be So therefore, we can extend the delta coloring of G minus e to G. This is why bipartite graphs can be delta colored. This is the strategy.

Finally, we will look at the Vizing's theorem. Vizing's theorem says like, of case bipartite graphs require only delta or delta plus one colors, while sorry bipartite graph require only delta colors but, in general it may require one more color. That is what Vizing's theorem says. Whatever it is, it will not require more than one color - just one extra color will be enough.

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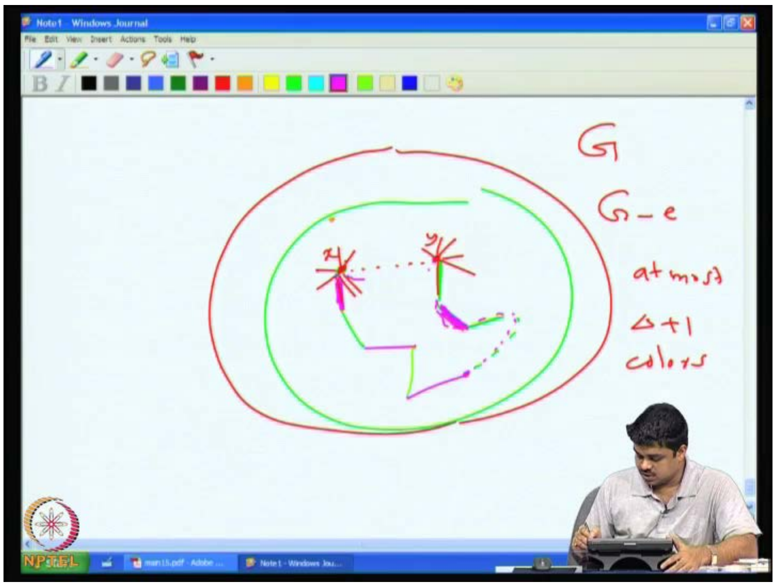

So how will we prove that with at most delta plus one colors, we can edge color any graph? We will again do an induction on the number of edges. To do this thing, we will consider a simple observation. We can, as usual, consider a graph G. Its maximum degree is delta. You can consider some edge. You can remove that edge by induction assumption. G minus e is, we can say x and y can be edge colored using at most delta plus one colors. Because, if the delta itself has reduced, then we are done. We will have an extra color to color. Therefore, we can assume that the delta did not reduce. The delta is the same. The maximum degree is the same.

Of case, there are - if you look at x - in fact, delta minus one edges can be seen here, like earlier. How many colors will be there, around? Only delta minus one. Two colors are missing here, not one. You can take, suppose some color is missing. Some orange color is missing here – orange, sorry - may be green is color is missing here. The green color is missing at this thing. That means, the green color is not there on any of this thing and here let us say, violet color is missing.

Now, what will happen if I trace like earlier, the violet? Suppose, there is a violet here and I trace the violet. If violet is missing here, violet is missing here and violet is also missing here, then we can easily give violet color to this. Suppose violet is here and then I try to trace the violet-green-violet-green path here, is it possible that it can abruptly stop somewhere?

If it abruptly stops somewhere without reaching y, then what we can do is, we can do the trick that we have done in the case of bipartite graph. We can just exchange the colors on this path. Violet-green; green will become violet and violet will become green. Then, the color will be released on this violet. So, the green color will be, sorry - here, it will become green and then violet color will be available for xy edge, because violet is missing at y also.

What we can infer is, whenever we start a path like this, it has to go all the way and reach y. 'Reach y' in the sense that it can only enter through (violet is missing here) only a green edge. Some violet-green path should be here, so it should be like this. In the bipartite case, we argued that it will never happen, because this is going to be an even path - plus one edge and it will create node cycle. But, if it is not a bipartite graph, that can happen. That can indeed happen.

So what we can just remember is, suppose if you have an edge, G minus e. If you remove xy and if you look at any coloring of G minus of e and if you cannot extend this coloring at G, then it is clear that if found color is missing at y and one color is missing at x, if you trace the corresponding path, that should start from x and reach y. Whatever is starting from x, it should reach y. Otherwise, we can do that exchange strategy along the path and then we will release a color for the xy edge. This is the thing.

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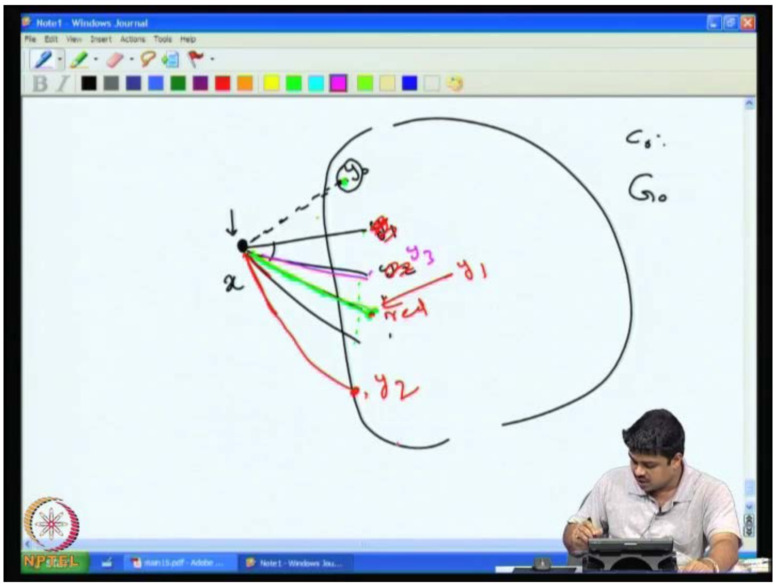

Now, the next point is, suppose I try to do the induction like this. Suppose, you get one vertex x. Now, you can think that there is a y zero here which I removed. This is the other neighbors of it - y one, y two…..

Here, I just decided to remove this edge, so that in the remaining graph we have a coloring. This coloring is say, c0 - that is the coloring of this thing. After removing this edge, we can call this graph G0. This coloring c0 of G0 will be such that, if you look at the missing edge here, because there is a missing edge here - because delta plus one colors are there. Then two missing edges are there. We can just say one missing edge is here and then the missing edge here will also be different.

Now, what we can do is, we can see that here also, there is a missing edge. Suppose green is missing here, say, among all these edges, one of them may be green. We can pick it up and then we can consider so we will rearrange the. For instance, this is a green edge and then here, there will be one color missing.

So let us say, some red color is missing here. Now what we do is, I will look for the red edge here. Suppose, the red edge is here. Suppose, this is something. It is possible that the red edge is here. So, this will be called y one, now. I will renumber this vertex as y one and this red edge and this will be called y two. Then, I will see what color is missing here. Suppose violet color is missing, suppose violet may be here. So, I will call it y three onwards.

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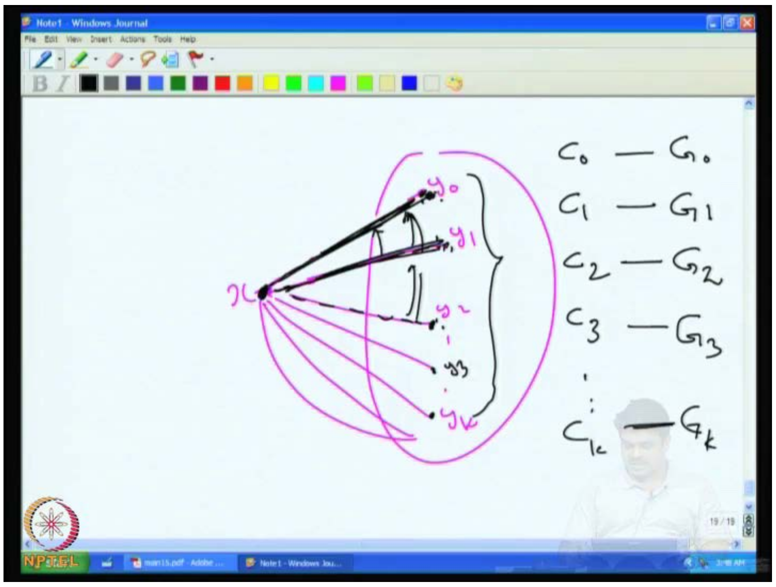

Then I will make this order of these edges, the neighbors and then, I can rearrange them. For instance, I can read out the figure now, like this. So, this is x, this is y zero, this is y one, this is y two,……. up to y k and so on. The point is, with respect to the original coloring, for instance, whatever color is missing here will be the color of this xy one edge. Then, whatever color is missing here will be the color of the edge xy two and whatever color is missing here will be the color of xy three and so on and it may not be possible to extend this sequence till it covers all the neighbors but, say y k. is such that

Now, if you look at this missing edge, it is not present anywhere. Therefore, in that case we will stop it, isn't it? This original coloring is c0 and then when I remove this edge, the graph will be called G0.

Now, what we will do is we can use the color on this for this edge. I can put back this edge. Whatever color is there for xy one with respect to c, I can give to this and remove this edge. Why is it possible? Because this missing color for y zero was available here and then x will not protest anyway. x had already this thing. I am only deleting this and giving it to this.

So, this new graph can be called G1 and the new coloring may be called c1. This c1 and c0 are only very slightly different. The coloring is more or less same. It is only just the exchange. All other edges will retain their color. Now, the third one. What we do is we can give the color of this edge xy two to here and then remove this edge. Then the new graph will be called G2 and new coloring will be called c2. And similarly, for y three.

So, like that for each of this thing, I can create a new graph Gk and new coloring. While these colorings can be seen as derived from the original coloring, this graphs and the coloring can be seen as related to the original graph like this. For instance, the original graph was G0 by the removed xy zero edge, so if G1 is the graph obtained by starting from G and then removing the xy one edge, so Gi is the graph obtained by removing the xy i edge.

Similarly, the coloring c0 was the coloring which we obtained by induction, when I removed this edge xy zero. But, then the coloring of xy G1 which I named c1 is the coloring, which I obtained by rotating this color - I mean, giving this color to this and making this empty by deleting that edge. Similarly, c2 is a color obtained by giving this color to this, this color to this and making this empty.

Similarly, G3 will be by three level rotation - this color will go to this, the color of this will go to this and this will go to this. Therefore, from c0, it can be obtained by three rotations – sorry, two rotations of colors. Like that, we can think of c1 c2 c3 etcetera as derived from c0 by some kind of rotating the colors on this thing. So, we will complete the proof in the next class. Thank you.