Graph Theory

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Lecture No. # 02

Matching: Konig's theorem and Hall's theorem

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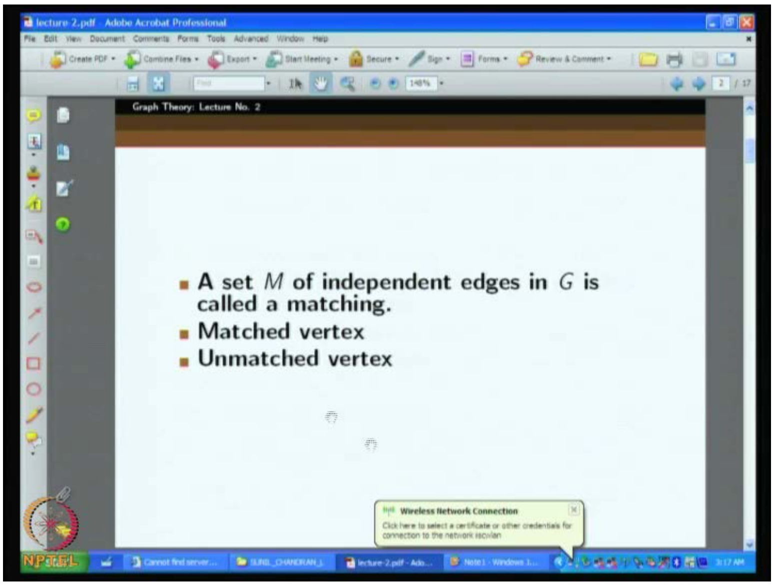

Welcome to the second lecture of graph theory course. Today, we will talk about matchings; a collection of independent edges in a graph is called a matching.

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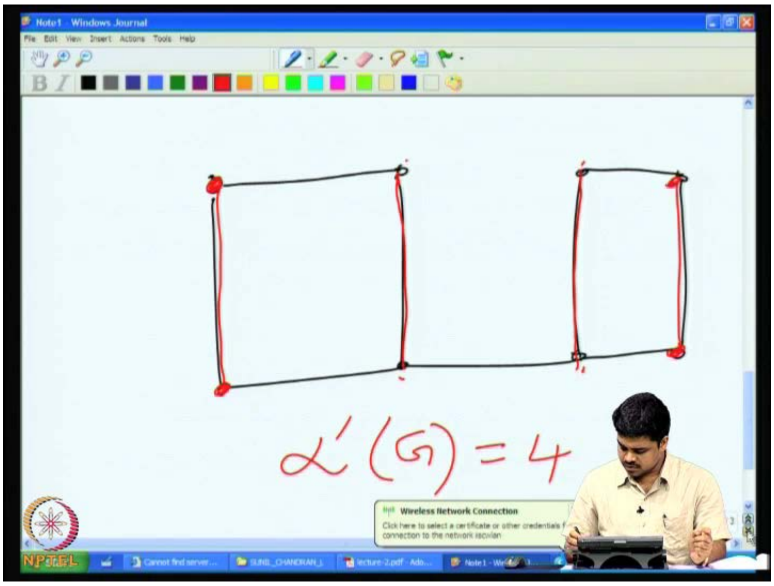

So, let see an example, so consider this graph (no audio from 00:41 to 00:51), so here I will mark some independent edges, which means say this one, say this one, what do you mean by independent edges? So, when you consider these two edges, they do not share any endpoints, the end points of these edges this and this, the endpoints of this and this; for instance, if I had selected this edge, then this would not be independent. So, Matching is a collection of independent edges.

Now, let say… See, given a matching say, this is a matching, the vertices, which are touched by those edges that means, incident on those edges are called the matched vertices, with respect to that matching. And those vertices like this for instance, which are not touched by or incident on these two edges are called the unmatched vertices.

Now, here like usual, we are interested in that cardinality of the maximum matching instance weak in this graph, we can get more than this; if you are looking for the independent say edges, so this one is here, we can in fact get this one also, this is also independent, because these three are not sharing any end points with each other. And I can add this also; so, now I cannot add any more things, because all the vertices are now matched.

So, the biggest independent edges, collection of independent edges that means matching that, I can get in this graph is called is of cardinality four, and this cardinality the cardinality of the biggest independent, set of edges, biggest matching in a graph will be denoted by alpha dash of G, and here for this graph it is four as we see.

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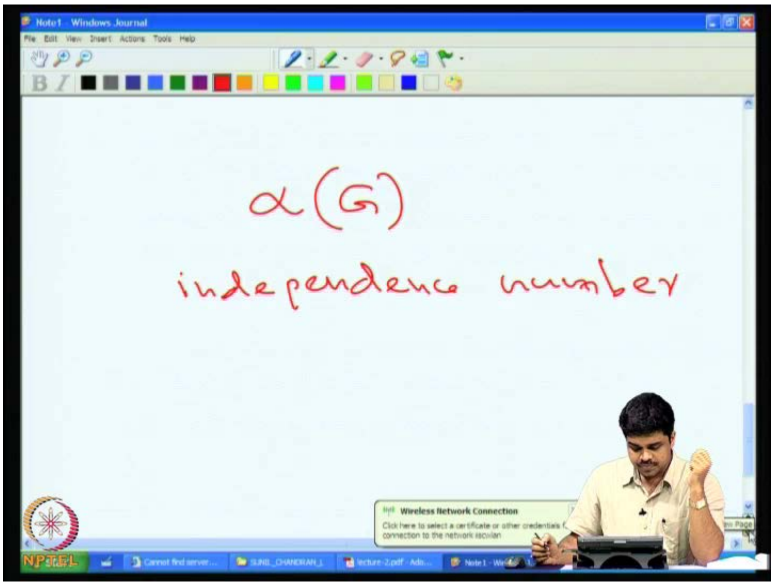

You can see that I have used alpha dash to denote this thing, while we had some concept, which was denoted by alpha of G in the last class, it was the independence number or the stability number, what was this? This was the cardinality of the biggest independent set in a graph, what was an independent set? Independent set was the a collection of vertices, such that they are not touching each other, which means there was between any pair of vertices in that set, there is no edges between them, between any pair of vertices in that set, there is no edges in between.

This is similar here we are talking about, when we are talking about matching, then their edges not vertices, the edges are not touching, each other that means they are not sharing any endpoints, somewhat similar concept; so to indicate that, we will use the notation alpha dash G, to denote these things.

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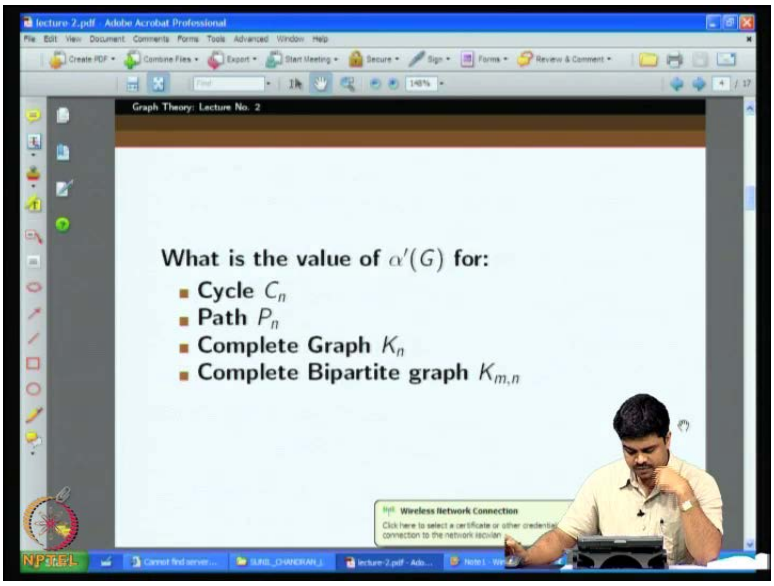

Now, let say go back to the slides, so here to understand what is the parameter alpha dash of G little better, we will take some examples; for instance, let us look at a cycle.

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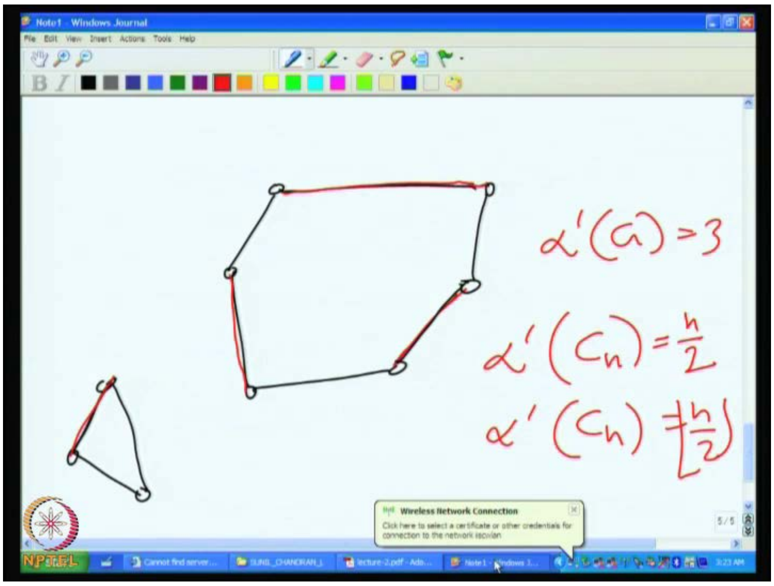

So, a cycle looks like this as say. So, suppose let us take a simple cycle; one, two, three, four, five, so 6 line cycle this is see 6. So, what is the cardinality of the biggest matching here? I can mark the matching like this, so here one. And then once I have taken this I cannot take this or this, because then they will share the endpoints, so I have to take this one now. And then once I have taken this thing, I cannot take this, then I have to take this thing.

So, we will get a alpha of alpha dash of G here is equal to 3 for if we had n vertices in the cycle, suppose alpha is considering the alpha dash of C n, then it will be n by 2, as we can easily see if n is an even number. If n is an odd number is not very difficult to see that this will be n by 2 floor; for instance, I can take a small example here. So, for instance I take three line cycles, then once I take this edge, then I can have take this and this, this is three by two floor.

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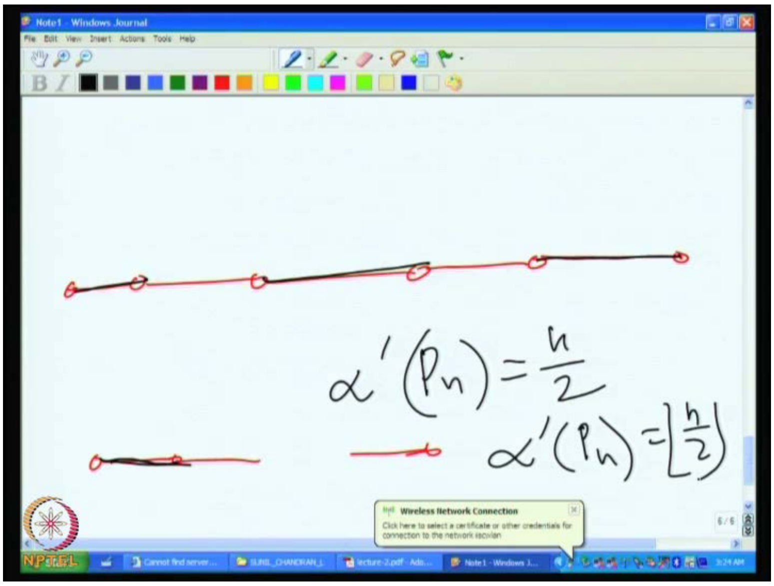

Now other example, I wanted to look is the path, and similar case the path, see path again suppose, if we consider this an even length path, the number of vertices in the path p even, so here I have six vertices in the path, then we can mark a biggest matching like this for instance right three. So, this is then if then n by 2, that is P n; alpha dash of P n is equal to n by 2. So, that is because we had an even number of matchings vertices in the path. Suppose, it was an odd length path, so for instance if I had taken say a five length path, then I cannot do this for instance, if I take this, and then I have to take this, then this will not get a partner, I will say partner. So, then I will get only two; so I will have to say alpha dash of P n in that case is equal to n by 2 floor. So, that is for the odd case.

So, after discussing this path and cycle of even case and odd case, we can easily see a certain property that if the number n is even, then only we can probably match every vertex is the there is only chance of matching all the vertices, is only when it is n is an even number, otherwise if it is an odd number, at least one vertex has to be unmatched it is not that all the vertex vertices can be matched, just because n is an even number, but it is obvious that if n is an odd number, you cannot match all vertices right the maximum matching cannot achieve that.

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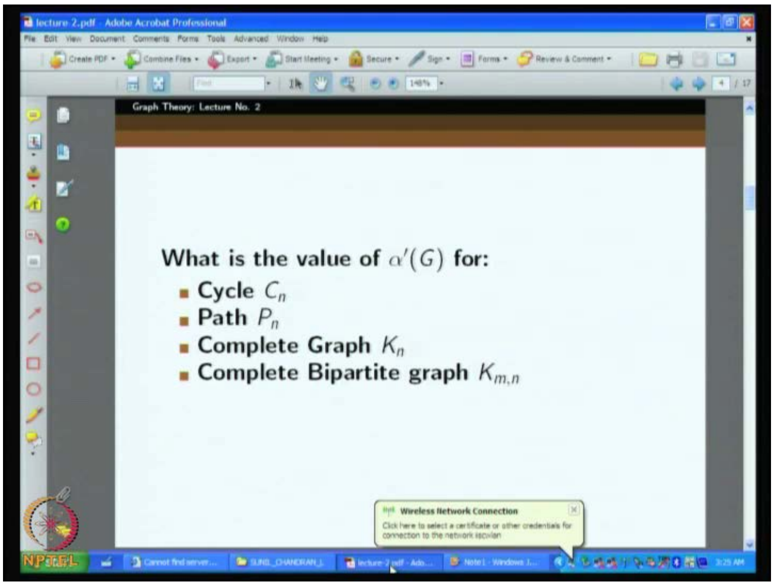

So, for a complete graph K n it is clear for instance, if n is even, then we can get a matching of cardinality n by 2 and that is the biggest possible in a graph of n vertices. Why is it so? Because each edge in the matching will touch two vertices, so if once we get n by 2 edges, we already have touched all the vertices right we cannot go more than that. Now, if it is as I told if n is an odd number, then K n one vertices has to be unmatched, and then we will get a matching of cardinality n minus 1 by 2, the biggest one. For a complete bipartite graph K m, n, where m is less than equal to n. So, what should be the biggest possible matching? It is not very difficult to see that.

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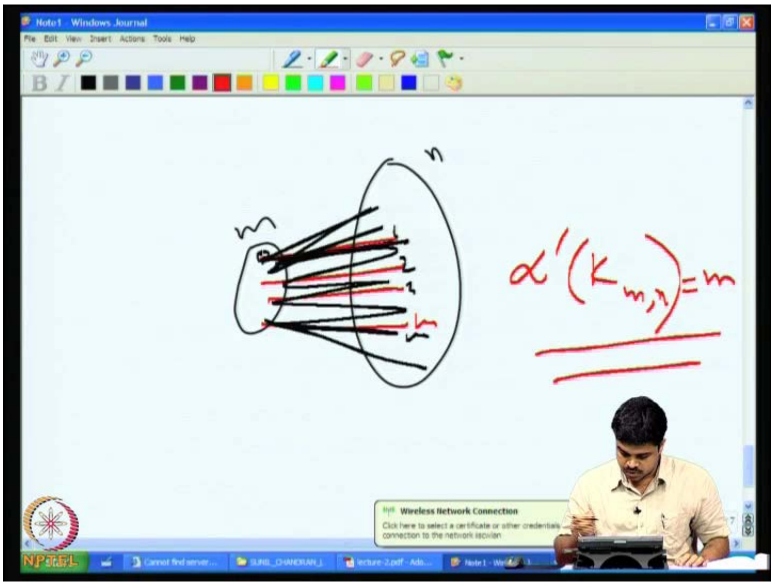

So, if we consider some the edges are like this between this, all the edges between this. Now of case, we cannot get more than m right, why we cannot get more than m? Because any edge in the matching should have one endpoint here and one end point here. So, if we have more than m edges in the matching, then here we should see more than m vertices, but it is not true. So, at most m only can we get, but we can indeed get m edges, because all the possible edges, we just have to mark one, two, three m edges, m vertices and they just take this with this say, this with this and this with this, this with this like that we can we can, we can indeed get m. So, for this thing the for k m, n the biggest matching size is equal to m.

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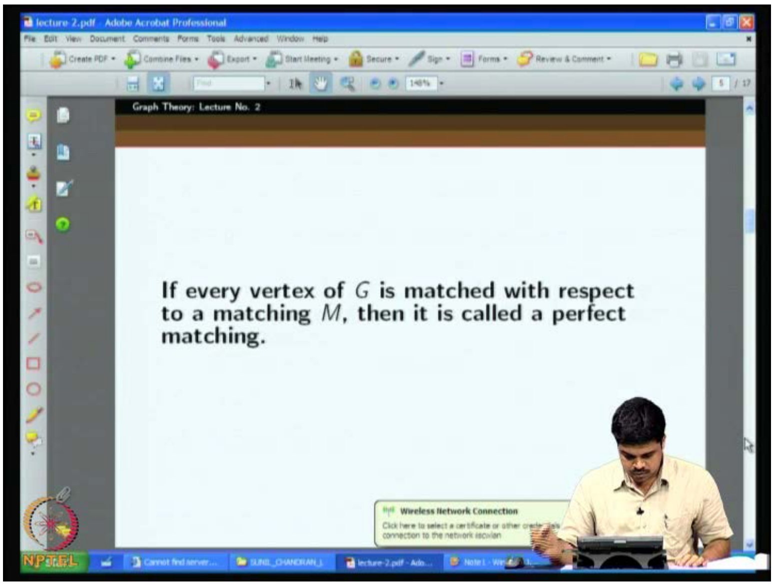

Now, again going back to the slides, so let us it is here is a new definition, so matching is a perfect matching, if every vertex in the graph is matched by that matching. In that case, it is easy to see that if there are n vertices in the graph, and has to be even number otherwise, as we show already we cannot get a perfect matching; and then n by 2 edges will be there in the matching, because if every vertex is matched two vertices can go to one edge, so which essentially means that in a perfect matching, we should have n by two edges.

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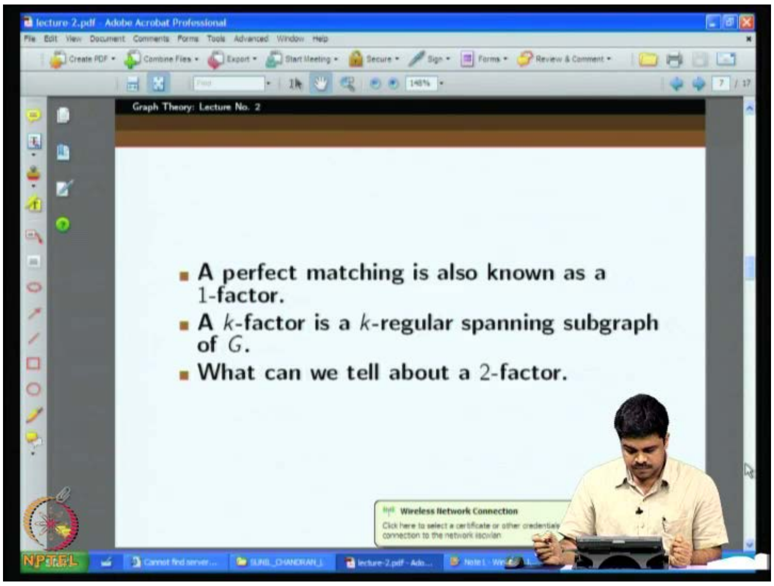

Now, that can we tell about n, so n is an even number, and then here is a information that perfect matching is also known as a one factor. So, it is another name, which is used in the literature, this one factor, so when I say one factor, so one you may be thinking this should be two factor also yes there is a two factor, there is a notion of three factor, there is a notion of k factor in fact ,what is the k factor? A k factor is a k regular spanning Sub Graph of G; so there are two words coming here regular k regular, what do I mean by K regular?

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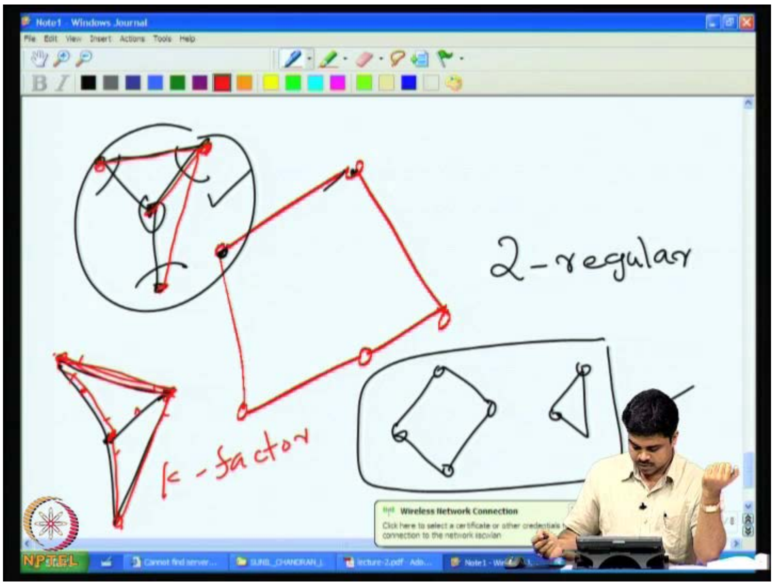

So, k regular means the degree of each vertex in the graph is K. So, for instance, we can consider say a cycle again; this is a two regular graph that means here, if we take any vertex. So, this vertex it is degree is two, three are two edges incident on it the degree of this vertex is also two, the degree of this vertex is also two. So, every vertex as same degree two that is a therefore, this is a two regular graph; another two regular graph, I can draw for instance like this; this is a two regular graph together though it is not connected this is also two regular graph. And if I want a three regular graph, let us consider this graph, so I am just putting like this is a three regular graph, so degree of this vertex is three, there are three edges here, incident on this thing here also three here also three, so this is a three regular graph.

So, regular means that and another word that we have used here is spanning Sub Graph. So, we already know what a Sub Graph is; Sub Graph means, we are collecting a certain number of edges, and certain edges from the graph G and certain vertices from the graph G, and then putting together these things. So, making sure that the edges, we select are I have both the endpoints in the collected vertices. So, this is a Sub graph, we had already defined it yesterday.

Now suppose the Sub Graph, the vertex set the sub Graph is the same as the vertex set of the original graph that means, we do not loose any vertices from the original graph and we considered the Sub Graph. Then it is called as spanning Sub graph, it spans all the vertices. So, and ours its if it so happens that our Sub Graph is not only a spanning Sub Graph, but it also k regular, then it is called a k factor; for instance, it is go and take an example here. So, here from this graph for instance if I try to take a spanning Sub Graph, say I can mark a spanning Sub Graph here, say this is a spanning Sub Graph, see because it this Sub Graph is has all the vertices of the original graph in it, but some edges are missing. So but this Sub Graph here is definitely not a regular spanning Sub Graph. So, in this case, for instance, I could have let us for instance, I could have considered this graph. So, I will redraw it somewhere is a graph like this.

So, this graph say, I modified this graph, and then in this graph suppose, I want to get a spanning Sub Graph and a regular spanning Sub Graph. So, one possibility is to collect of case I have to collect all the vertices, but I can collect this edges that means, I will skip this edge this is a not only a spanning Sub Graph spanning, because it touches all the vertices here and then if we look at the degree of each vertex, and the red see there are red to red edges, here two red edges. So, this is true regular spanning Sub Graph of this thing.

So, like that when a spanning Sub Graph is both K regular and a Sub Graph is both the spanning Sub Graph and k regular, then it is called a K factor. So, you can see that if we have talking about one factor here, then we will be selecting exactly a matching in fact the perfect matching from the graph right, because every vertex should have degree so here if it is select this one then we cannot select this or this. So, therefore otherwise the degree will go by one therefore, the one factor, the word one factor is the same as that denotes the same concept as perfect matching and now. So the more general concept is k factor anyway that was just passing remark, and then it is ok. To understand this, we can ask a question, what can we tell about a two factor? So, we show some two factors here for some graphs. So, those who are those who are (( )) for instance that was the cycle here for instance this is a two factor of this graph this is a cycle, but is it necessary that a two factor is always a cycle.

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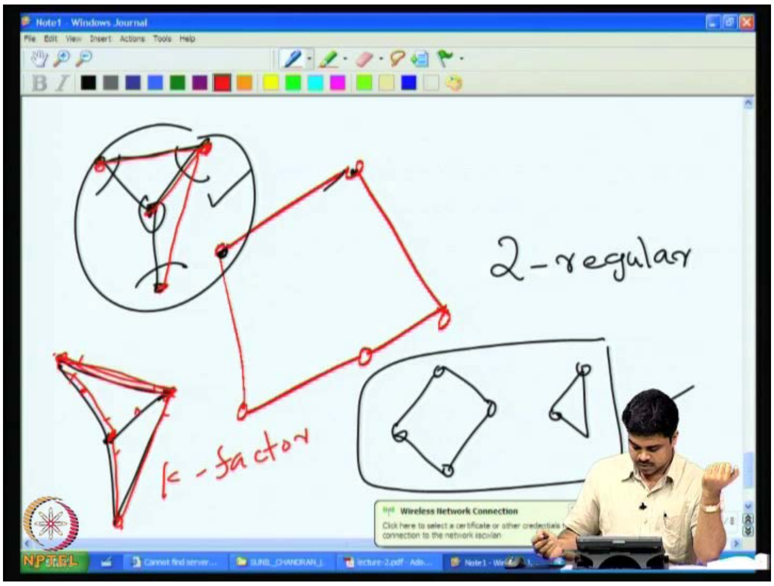

So, let us take another example; consider this graph does this graph has a two factor? Yes; the question is do we have a spanning Sub Graph of this thing, which is also two regular; yes. So, this is the way we can mark it. So, we mark it like this; so this one you take, then this one you take right, and then this one you take. So, you see that this is indeed the spanning Sub Graph, because every vertex is include in Sub Graph, and then if we look at the Sub Graph, it is a two regular Sub Graph, but of case it is not connected we do not need connectivity so but then this is not one cycle in fact it is a collection of cycles, here we have a triangle here, we have another triangle this is and here we have a four cycle. So, this is called a cyclic cover, because this is a collection of cycles, which covers all the vertices; so this is called a cycle cover of the graph. So, the another name for two factor is cycle cover.

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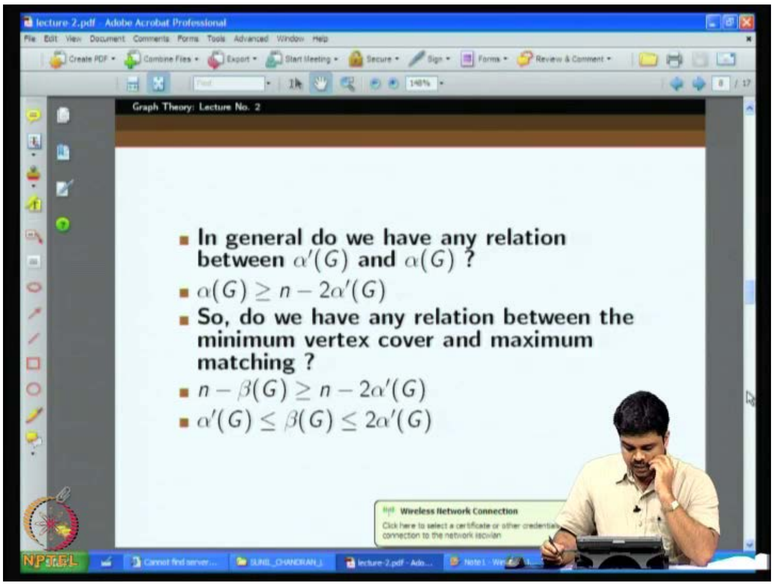

And now, going back to this; in general, do we have in a relation between alpha dash of G and alpha of G, we have already told this names alpha dash of G and alpha G, because of the similarity of the definitions, it is a it is a more or less similar concept; one is about edges, one is about vertices. So, we use that notation for that. But then is there any connection between how can I relate both of them together.

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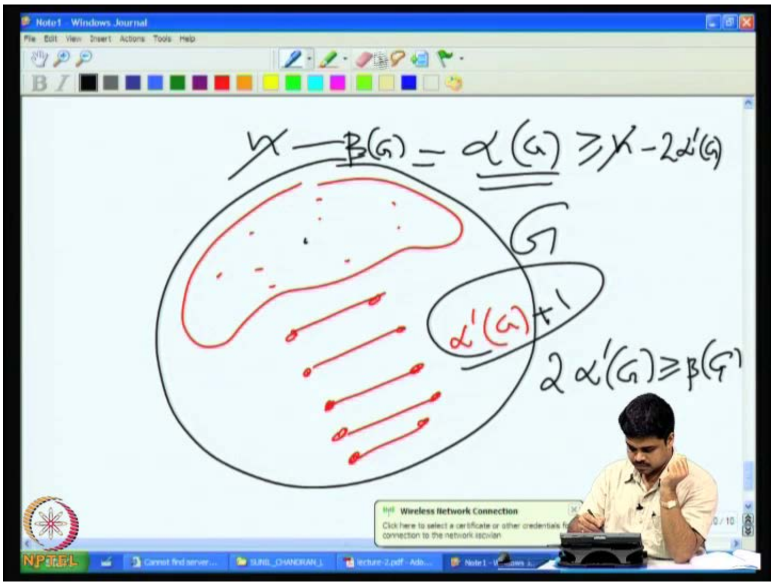

So, here is one relation we can see that alpha of G is always greater than or equal to the number of vertices in the graph minus 2 minus alpha dash of G. Why do I say like this?

So, let us explain this things suppose, this is a graph. I am just abstractly, I am drawing the graph like this is the green graph. And now if I suppose, if I identify some the biggest matching here, so how many of them are there? Alpha dash of G of them see this alpha dash of G, this number is alpha dash of G. So, this is somewhat there are other edges, but I am just marking the matching, which I have considered.

And now, what can I tell about the remaining vertices like yesterday, also we had consider this kind of a question; what about the remaining vertices after removing these things? If I remove these two times alpha dash of G vertices alpha dash of G here, alpha dash of G here, two times alpha dash of G, what will remain say this remaining vertices, what can you tell about? Can we have any edge here? For instance, is it possible for need to have an edge here like this? It is not possible, because suppose if there is an edge like that, then we could have added this so edge also along with this collection and we would have got a bigger matching that means, these edges and this also together would have made a matching of cardinality alpha dash of G plus 1, which will be a contradiction, we know that the biggest cardinality of a matching in a graph is alpha dash of G.

So therefore, this edge will never happen never occur. So, we can say that this will not be there. So now let us, what we understand is, if you remove these vertices, the remaining thing what we see here is an independent set. Therefore, so this the biggest if I am looking for the biggest independent set in the graph, the cardinality of the biggest independent set is a alpha of G, and that number is going to be bigger than this number that means that is going to be greater than or equal to n minus 2 times alpha dash of G, 2 times n minus n minus 2 alpha dash of G. So, yesterday we had learned one connection between one relation between this parameter, and the minimum vertex covered beta of G is so that it was n equal to n minus beta of G, n minus beta of G. So, n minus beta of G is equal to alpha of G greater than or equal to this.

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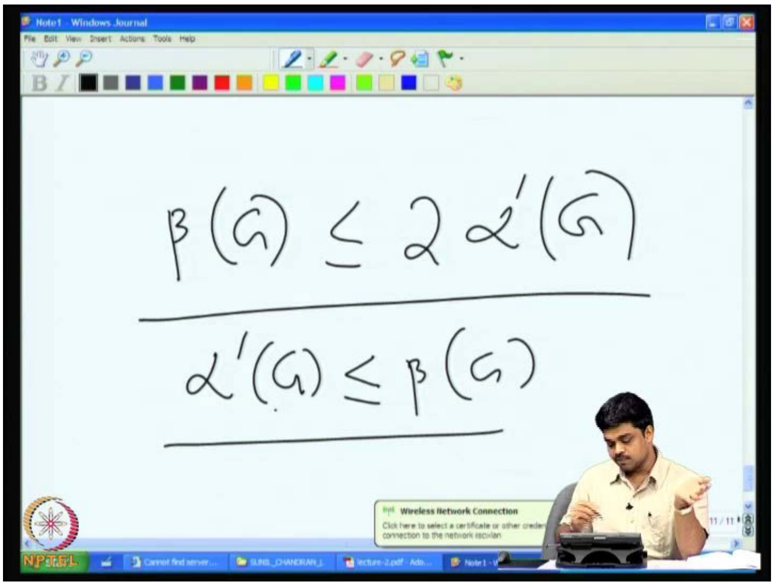

Now, if I cancel this and then rearrange this we will see that two times alpha dash of G is greater than or equal to beta of G; this is from this is what we are what we studied in the last class. So, what we get is beta of G is less than equal to 2 times alpha dash of G, this is what we get right alpha dash of G. So, this is what when I rearrange this thing, when I take this thing, I take this thing here. So, what I get is two times alpha dash of G is greater than or equal to this, but is there any relation between actually, a natural relation between this beta and this thing? So, alpha dash it we are saying that it is less than equal to two times alpha dash of G, but is it possible that it is even less than alpha dash of G or bigger than alpha dash of G. Some thought will reveal that this thing beta dash of beta of G has to be always greater than or equal to alpha dash of G, because if we consider any matching, let us go back to the matching.

Here if I want to consider a vertex cover, I have to select at least one end point of each of these edges, why because if I do not select this edge or this edge, then this vertex or this vertex in my vertex cover, and then this edge will not be covered; I hope you remember what vertex cover was suppose to do. The vertex cover was suppose to be a set such that every edge has at least one endpoint in the in this vertex cover right. So, suppose if I do not add this and this both of them in the vertex cover, then clearly this edge will not be covered by anyone. So, it is a must that either this should going to vertex cover or this should going to the vertex cover.

So therefore, the vertex cover cardinality has to be at least as much as the number of edges here that means alpha dash of G. So, that is what I have written here. So, the cardinality the vertex cover has to be at least alpha dash of G. But then not only that it has to be almost two times, sandwiched in between alpha dash of G and the double of it. So, this is an interesting relation.

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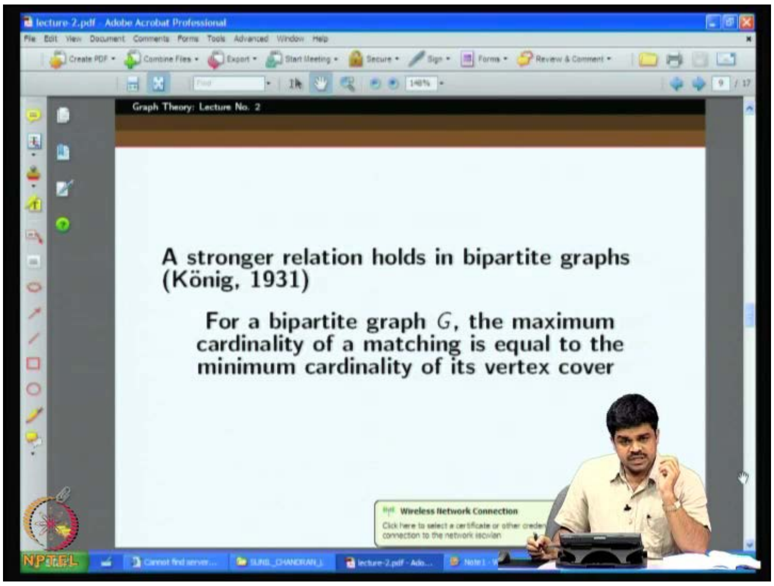

And but this is for general case, it so happens that there are some situations where this is we cannot improve with much that relation. But then if we consider only bipartite graph, we are not interested in all the graph, but only bipartite graphs, then we can get a better results stronger relation between these too. So, it is a theorem by Konig; this is theorem says if G is a bipartite graph the cardinality of a maximum matching, that is our alpha dash of G is equal to the cardinality of the minimum vertex cover beta of G.

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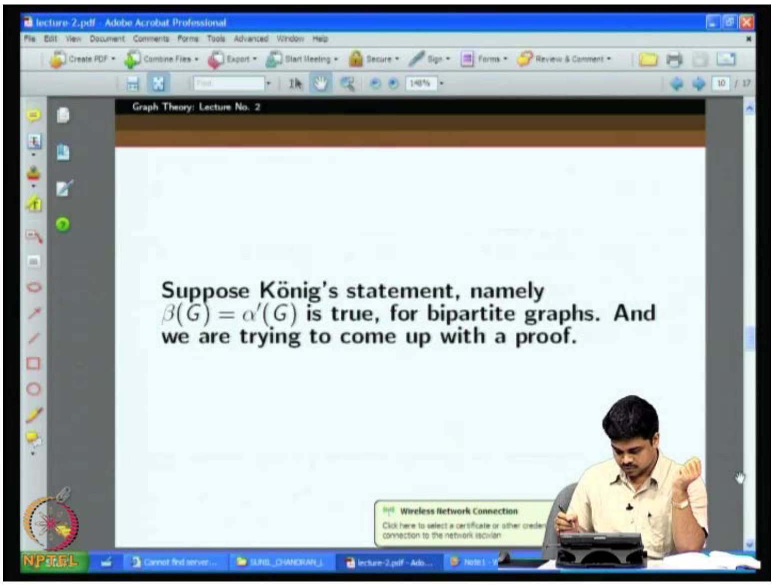

So, alpha dash of G is equal to beta dash beta of G what exist. So, why it is true? So, if we consider for instance you can consider some simple bipartite graph, you can check.

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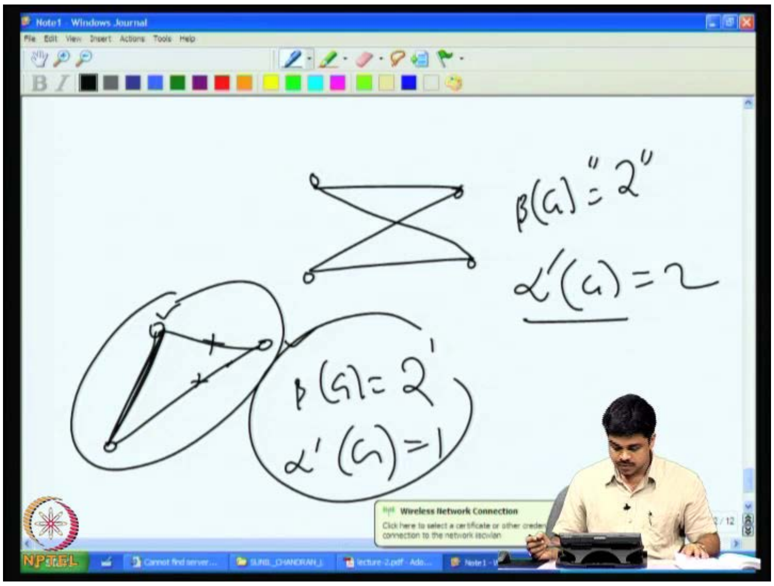

So for instance, if you take say here is a bipartite graph this circle. So, what is the cardinality of the minimum vertex cover here of case two? It is a cycle you have already seen that it is beta of G is equal to 2, what about alpha dash of G that is also equal to 2. So similarly, we can take some other examples and see. So as long it is a bipartite graph it is true, but you can try on a non-bipartite graph for instance here the triangle, it is a minimum vertex cover cardinality is two, why because if you want cover all the three edges, you have to take of them at least.

But on the other hand, if you look for the alpha dash of G, beta of G here is 2; if you are looking for alpha dash of G it is only one, because you can only take one if once you take this thing, you cannot take this or this right. So that is, but not, this is not a bipartite graph; so this is a theorem for bipartite graph. Let us now see, how we can prove it, the next aim is to come up with the proof for this result.

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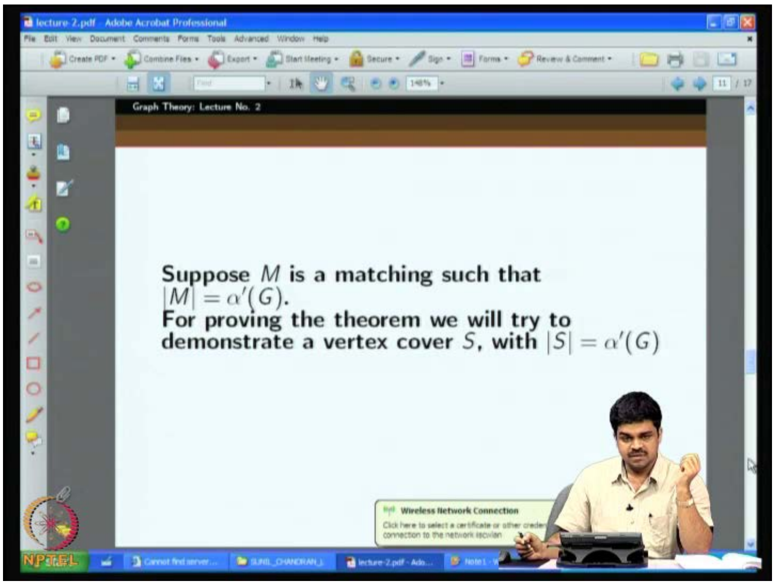

So, let say suppose Konig told as this statement, beta of G is equal to alpha dash of G. Now, when we if you are trying to come up with a proof. So, this is our approach; this is going to be our approach. We will first consider a Matching the biggest Matching in the bipartite graph. Let us say it is M, and the cardinality of M is in fact alpha dash of G. For proving the theorem, our approach will be to demonstrate a vertex cover S with the cardinality alpha dash of G, we will try to come up with a vertex cover such that it is cardinality matches the cardinality of the maximum matching.

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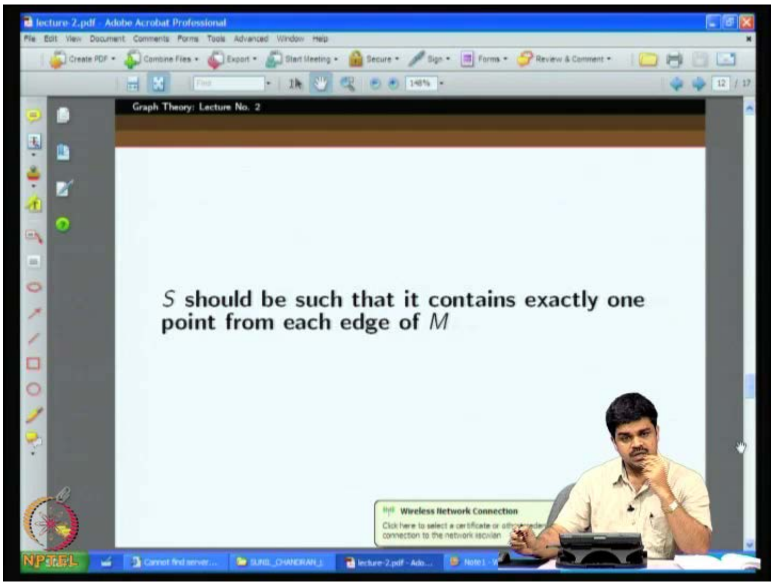

And as we have already seen, this cardinality of the vertex cover can never go below alpha dash of G, the cardinality of the maximum matching; why because the the matching is a collection of independent edges, each edge should have at least one of it is endpoint in our vertex cover, otherwise that it will not be covered. So, definitely the cardinality of any vertex cover has to be at least alpha dash of G. Now, but it can be more than that, we have to someone demonstrate a vertex cover of cardinality exactly equal to alpha dash of G; this is our aim. To do this thing, we will do the most straight forward thing, we will start with a matching in the bipartite graph, I will draw figure here.

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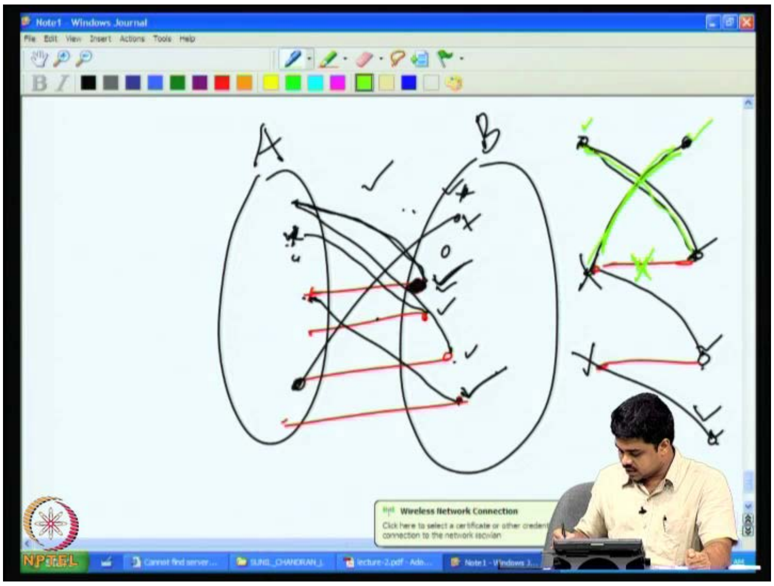

So, suppose I draw this. I will draw the bipartite graph like this, and then first I will mark a matching say, let this be the biggest matching here. So, there are many other edges, I am just marking the matching edges. Now, as we have already seen, if we are looking for a vertex cover, then it should contain at least one endpoint of this edge, this or this; similarly, one endpoint of this or this at least one this or this, may be both, this or this, this or this. Not only that, but we are saying the Konig theorem states the cardinality of the vertex cover is going to be exactly equal to this number, the number of the edges here, the matching, cardinality of matching. In that case, we know that if you take this and this for an edge, then definitely we are going to have a set S, whose cardinality is greater than the total number of edges here, greater than the total number of edges here.

So, the only way is to have exactly one endpoint taken from each of these edges; then we have a possibility of keeping the cardinality of the S, set S that we are selecting equal to the cardinality of the perfect matching. But then how do we make sure that it is going to be if I arbitrarily do that it need not to be a vertex cover, how do I make sure that we will finally, end up with a vertex cover to do this thing we have to observe certain things. So, let us say suppose, we see an edge so there is there is an unmatched vertex here, there are some unmatched vertices here; suppose, this is the A side of the Bipartite graph, this is the B side of the Bipartite graph.

So, considering an edge starting from here and suppose, these edges like this then clearly we are not going to select this in our vertex covers in our set. So, we have to select this one in S, because we have to cover this edge right at least this endpoint or this endpoint should be there in the in the vertex cover. So, this one should be selected.

Similarly, suppose there is another edge, if I can see an edge here like this, then since I am not going to select this one, I have to select this one. So, what would be an initial approach? I will go ahead, I will search for each of this edges coming from the unmatched vertices and the other end points will be selected. So, one may wonder whether, what happens if there is an edge like this starting from here to here is it possible, then which one will I select? If I select this or this both are unmatched, then I am going to get a set which is more than the cardinality of the perfect matching that is not possible that is not allowed. So, what should we do?

So, we cannot do anything because we have to since, we have to cover this edge either this or this should be there in the in the vertex cover, but then the good thing is that we will never have such an edge; suppose there is such an edge in the graph, then we can are you that there is a problem, because we told that this is the biggest matching ,then why do not we add this also to the biggest this matching, and then make it bigger that will be a contradiction therefore, this such an edge will never come here to here; that is why we want have any such problems. Whenever, we seen edge only one of the endpoint will be unmatched, the other end point will be matched.

Now, the next question is, so we have selected such things, is it enough? For instance, I have selected say this thing. So, I have selected this thing, so like that whenever, I show an edge going from an unmatched vertex using selected those things. See may be I will consider, because I am going to take one end point from each edge of the matching, then I will see which are the edges which are the endpoints, which I am taking from this side where from the B side and which are the endpoints, I am going to take from the remaining things, I will take from the A side. So therefore, so is it enough that I take these kind of vertices, which are at the endpoint of some edge which is coming from an unmatched vertex. Of case, it is not enough, because so for instance you can consider, an edge like this suppose this vertex is such that there is no edge going from this to an unmatched vertex on the a side but, there is an edge like this. And now since, I have selected this and not suppose to select this, then how will this edge be covered the only way is to select this one them right.

So, if I have an edge starting from here, and then I select this one and suppose I go via the matching into the other side and suppose, I see an edge going from here and reaching another matched vertex then that match vertex is to be selected, because this I cannot select, because I am planning to select only one from each matching edge each edge of the matching one end point from each matching. So, it happens that whenever I see a path starting from an unmatched vertex and reaching a matched vertex and then, I follow by a matched edge to here then again, I take in an non matching an edge from outside the matching, then I reach on the other side, and then I follow this thing. So then I this is called this kind of a path is called an alternating path the definition of the alternating path. So, we start from the A side, and then first it is an unmatched edge then it is a matched edge then is an unmatched edge then is a matching edge, unmatched edge like that.

So, this kind of a path is called an alternating path. So, I can from this thing carefully looking I can see that whenever I have an alternating path ending at a vertex of B, I will have to select this, because this is to be selected, then this cannot be selected, this is to be selected, this cannot be selected and this is to be selected then. So, all the vertices of B on a matching edge the end point, which is an endpoint of a matching edge, and is reachable by an alternating path starting from a vertex of A is to be selected otherwise it is not possible.

And there is another issue here, why do we say that? I say, because I selected this vertex I will not select this, but is it not contradictory, because I was telling that suppose there is an edge of this sort here, suppose this is an unmatched vertex, and this is a matching vertex. So this I am not going to select. So, I will have to select this right, so then how can I say that I will never select this thing.

So, here again interestingly this will never happen, why suppose this happens then I could have added this edge and this edge so may let us say let us use green. So, this edge and this edge instead of this edge, I can remove this edge from the matching and I can add this and this in the matching, so instead of one, I have added two. So, you have increase the cardinality of the matching, but that is not possible, because we have initially selected the biggest cardinality matching the biggest matching.

Therefore, this will never happen but this is an interesting kind of alternating path as you see. So, this starts with a vertex in the a side follows an unmatched edge then is a matching edge and goes to an unmatched vertex, this kind of alternating path is called an augmenting path. Why is it called an augmenting path? Because if you identify an augmenting path of this sort you can drop the matching edges in that augmenting path as path for instance this one here and then instead of those edges, we can put the other remaining edges that means the edges from the from outside the matching. So, that will be one more than the edges from the matching for instance here, one matching edge to non matching edge for instance, it can be in a more general case, it can be say for instance you can start with an unmatched edge, then it will be a matching .

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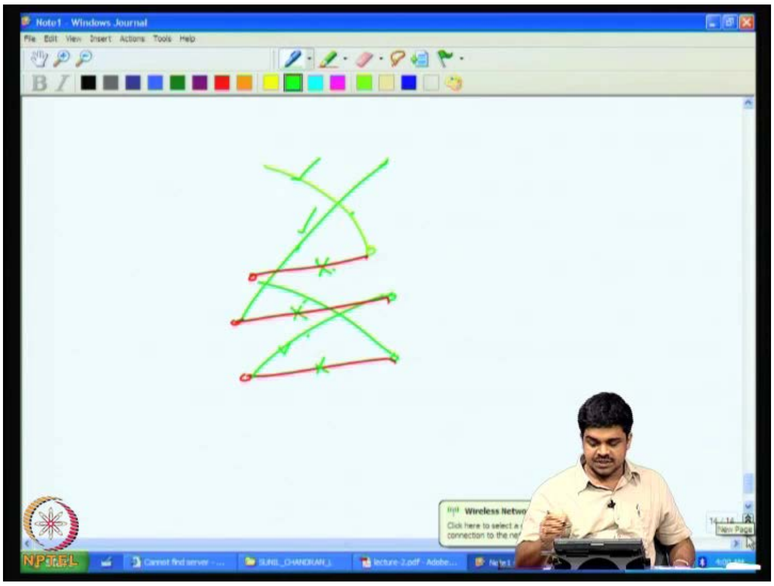

So, this is an alternative path that is why this is a matching edge always and then an matched edge, and then a matching edge then may be an unmatched edge, then there is a matching edge and then you go back to an unmatched vertex. Now, you see one, two three, four unmatched edges from outside the matching and then one, two, three ,only three edges of the matching now instead of this, this, and this if I had put this, this and this, this three four edges, instead of three edges. I would have increase the cardinality of a matching, which should be a contradiction if I had selected initially maximum cardinality matching, but if you just started with some arbitrary matching it is possible.

So, this kind of alternating path is called augmenting path, we have two kinds of some two notions which we will later also used alternating path, which is essentially a path is starts from the A side and then follows an unmatched edge and then matching edge, and then unmatched matching like that alternates between the unmatched and Matching and augmenting path is the special kind of alternating path, which starts and ends in an unmatched vertex therefore, as we can see the number of edges from outside the matching will be one more than the number of edges in the Matching edges. So therefore, you can augment by switching between the matching edges, and so once we have putting those other edges in the matching and removing the edges currently in the matching that is why it is called an augmenting path.

Now, you see this two notions this alternating path and augmenting path. So, using that we can complete the proof we are looking for that for the bipartite case, the point is you start from the A side. So, the our intention is to select one endpoint from each of the matching edge of the matching, each edge of the matching, but I will decide so the then it is either on the b side or on the a side so for which all match edges in the matching, I will collect from the b side is what here this how do I select that.

So, the rule is I look at the alternating path that is start from the some unmatched vertex of A and if it ends, in some of this vertices on the B side, then I will select that so which are also the endpoints of the matching edges, as we have seen that we have to select that right by our this think and all the remaining edges will contribute the endpoint from the a side to our vertex cover. So, the set s now you can easily see that this collection is going to give as a vertex cover why is it.

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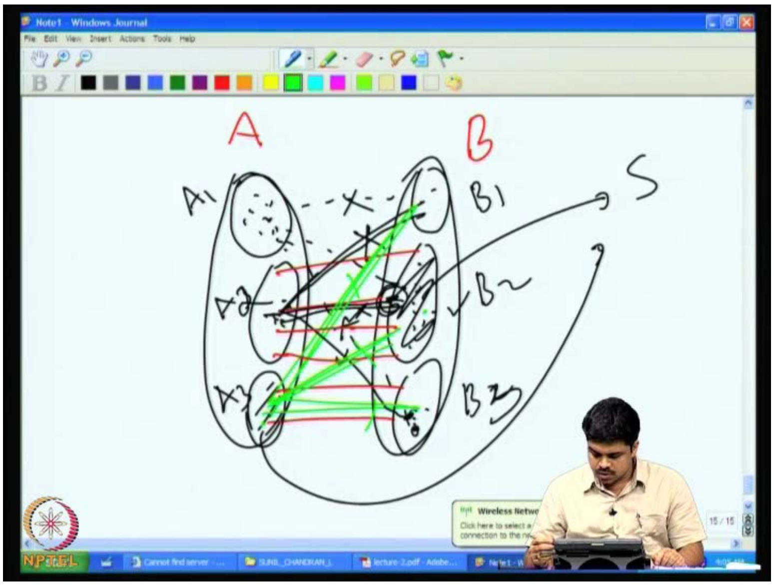

So, let me group the bipartite graph vertices into these categories. So, let us say this are the matching edges, we considered now these are there are three kinds of vertices: here on both sides this is the A side, this is the B side and here so one categories, I will call a one this is the unmatched vertices and the other category. It is the other category is the partners of the vertices from B, this is also B one unmatched vertices here this B 2 B 2 is the vertices, which we selected to be added to S right to go to the S or in other words they are the vertices, which appear at as the endpoint of the some alternating path starts from here and also the endpoints of the some matching edge and the now remove the partners. So, other than the partners is whichever endpoints are remaining here that is a three and this will be also added to s.

Now, we can see that this and this together covers all the edges quickly you can see that, as we have already mention there are no edges of this sort. Now, what about this edges between this is covered by B 2. Now what about edges between, so here also that is I said b three what about edges between this and this there are no such edges, because is no if there is an edge then it is an alternating path and this will, it will essentially this vertex has to be here in b two this vertex has to be in b two so this is b three.

Therefore, there is no alternating path between a one and b three. So, no edge between there is nothing to cover. Now, let us consider this so, what about this kind of edges is it possible to have an edge like this as we are already mentioned some alternating suppose, there is an edge like this some alternating path ends here starting from here then. Now, following like this you will get an augmenting path. So, that is also not possible. So, these edges are not there. So, we do not have to cover anything here and then, what about this thing, this is already covered by this and what about this thing? So, the edges between A 3 and B 2 we are already covered, what about the edges of this sort they are also covered there also not there, because if such a thing is there an alternating path reaches in a vertex here and then it goes to from follows the matching edge, and then goes to the B 3 therefore, between A 3 and B 3 also we do not have any edge the finally, we have to verify the edges which are between A 2 and B 3 also we do not have any edge.

Finally, we have to verify the edges between say this kind of edges so what about are we covering all of them so of course, these are all covered by. So, here what I have done is I have grouped the A side into three parts and B side into three parts and based on this three parts. I have three into three, nine category of edges in the bipartite graph. I have make sure that some categories are not possible some edges are nonexistent and whenever their existing then they are covered by either B 2 or A 3 that is what I have argued in.

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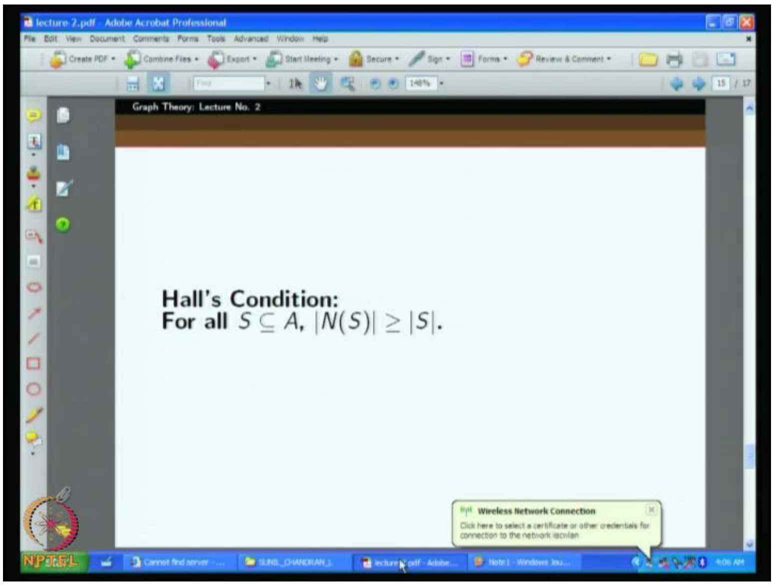

Now, what we have done is to get, a whatever done is to show that there x is a subset of vertices, whose cardinality is equal to the cardinality of maximum matching in a

Bipartite graph such that is subset is a vertex cover.

So, our minimum vertex cover has to be less than equal to alpha dash of G, but then we know that no minimum, no vertex cover can go below alpha dash of G. So, it has to be alpha dash of this is what the proof of the Konig theorem.

Now, the next thing is called a next result is called Hall's theorem, this is to in a bipartite graph, when can I say that a matching x a matching so for instance, if it is a bipartite graph with one of the parts being a and other part being b, when can I say that there x is a matching for the A be the side A side and B side is the two sides, and then one can is say that there x is a matching for this bipartite graph, in this bipartite graph which matches all the vertices of a matching of A when it is possible for instance if A equal to cardinality of A and cardinality of B is equal to equal, then it would be equivalent a perfect matching if A matching of A will be a perfect matching, but it need not be the case so the A may be smaller than B the cardinality of A may be smaller than B, in that case we are just looking for the cardinality of a matching, which can match all the vertices of A.

So, here is a result it is called hall's theorem which gives a condition for the existence of such a matching. So, hall's condition says you consider any subset of A, and then look at it neighborhood. So, what is a neighbourhood? So, for instance so let me draw the graph suppose this is our bipartite graph.

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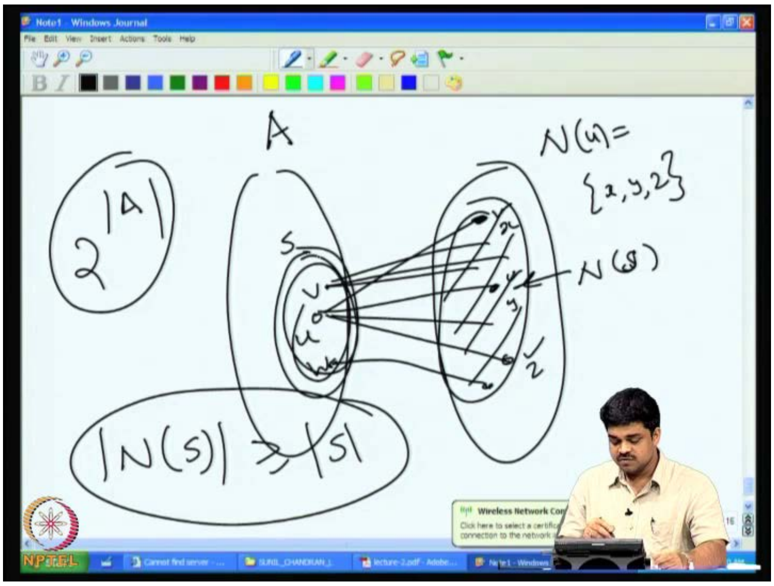

Now, this vertex has neighbours here see, if we look at the neighbors these are the neighbors of it if these are the edges going out of it this is the neighbourhood. So, if this vertex is named u, and then this is x, y, z, then we say that x, y, z the set of is the set of neighbours of u this is N of u, now we can also take a collection of vertices here may be u, v, w some and then look at the neighbor set of all these things for instance a vertex here which is a neighbor of at least one of them will be collected together and this will be called, the suppose this is s this will be N of S. N will be the set of vertices, on this side such that it is an neighbor of at least one of the vertices from this thing this is called N of S.

Hall's condition says suppose you consider any subset S of A and if it. So, happens that the cardinality of the neighborhood of that set is greater than or equal to the cardinality of the set itself then, we say that the Hall's condition is satisfied and Hall's theorem says if the Hall's condition is satisfied then there will be a matching that matches every vertex of a to checking, whether Hall's condition is satisfied or not may not be very easy because how many subset are therefore, A so definitely to the power a subsets are there right is it not? 2 to the power a subsets are there each of them we take and consider the neighborhood and count the number of neighbors and ask, whether it is bigger than the cardinality of A cardinality of that particular subset this may take exponential time, but again but, the condition is satisfied or not sometimes is it very easy to check, as we seen later in an example.

So, the Hall's theorem says if this Hall's condition namely if for every subset S of A its neighborhood N of S, as cardinality as needs at least as much as the cardinality of the set itself then they will be a matching it matches every vertex of A, we will look at a quick proof of this.

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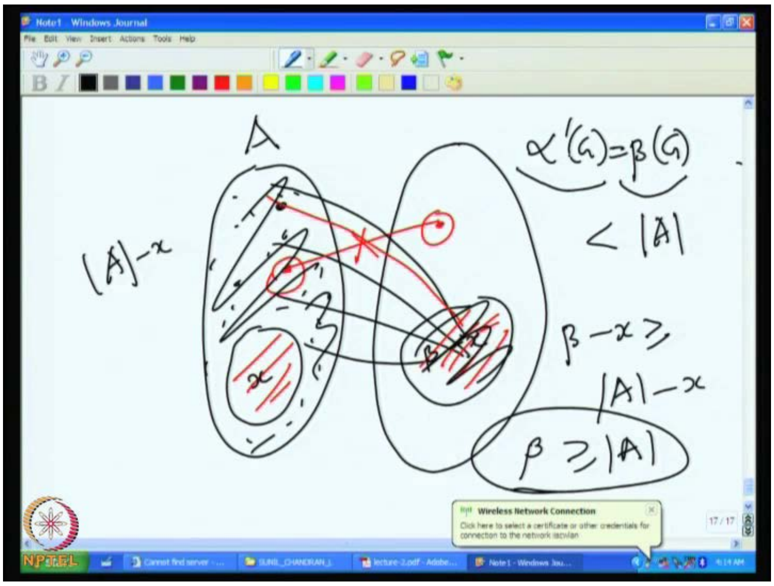

So, we will use the previous theorem the Konig's theorem about the cardinality. So the vertex cover and matching, this is the way we do this suppose not the Hall's condition is true, but we do not have a matching that matches every vertex of A in that case so let say there are some unmatched vertices, we will consider the biggest minimum vertex cover of this bipartite graph, we know that minimum vertex cover cardinality beta of G is going to be equal to the cardinality of the biggest matching, the minimum vertex cover cardinality is going to be equal to the cardinality of the maximum matching.

So, since by assumption our maximum matching is smaller than the cardinality of A that means, it is not matching all the vertices of A, this thing also should be smaller than A that means, this should be less than cardinality of A. Now we identify the vertex cover minimum vertex cover here suppose a little bit of it is here. So, let us x of them is here, and then how many of them are here beta minus x of them are there.

Now this being a vertex cover what can I tell about the edges, which are starting from a vertex outside, this how many of them are here here there are cardinality of A minus x vertices here outside x, because x vertices are in this vertex cover remaining number of vertices is a minus x and of course, this x cannot be equal to a there should be something left a minus x.

Now, all the edges starting from here has to go into this why suppose, you have an edge like this then who will cover this edge, because this is a vertex cover this is a vertex cover this has to cover all the edges if I have an edge like this, this two end points does not belong to the vertex cover so that is not possible. So, all the edges starting from the A side has to go here. So, it is happens that the neighborhood of this a minus x vertices outside this x this vertices is only this much. So, what do we get by Hall's condition beta minus x has to be greater than equal to A minus x A minus.

So, beta has to be greater than equal to A is it not? But this contradiction, because beta has to be equal to alpha dash and alpha dash we have already told is less than a therefore, we get a contradiction. So, by using now we conclude this lecture in this lecture, what we have done is mainly two theorems about matchings and bipartite graph, one is Konig's theorem, which says the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover. And then we have shown the Hall's theorem, which says that if the Hall's condition is true, and then we will get a matching of matching, which matches every vertex of A. In the next class, we will consider some applications of Hall's theorem and some more ideas about matching, thank you.