Graph Theory

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Module No. # 03

Lecture No. # 20

Adjacency Polynomial of a Graph and Combinatorial Nullstellensatz

Welcome to the twentieth lecture of graph theory. So, today we will look at a slightly different topic of case about the coloring problem, the graph coloring problem. So, the material for this will be available in the graph theory text book by U.S.R. Murty and bondy

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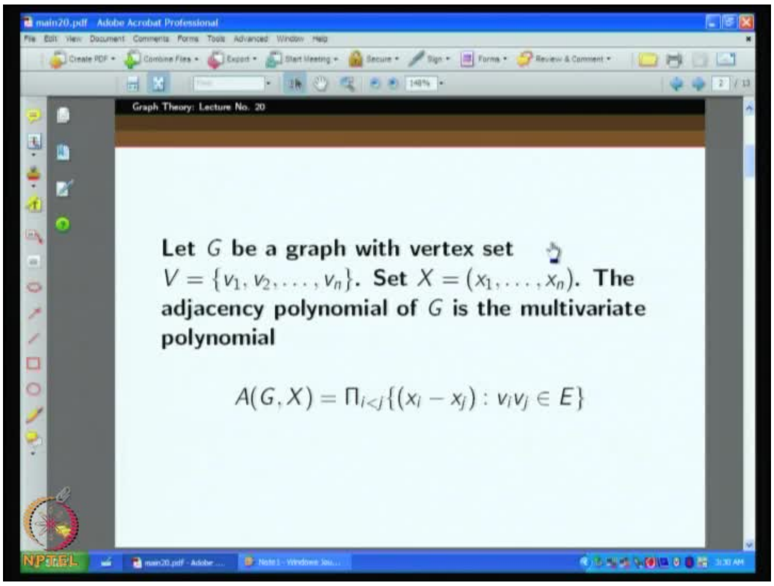

Earlier I was mostly following reinhard diestel book; this is taken from Murty and Bondy… Bondy and Murty’s book. Here, we are going to study certain polynomials which will help us understand the graph coloring problem. So, we are going to associate a polynomial with a graph. So, let G be a graph and the vertex vertex set is v 1, v 2, v 3 up to v n. And, we will define some variables, x 1, x 2, x n; corresponding to each vertex, we will associate a variable. Then, the adjacency polynomial of the graph G is the multivariate polynomial, the polynomial in these n variables. A G comma x equal to the product of x i minus x j, where these i j are… v i and v j is an edge in G.

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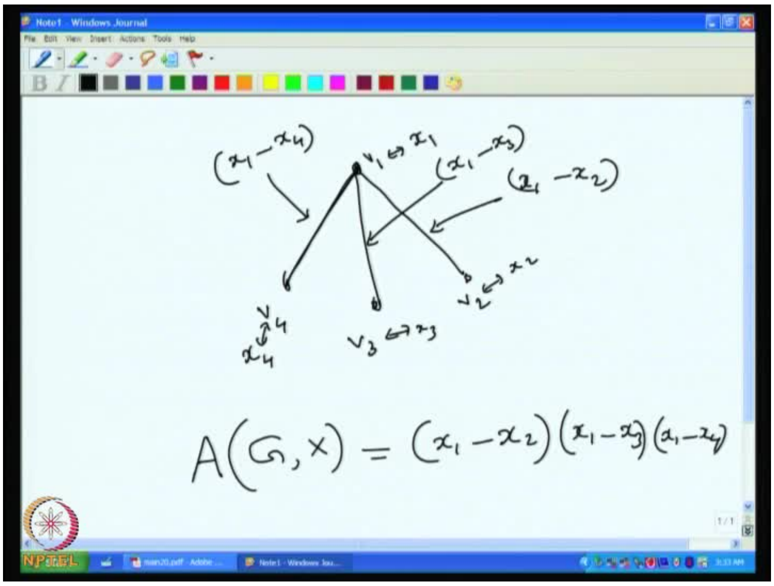

So, what it means is… so you can take a small example. For instance, let us consider this simple, very simple example (( )). So, let me say this is v 1 and I assigned the variable x 1 to it, right. This is v 2; therefore, x 2 is assigned to it. v 3, so I have a variable x 3 for that and v 4, variable x 4 for that. Ok?

Now, we have these edges v 1 v 4, v 1 v 3, v 1 v 2. So, our polynomial will correspond to these terms. First, for this edge we will have the term x 1 minus x 2, not x 2 minus x 1 because we are taking the smaller index first in the the higher index next; i less than j is what…So, this is this edge will correspond to the term x 1 minus x 3 and this will correspond to the term x 1 minus x 4. So, what will be the polynomial, adjacency polynomial for this thing? A G of x vertex will be equal to x 1 minus x 2 into x 1 minus x 3 into x 1 minus x 4. This will be the polynomial for this, right?

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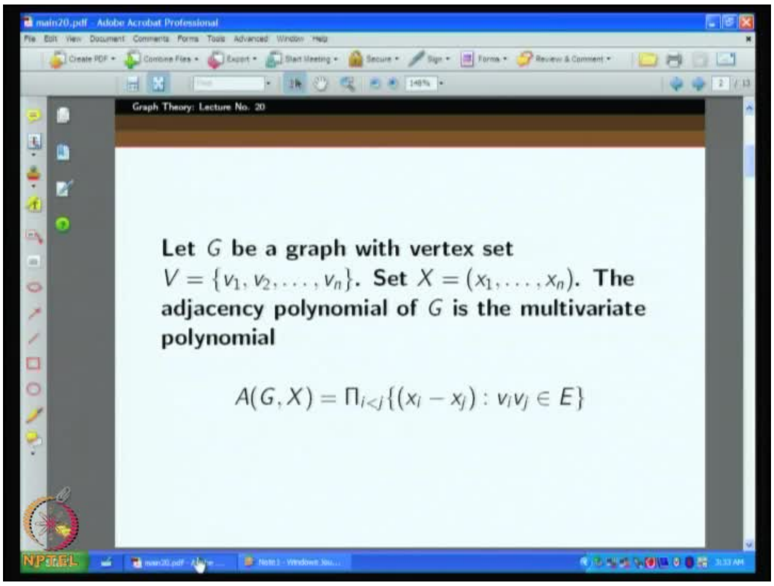

So, going back to this… this is the weight is defined. So, we our terms correspond to the terms this, x i minus x j terms correspond to edges v i v j and then, of case, x i comes earlier in x j if i less than j; that is the way this.

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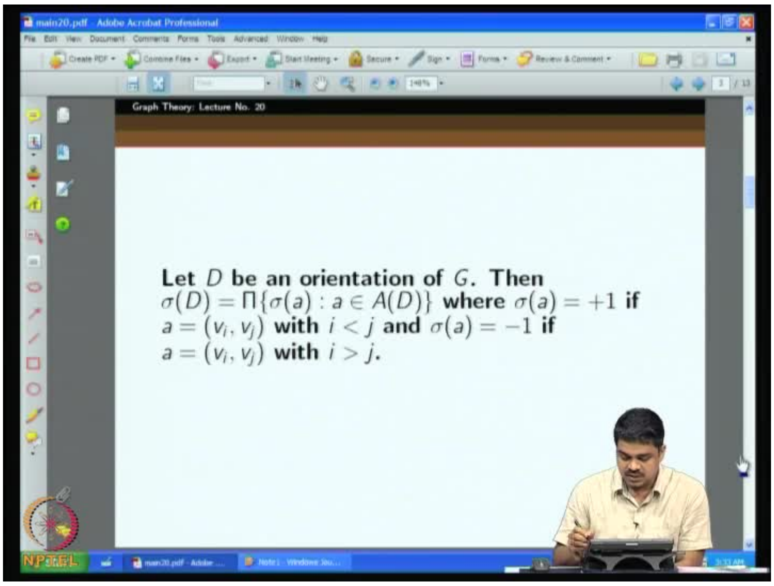

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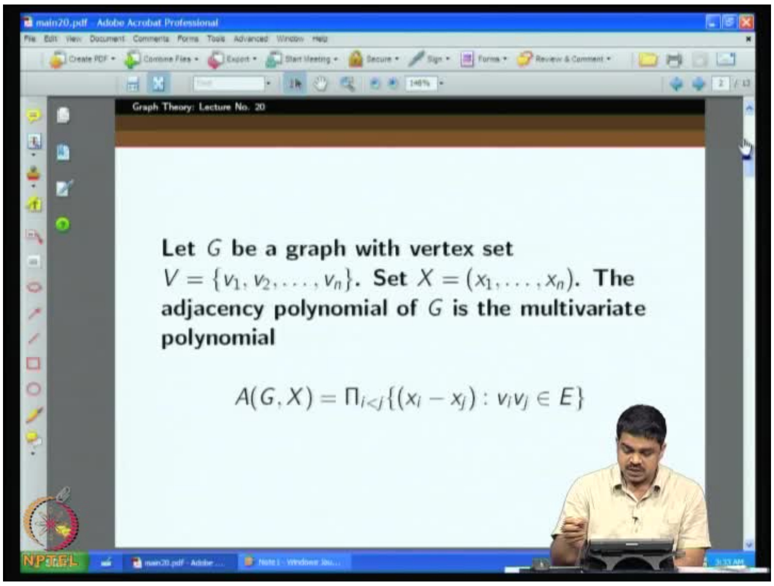

So, how does this polynomial help? Why do we defined this and we say that… why do we say that this is going to help as in understanding the graph coloring problem. See, you can see that if there is a coloring, proper coloring, vertex coloring of the graph, then… if there is a proper vertex coloring of the graph, then if we consider these colors as numbers, right and then this if we substitute x i with the color given to, the color given to the i th vertex, then we know that because whenever i and j is an edge. x i and x j will not get the same color; x i minus x j will not become zero

So, no term in this polynomial x i minus x j will becomes zero. Therefore, if we multiply these terms, what you get will be a non-zero value. Right? This will evaluate to non-zero if you asign the value of the color to the corresponding vertex. So, coloring would corresponds to a a solutions of s… the values for x 1,, x 2 x 3 etcetera such that the when we substitute these values the polynomial evaluates to an non-zero value

So, if it evaluates a zero zero value, then is not a coloring. On other words, any solutions, any assignments of values to exercise such that this polynomial evaluates to a non-zero value will be a valid coloring, by proper coloring. Why? Because even in one case x i minus x j happens to be zero, the polynomial will have to evaluate zero. That means x i can never be equal to x j if i j is an edge

So, this is why the this polynomial becomes interesting for the case of the graph coloring vertex coloring problem. So, that is why we decide to study this polynomial, right?

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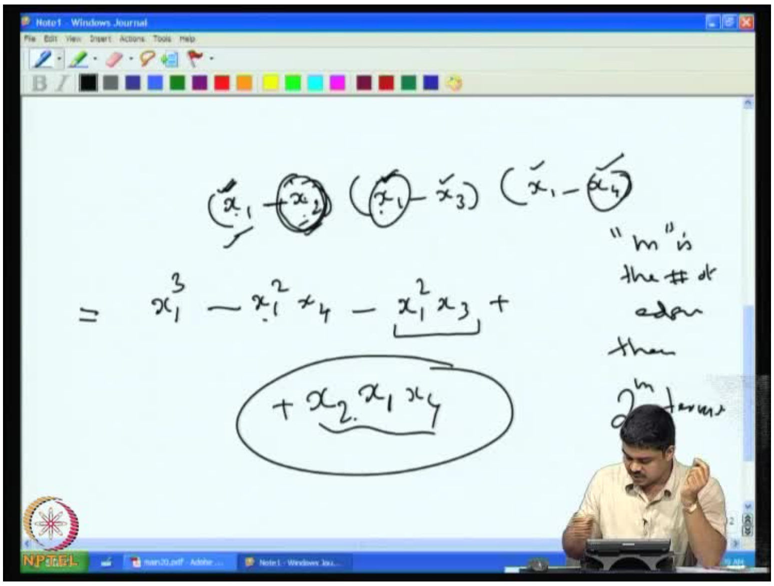

Now, so that much is… but this is more or less keep observations but, how can we understand this polynomial better? Of case, one obvious way is to expand it. Right? So you want to expand it and look at what kind of terms will come, right? So, for instance in our case, let us look at x 1 minus x 2 into x 1 minus x 3 into x 1 minus x 4. This can be expanded… so like, so when you expand, how do how do we get it? You take one term from here, one term from here, one term from here. This is (( )) x 1 cube plus… then one term from here, one term from here, and then other term from here. So, x 1 square into x 4. What is the sign of that? Because this is two pluses and this is minus, therefore, this will be minus; this is minus term. And then, now one term from here, here it (( ))…the second term can be changed and then third term. So, again x squared x 1 square into x 3; so minus. This is minus, right?

So then, we can take safe. So, like that so you can generate all the terms. How many terms you can generate? Because there are two possibility of taking terms here, so taking, picking up the the multiplier from here. So therefore, and also there are how many equations equal to the number of edges terms are there? So m is the number of edges. If m is the number of edges, then we will have two to the power m terms; two to the power m terms will come. So, for instance, you can, you could have picked up here this minus x 2 and you could have picked up this from here and you could have picked up this one here. That is a minus x 2 and minus x 4 and plus x 1 will make a total plus. So this will be x 2, x 1, x 4; this kind of terms.

So, you see that. So, corresponding to each edge, we can pick up a term and then we get monomials; two raise to m different monomials like that. So, what does this monomials correspond to? So each monomial in the expansion has this sequence x 1, x 1, x 1 here and x 4 here, x 2, x 3, x 4; this is x 1, x 1, x3. So like that and also a sign. There are two things, the sign and the term itself- the constituent things

So but, first let us try to explain this monomial. The the the sequences of variables that we see there, like x 1 square x 3, x 2, x 1, x 4 like that. So, for instance let us look at this term x 2, x 1, x 4, that means we picked up this x 1, x 2, the second term. You know always these terms are written, the first term is smaller and the smaller index. The the first term has the smaller index and the second term has the higher index. So, I picked up the second term.

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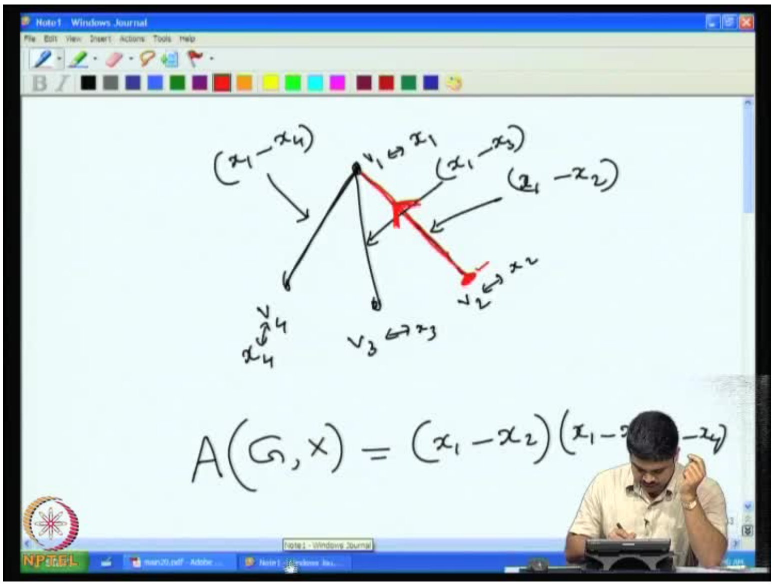

So, we can try to… so we will go back to other picture. Here, in this, for this edge, we decided to pick the second term. So, we can orient this edge from the second term to the first term, right? If the... this may be we can give a direction to indicate that essentially, this was picked up, right? The second term was picked up.

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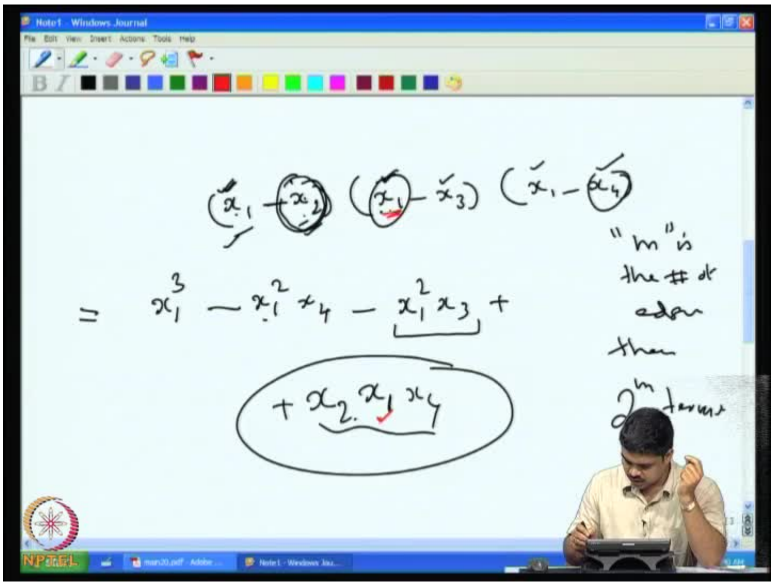

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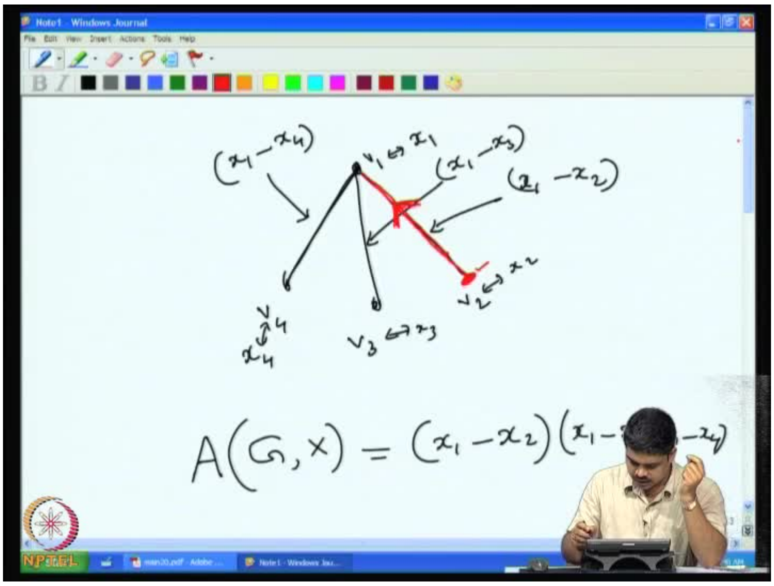

So, if on the other hand, for the next the next case… so here, the this is… this is the first term picked up, right? So, in that case, we can say here the arrow can be put like this. It means that the first term was picked up; the arrow is… the tail of the once again arrow to this. So, the tail is essentially the term which have picked up. The first and second or in other words the lower or the higher, right?

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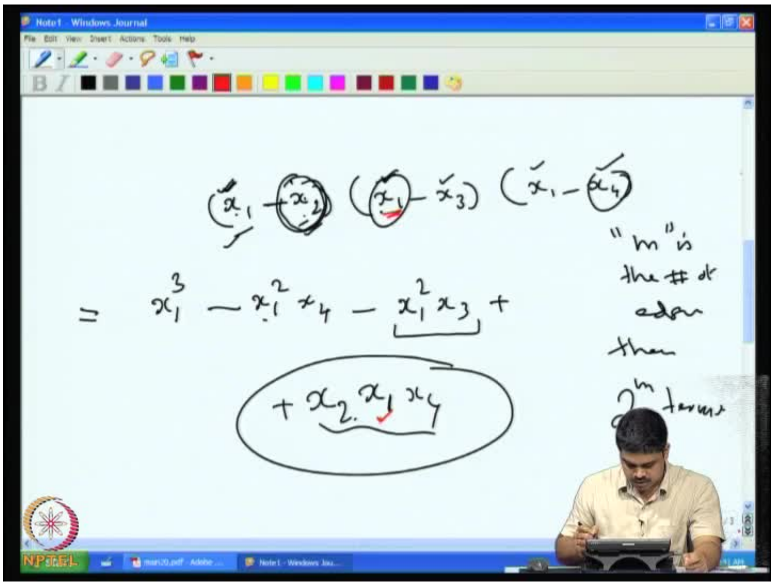

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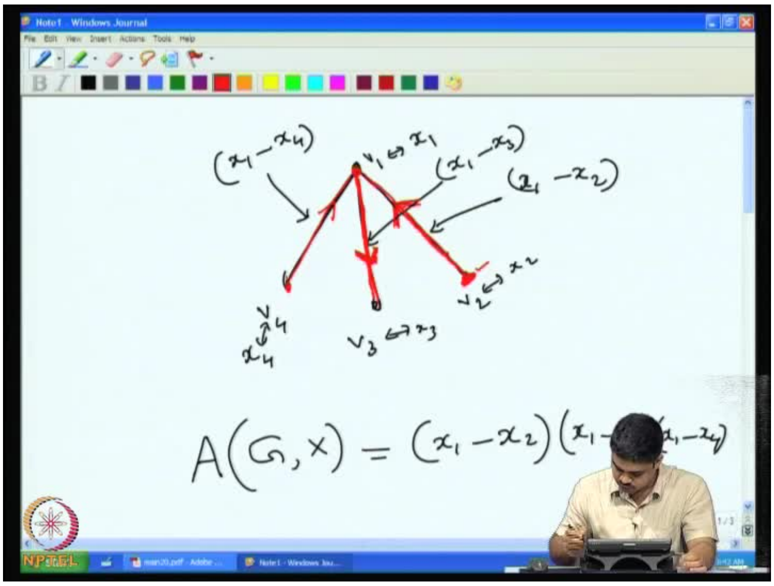

And, now this one x 4. So, if we go to the third term… so here, this third edge so here x 4 is picked up. That means we have to orient it this way, direction backward. So, essentially you can see that the way we have picked up the variables from each other terms, each other terms in the product corresponds to giving an orientation for each edge. So finally, when you make a monomial, that monomial corresponds an orientation of the total graph, giving some directions to each edge of the graph. That is what it happens.

Now, how will we interpret the the degree of a vertex for instance? Here we see that the outdegree is one. So, that means x 1 has occurred just once in that term. So, here we see that the outdegree of this thing is one, so that means x 2 has occurred just once. Similarly, here the outdegree is one.

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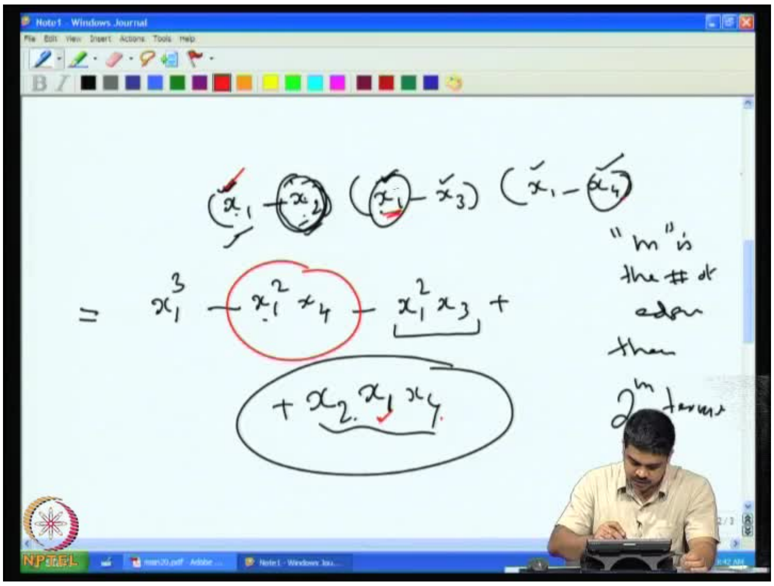

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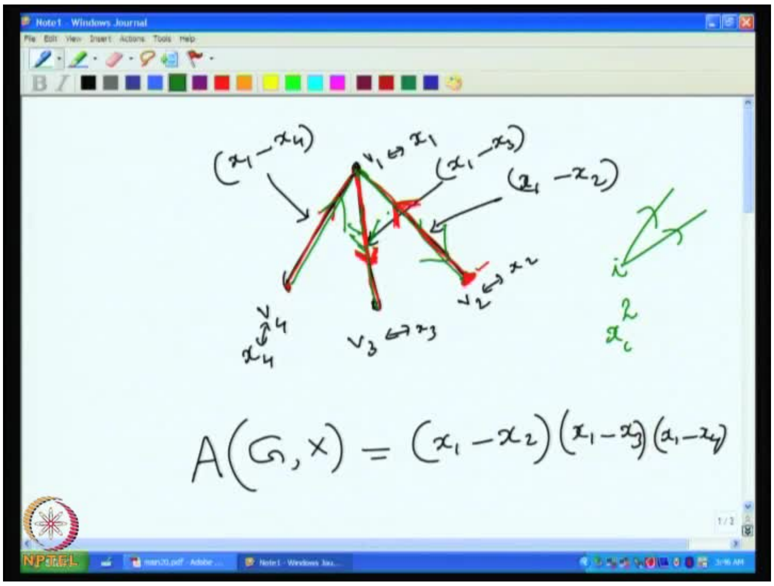

So, on the other hand, if I have picked up another term… say for instance... So, if I had picked up x 1 square x 4. So, x 1 square x 4, then, so that means here x 1 is picked up, x 1 is picked up, x 4 is picked up in the third one. So, that will look like… so I can mark it with green. So, here x 1 square, first x 1 picked up there, that means the directions is in the opposite way here and then here it is picked up this way; here it is picked up this way. So, x 1's outdegree with respect to this orientation is two, that is why x 1 square has come and x 2… right, x 2 did not … yeah, x 2 did not get any degree. So, there x 3 did not get any degree, outdegree. Therefore, it is it is not appearing in that term. Here x 4 has one appearance therefore, x 4 has a degree one

So, essentially to summarize, so each monomial in the expansion corresponds to an orientation. And, this orientation is with respect to how we picked up terms from each constituent term of the product, either the first term or the second term. We give the direction to the edges based on that. Right? So, now the… that is the way the orientations are… all possible orientations, two raise to n possible orientations are there. They corresponds to the two raise to n possible monomial that appear here, right.

Now, the next thing is to understand what signs we can get. We can see that the the same term, say, x 1 square x 3. So, may be x 1 square something else can come several times irrespective of… see, in this graph it may not happen but in some other graphs you could have observed that the same term can repeat several timess. Because what matters is if with respect to orientation, a particular vertex, say i th vertex has two outgoing edges from that, it appeared two times in the monomial. So, x i square will come there, right?

So, if the degree, outdegree sequence with of two different orientations were same, then the terms will read same. They will read same; they will be same. So… but then only thing is the sign may not be same; the signs may be different. It is possible that one of them may get a positive sign, the other may get a negative sign.

So, though there are two raise to m terms, some of them are same terms but possibly with different signs, then they may add up to make a zero. That means they may cancel out. So finally, when after even if this, if we really do the cancelations, then the expansion may not have all the two raise to m terms. Right? So… them, some of them terms may be lost. It is also possible that in some cases they may be add up and make a term like two times x 1 x 2 x 3 something. It is also possible.

So, it is important to understand that these terms also have signs. So… but then now we have to understand what the sign of a monomial may be. So, in terms of the orientation which corresponds to that monomial monomial. So, that is also not very difficult to understand because you see, whenever we are orienting the edge from the lower to the higher, that means we are working of the positive term. When we are orienting the edge from the higher to the lower, we are actually picking up the negative term. Therefore, in the orientation we will look at each edge, each directed edge and see whether the direction is from the lower to the higher or higher to the lower. Accordingly, we will say that that edge in that orientation, that edge is a plus sign or a minus sign. And now, we will multiply all these signs together and then we say that that is the sign of orientations that will definitely correspond to the sign of the monomial in the expansion. Right?

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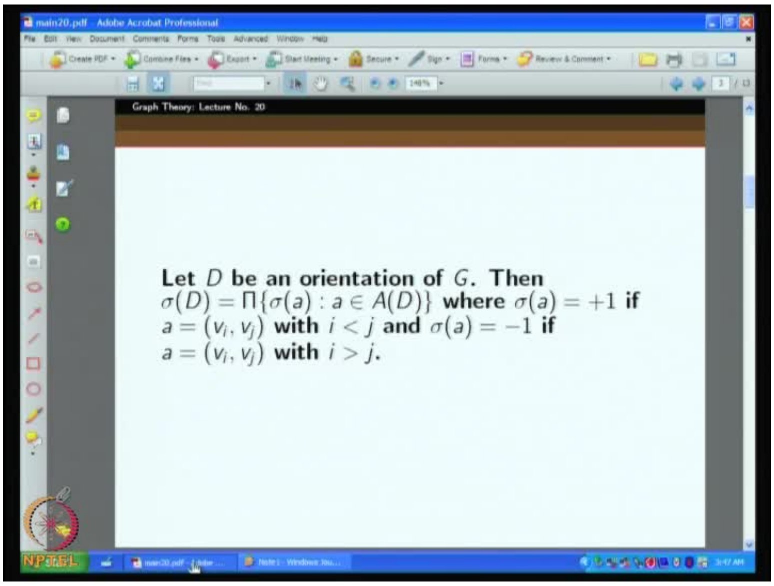

So, we can define it this way. d be an orientation of g, then sigma of d that means the sign of the orientation can be defined as the product of the signs of the a direct edges in that orientation; for each edge a in the directed edge set, edge set of the directed graph. So, you have to define a directed, define a sign and then multiply the signs. That is all. So, how do we define the sign of a a directed edge? So, sigma of a, a be in a directed edge, is equal to plus one. If a is equal to v i v j, where i less than j. It is directed from this smaller index to the higher index. Similarly, it will be negative if it is in… the edge is going from the higher index to the lower index.

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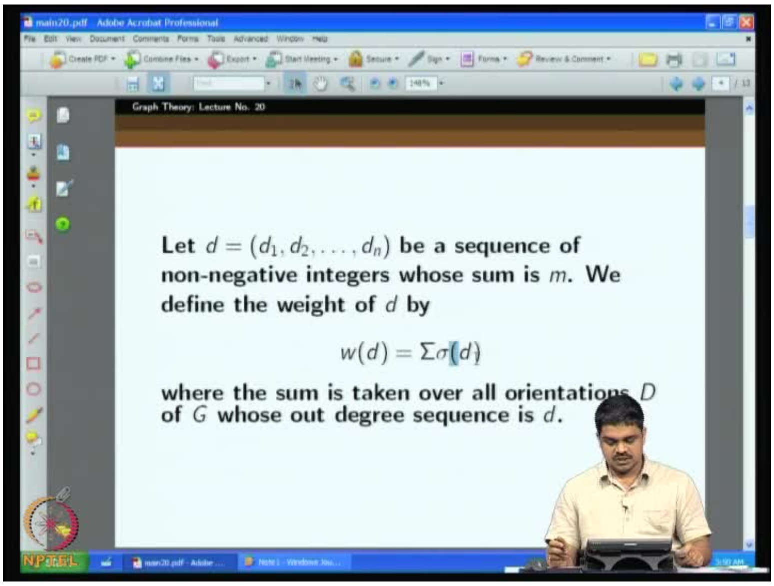

So this is the thing. Now, so you can pick up a particular degree sequence, outdegree sequence. That means the first vertex v 1 has degree d 1, outdegree d, second vertex has outdegree d 2, second third vertex has outdegree d 3 and n th vertex has outdegree d n. That is the degree sequence, outdegree sequence of a this thin.

It is possible that given given a particular orientation of the given graph and we are given an undirected graph g. So some orientation is given to the edges, some direction is given to the edges. Now, with respect to this orientation we can write on the degree sequence; v 1 (( ))degree d 1, v 2 (( ))degree d 2 like that we can write down. But, it is not that not true that each different orientations have different degree sequences. It is possible that two different orientations may have the same degree sequence.

So we collect, given a particular degree sequence d 1 d 2 d n. We collect all the orientations corresponding to that. That means all the orientations with that particular degree sequence and then we consider the signs of each of them, we sum up. This is the sign of the degree sequence, that particular outdegree sequence has this w of d equal to sigma. Sum of this signs of the, sorry, signs of the here… this is the mistake. So, it is say that signs of the orientation sigma of big d here.

So, signs of the orientations with that particular degree sequence. What is corresponds to in the expansion of that polynomial that we have these particular terms that x 1 raise to d 1, x 2 raise to d 2, x 3 raise to d 3, so x n raise to d n; this is what correspond to the degree sequence. There are several such terms so finally, what will be the… when you add them of what will be the coefficients? Because some of them are negative, some of them are positive. So you are just adding those coefficients together to get the total coefficient; that is the… that is the w of d. Because for a given term, the term can only be distinguished by looking at the powers on size, right? x 1 raise to what, x 2 raise to what? That will only be distinguishing between two different monomials monomials

So, we can collect them together and then the the coefficient added together and that corresponds to summing of the signs of the corresponding orientations. That is all it says.

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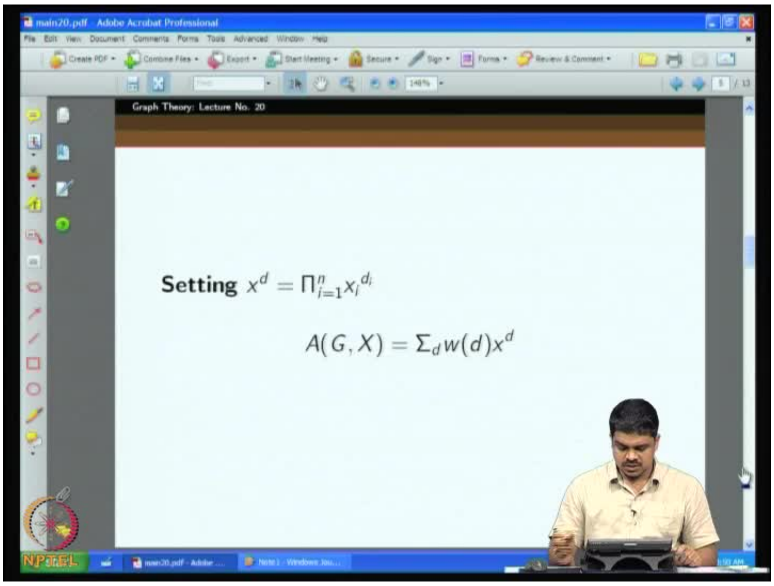

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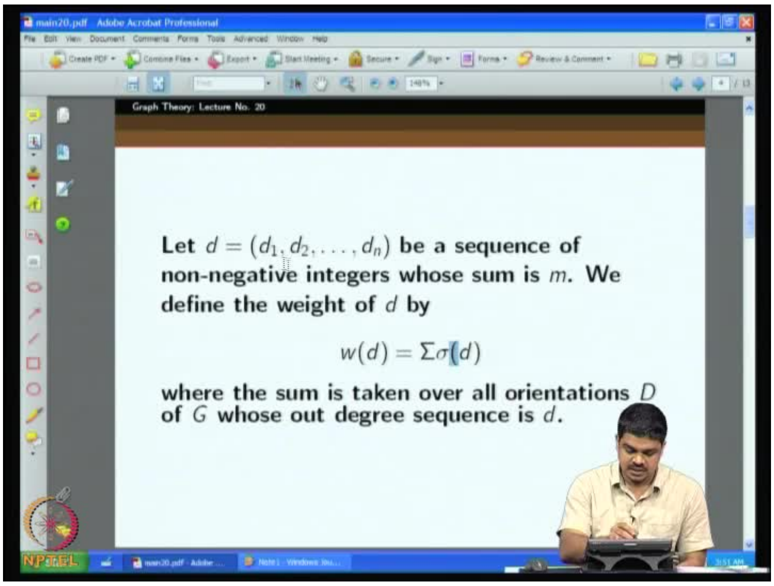

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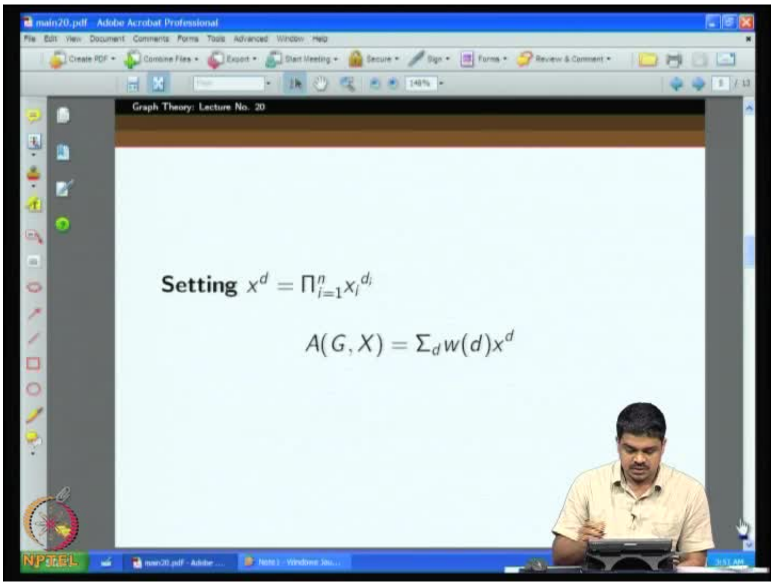

So now, the next thing is to say that for instance, you can say x raise to d, d being a degree sequence. d is equal to… yeah, so d is essentially this d 1, d 2, d n, right? And then, this x is sum the variable x 1, x 2, x 3, double double of the variables. So, that can be written, that can, that is a short form of a monomial in fact, x 1 raise to d 1, x 2 raise to d 2, x 3 raise to d 3, x n raise to d n. These monomials short form is x raise to d. Then we can write this polynomial a g of x has the sum over all possible degree sequences, the weight of the degree sequence, that is, sign of the degree sequence into… sorry, its weight of the degree sequence, it need not be plus or minus one. It can be something else also, weight of the degree sequence into that x raise to d. That is the monomial. This is the way we can write.

See, what is the good thing about writing like this? Because it will be much easier to understand when this can be non- zero with respect to with respect to as the orientations …when you want to study this polynomial with respect to the orientations. See, as we have seen, what we are interested in is in some way when these polynomials evaluate non-zero values, right? That is what correspond to the proper vertex coloring of the corresponding graph, right?

Now, to understand that we have expressed this polynomial as a sum of, see this with respect to that degree sequences we have expressed as a sum… the weight of degree sequence and x raised to d. And so the now we can we can always ask, suppose given a degree sequence, will… you can, can can you can consider all the possible orientations with that degree sequence. And if I sum up the signs of that degree sequence, will it be negative or positive? Such questions… or it will it be zero? Such questions can be asked, right?

So, it is, this is what we are going to make use of in the later proofs. So, now we want, see our intention is to use this polynomial, to show a result for the lists coloring... for the lists coloring of graphs. What we will show is, if g is given and it so happens that there x is an orientation of the graph, the edges of the graph in such a way that there are no directed odd cycles with respect to this orientation. And also that the outdegree of each vertex is strictly less than the list size of…list size, list size of that vertex. So, given a vertex, the cardinality of the list, the size of list is strictly bigger than the outdegree. In that case, in that case, we can always come off with a cycling edge coloring.

So, we have studied a particular, some theorem like that earlier. We have seen that if degeneracy is small, we can do that. For instance, if there is an ordering of the vertices and the higher number neighbors are small, then we can do a coloring, right? We we can do the coloring from the less.

And also, in the last class we studied that if this second condition is again same, suppose the outdegree of each vertex is strictly smaller than the cardinality of the list plus the property that every induce sub graph has a kernel. Every induce sub graph for the directed graph has a kernel then also like that if you can get an orientation for the graph, then also we can get it. Here there is different slightly different thing. We are saying that the same conditions, that is, the list should be such that the cardinality is strictly more than the outdegree. Plus the other condition is just that there are no odd cycles; odd cycles with respect to that orientation. If you can find even one orientation which avoids odd cycles, that is enough.

So, that is pretty interesting much, may be much more interesting than the statement we came up with yesterday. So, though we used it for proofing the bipartite graphs, right, in the in the this thing. So, here also, because the… here also we only want to avoid the odd cycles. So, the of case we can think about because bipartite graphs does not even have an odd cycle. So an orientation cannot bring odd cycles there, right? So, that would lead to that, this also will lead to that. Right?

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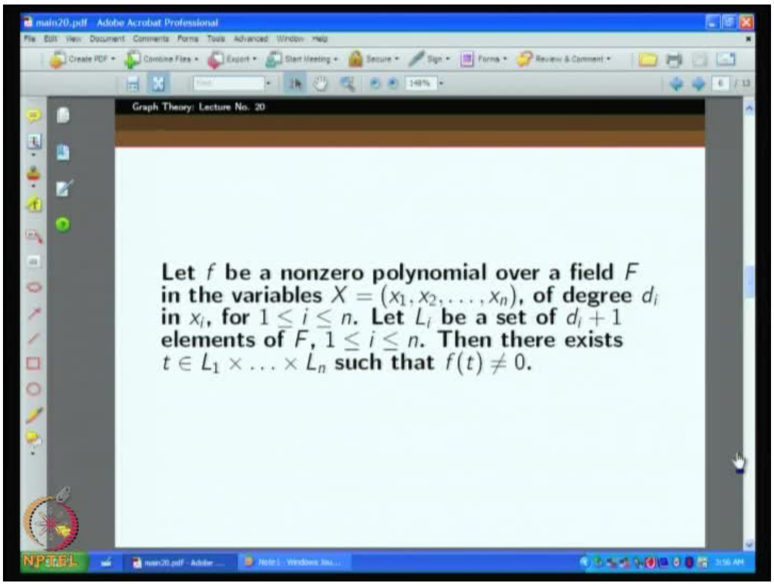

So then, here we are going to use some techniques. So, we have to develop these tools first. So, the look at this thing; let f be a non-zero polynomial over a field F, right? Let f be a non-zero polynomial over a field F in the variables x 1, x 2, x n of of degree d i in x i for each i. Let l i be a set of d i plus one elements of F for one less than equal to i less than equal to n. Then there exists a t element of l 1 cross l 2 cross up to l n, further f of t not equal to zero, right?

See, this statement is very familiar when we restrict our x to be just one variable, I mean if the polynomial is just in one variable, that is, there is only one variable, it is like saying that if you give me d plus one, d being the degree of the polynomial. That means x appears up to x raise to d and not x raise to d plus one or more, then you can only have d distinct roots. If there are d plus one distinct value, one of them when substitute for x will evaluate to a nonzero value. That form be a root because there are only at most d distinct root. This is a famous theorem, the fundamental theorem of algebra.

So therefore, we can… so we are familiar with it. But the only thing is we want to generalize it to a several variable case.

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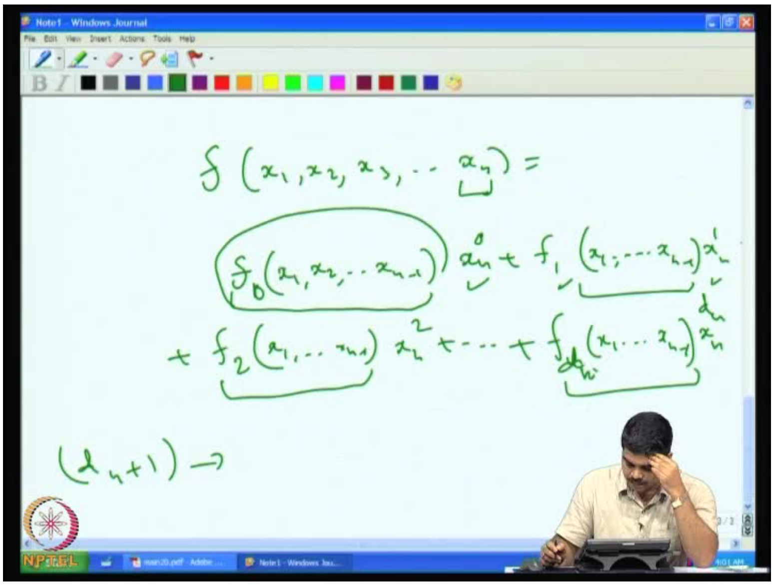

So, to do this thing what we can do is… yeah, so you can consider this function, sorry this polynomial f in x 1, x 2, x 3, x n. So, this polynomial, in this polynomial we can separate this x n and in fact it can be expressed as a polynomial in x n alone where the coefficients are essentially polynomials in the remaining variables x 1, x 2 up to x n minus 1. So will say f 1 is a polynomial in up to here and then, so x 1 x n raise to 1, right? So, and so we can say say f 0 into plus f 1 of x 1 comma x n minus 1 into x n raise to 1. So f 2 of x 1 to x n minus 1 into x 1 raise to 2 and so on. So f n into… sorry. So the essentially we are saying that the degree up to which it can go is say f d i… d d n, right, f d n. So x 1 to x n minus 1 into x n raise to t n. So, we are restricting the degree of d n to be… degree of x n to be d n degree of x 1 is d 1, degree of x 2 is d 2 and so on, right?

Now, we know that by because the one variable k is familiar; its known. Therefore, we can use this to infer that. So, if x 1 has d n plus 1. So we are given d n plus 1 distinct to values, then x n can get a value such that if this polynomial was non zero, if the total polynomial was non zero then it evaluates a non-zero value

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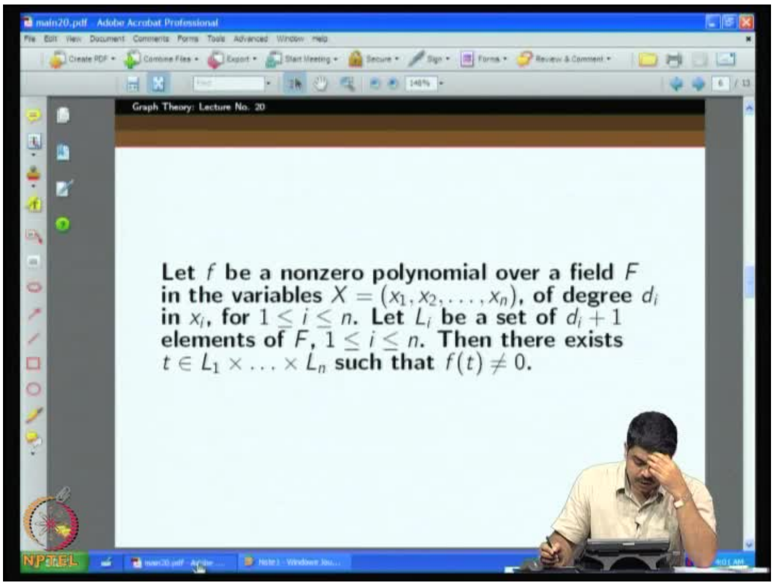

So, so why this polynomial non zero? Because you know this f 0 x n minus 1. So, they they all will become non zero for some values, right? By induction, right! Therefore, so since f 0… so each of this polynomials will get certain values, so we can make use of the induction hypothesis and so attach the. Therefore, by induction we will get that if there is a list of values for x 1 l 1 and if there is a list of values for l 2 for x 2 and if there is a list of values l n for x n, then we can pick up the some value from this d l 1 cross l 2 cross l n such that the when you substitute those values in the polynomial that evaluates to be non-zero. That is that is that is what will come. So now we can see that… so this is this is the by this is just proved by induction by just using the one, then we will generalized it to two and then it is to three. We know that then we can always make one of these polynomials non zero f 1 f 2… therefore, the total polynomial will become non zero and then we can add this selection from l n also with respect to that and then we will get total.

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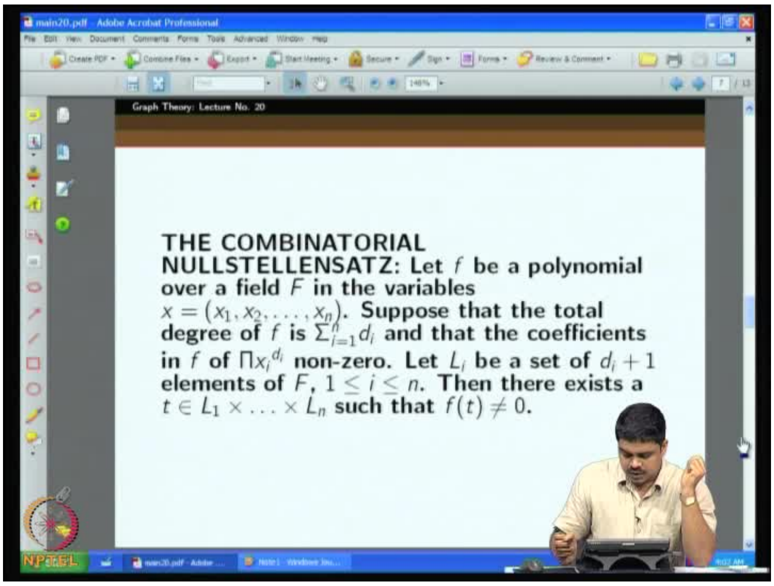

So so then the next one is a generalization, not a generalization. We want to… if you want to use it, we want to convert it into a convenient form. So we will make some minor modifications called the Combinational Nullstellensatz. So the …it is like this…so let f be a polynomial over a field of F in the variables, that is, x 1, x 2, x n. Suppose that the total degree of f is I equal to I equal to 1 to n d I and that the coefficient of f of xi di is non-zero. Here the difference is that we are not committing that this x 1's highest possible degree is d 1. Neither are we saying that x 2's highest degree is d 2. In the earlier case we had told like that. Here we are only saying that that total degree is going to be when you sum of the degrees of which monomial, it is going to be at is i equal to one to n d i so that is the some of the d i. But individually some of them can be bigger than the corresponding d i though over all the total will remain to be i equal to one to n sum d i.

And the the the another thing is we are not saying that the polynomial has polynomial is an non zero polynomial but, there is one term in it namely the x i raise to d i term. That is x 1 to the power d 1 into x 2 is to the power d 2 into x n to the power d n. This terms coefficient will be non-zero. These are the two conditions you are seeing so… And then we also need to understand what is d i. So d i is some parameters associated with each variable. x 1 has an associated parameter d d i d 1 and x 2 has an associated parameter d 2. I is an integer. So then what we are going to do is, so we have to pick up a list for each variable l i. For x i we will have a list l i. So, that we want to pick a value from this list for x i. And the cardinality of l i will be d i plus one: that is the relevant of d I, right?

So that is the way we are associating d i to x i. So the x i is the variable. We are saying that it has a list of values associated with it where… which are allowed for it and its cardinality d i plus one. Now, the compared to the earlier statement, here we are actually in some terms x i may have a degree greater than d i. But the total degree will remain same. So i is equal to one to n. Sorry, the total degree sum of i equal to one to n d i.

And the thus other condition is just that if you know this value d 1, d 2, d 3 etcetera and if you consider this particular term x 1 raise to d 1, x 2 raise to d 2, x 3 raise to d 3, namely x n raise to d n. This term coefficient will be non-zero

So, these are the only conditions. If these two conditions are satisfied then we can again pick up values from l 1 for a value from l 1 for x 1, a value from l 2 for x 2 and so on such that the total when we substitute this value, the polynomial evaluates a non-zero value. Then x is a t element of l 1 cross l 2 cross l n, such that f of t not equal to zero. This is what it says.

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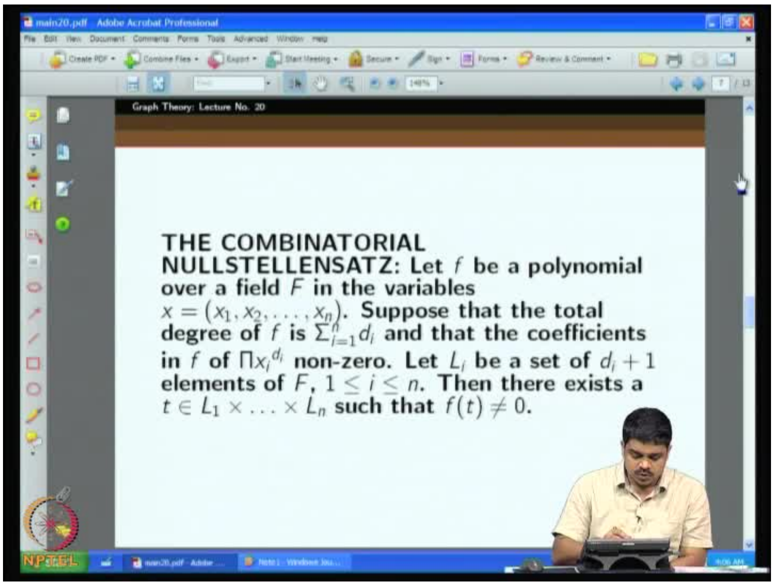

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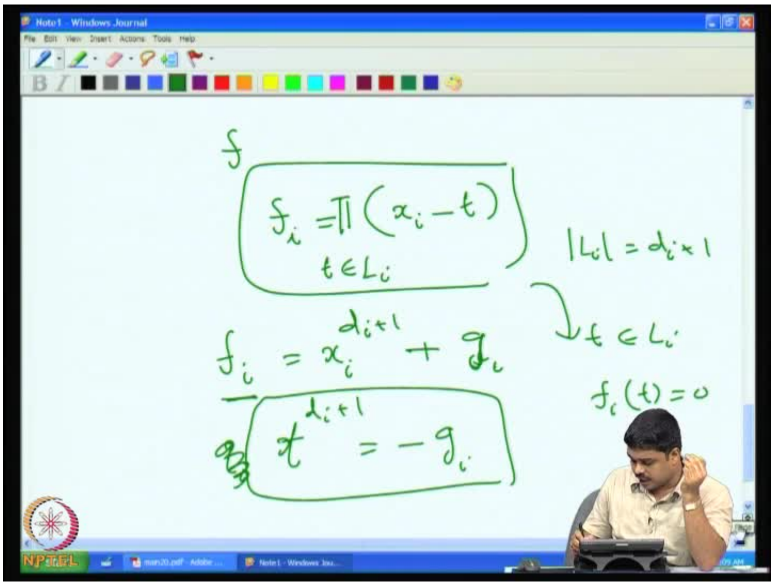

So, to prove this thing we can, what we can do is… so we will use the earlier statement, we will reduces to that. So, this is the way so our problem is it. So f is not a polynomial as we described in the earlier statement. So, it may have higher terms in x i.. x i has degree may be more than d i. We have to reduce it. So, we first will define some terms called some new polynomials called f i. So, what will be f i? f i will be just… from the list l I we will take all the values. So, we will define x i minus t sigma t from l i, right? So, we will create these terms x i minus t. So, there are essentially cardinality. So cardinality of l i is essentially d i plus 1, right? So, this will be a polynomial in x i raise to d i plus 1. The degree of this polynomial will be x i raise to d i plus 1.

So… and the first term, it will be the first term, it will be a long term. So, we can write f i as x i raise to d i plus 1 plus sum g i. So g i and g so such that the g i will be of degree d i, right? So therefore, we can say that f i equal to g i raise to d i. And, now the now the good thing about this is, if we substitute any value from l i in this f, in this, so for any t element of l i. So f i of t will be equal to zero. That means this this side will evaluate to zero. So g i of… so t raise to d i plus 1 will be equal to minus g i minus minus g i. So, this will come from any t we selected from this thing.

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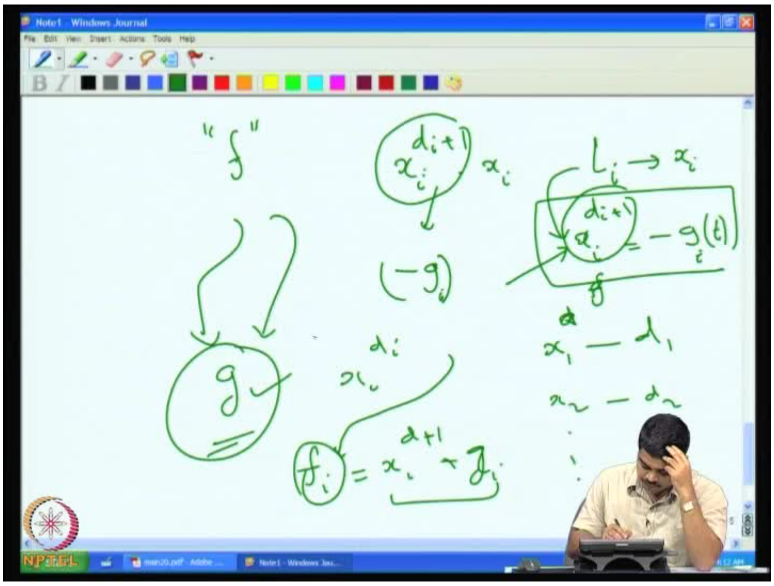

Now what we will do? We will go to f, we will we will consider the polynomial f and whenever we get we see x raise to, suppose some x i raise to d i plus 1, we will. So it can be right x i x i raise to d plus 1 into x i raise to something else, right, some smaller term. So, we can pick up this, separate this and substitute by minus g. So, then we will repeatedly substitute by minus g and multiply again, so every… because we are substituting the lower polynomial with lower degree, the degree will reduce because see other the because initially it has d i plus 1 degree, then only lower order. lower Lower degree terms are x i raise to d i and below or the degree will reduced.

So finally, if we keep on repeatedly doing it, finally we will end up with a polynomial g which is produced… so this is g I, so produce from f by this repeated substitutions and we will end up with by by getting a polynomial where all the degrees of x i is d i. Similarly, that means x 1 degree will be at most in this polynomial g i. x 1’s degree will be at most d 1 and then x 1’s, x 2’s degree will be at most d 2 and so on. This is what we wanted to… if you want to make use the other theorem, we want these conditions.

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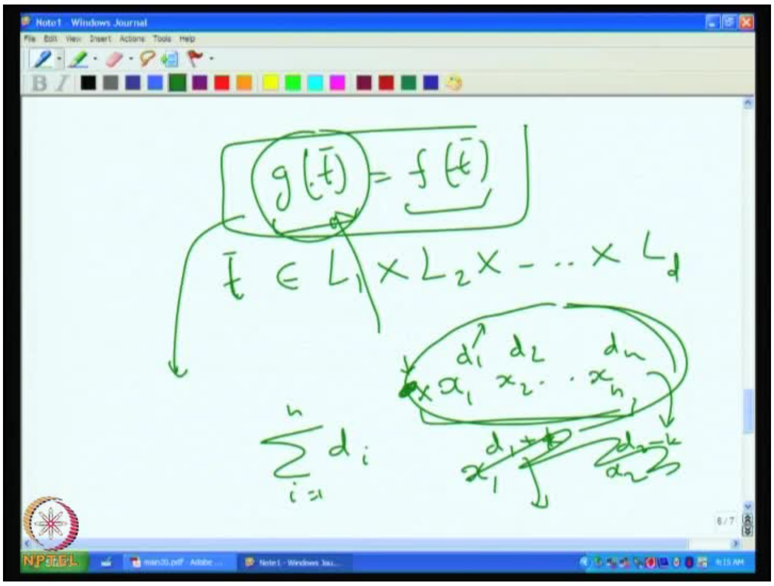

So in this statement we were not impressing that x i should is degree should be d i alone but, become we are allowing a higher degree also but, now we reduced it to d i, right? Now the polynomial the final polynomial will have degree only this much. And the the second point is, in fact, we only need to worry about this new polynomial g. As for as substitutions, when the values from this lists l i is substituted for a x i because because you are always substituting x i to the power d i plus 1 with minus g. But, we know that essentially if we substitute any value from l i to this x I, then essentially that current, that value will be equal to g of, g i of t also. That is what we substituted. Why? Because you know we know that f i is equal to x i raise to d plus 1 plus g I, so g i. And then for any value of t from the list l I, this will vanish, zero and this will be equal to minus g i. So , then we can we can when we when we evaluate in g by substituting a value for x i from the list l i. So g and f will evaluate same so we will we will have g of t equal to f of t. If t is picked up from, say picked up from l 1 cross l 2 cross so l d. This is what if if each value, if each x i got a value corresponding value from l I, then if you substitute g I, g or t does not matter, right?

And now so in other words to show that f of t can become non zero for a selection of values from l 1, l 2, l d for the corresponding variables, we only have to show that g of t g will become non zero for a selection of values. For x i get its values from l I, right, because these are for those such value these are same, right?

Ah so that is but, then now can we use the earlier statement? Even now we cannot use because we have to also show that it is a non-zero polynomial. Is it a non zero polynomial? So it is a… yeah, because the the the non non zero polynomial polynomial comes because if you consider this particular term x 1 raise to d 1, x 2 raise to d 2 and x n raise to d n. This is not at all changed, right?

So therefore, they will so and this, what is that total degree of this thing? This total degree sigma sigma d I; i is equal to one to n, right? And, if you look at every other term, the degree has reduced, right? The degree has reduced because originally the in f we had a nonzero coefficient for this thing and all other polynomials got its degree reduced because in some cases it may be d i either reduced or retain same because if it is all already small then we do not have worry but, on the other hand if it was like x 1 raise to d 1 plus something a some k and then for somewhere else x 2 raise to d 2 minus k, adjusting like that. Then we know that this got reduced therefore, the overall degree came down for that term, so for each term,

So therefore, this is the highest degree term and therefore, it means that suppose this is nonzero term, therefore, the the polynomial has to evaluate a nonzero nonzero value for some selection of x i because anyway there is a term which which dominate, right?

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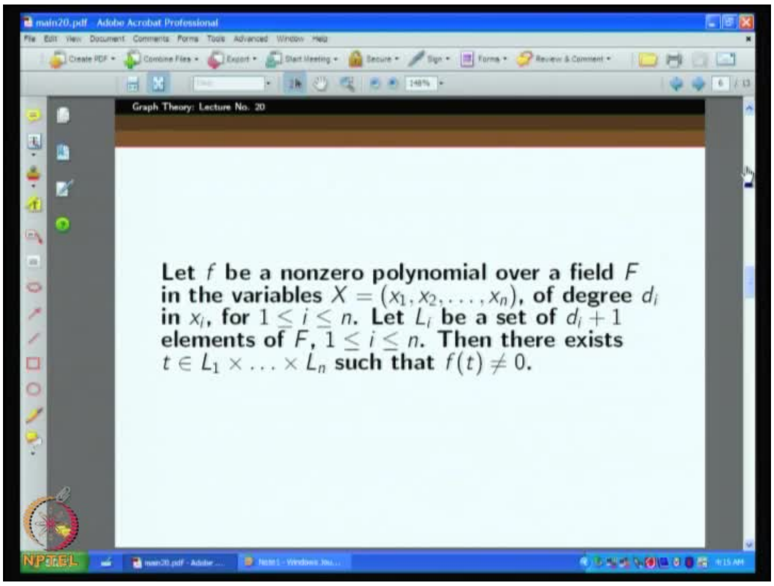

So therefore, we can say that the resulting polynomial is a nonzero polynomial. So why did we do that? Because if we want to apply the other theorem, the earlier theorem, we need earlier theorem means this theorem, the earlier theorem, we need it to be nonzero. So, we have g as a nonzero polynomial and also the degree of each x i is d I, right?

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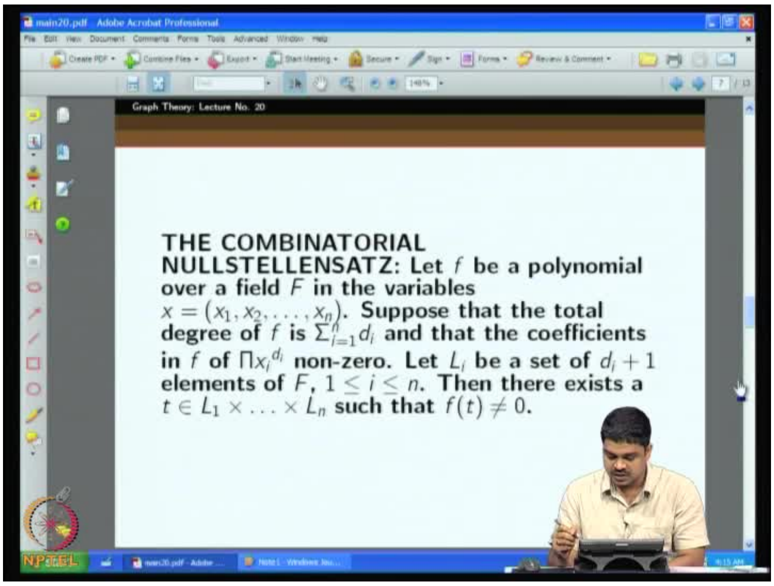

Therefore from the lists l i we can select to values for x i such that f of t will, g of t will evaluate not to zero. And therefore, the corresponding f also will not evaluate to zero. That is the way we constructed. So because we constructed g from… f in such a way that when if you are selecting values from the l I, the the g of t and f of t will be same. Now, we got it t from selecting from l i such g of t is not equal to zero. Therefore, f of t also has to be nonzero. So… this is it.

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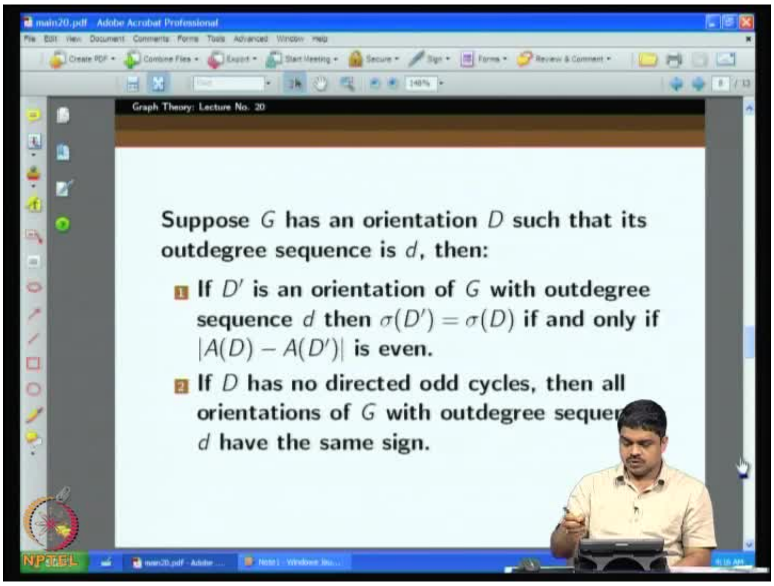

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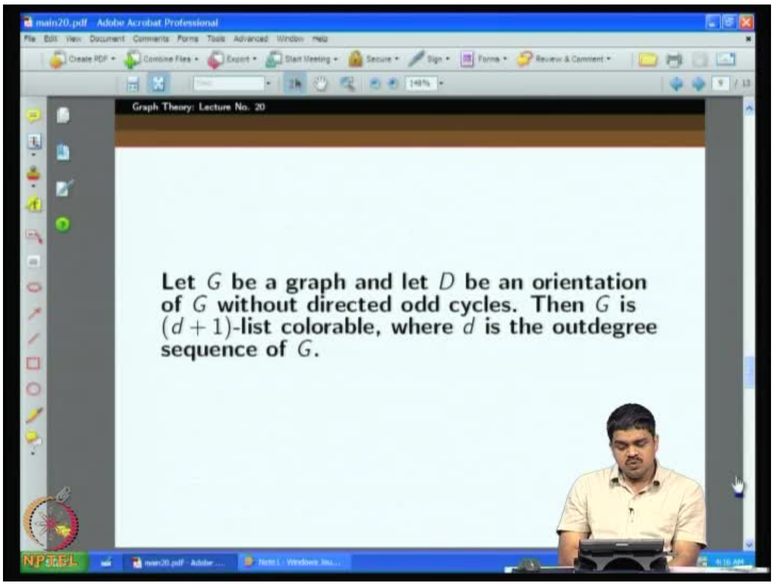

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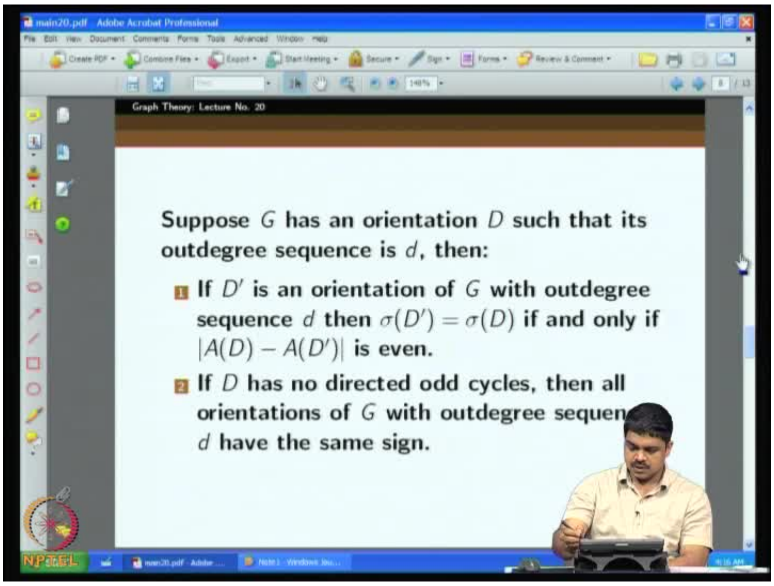

So, why did we do all these things? How are we going to make use of these things? So as I mentioned before, our intention is to show that if this is the theorem we want to show finally, if g is a graph and if there is an orientation of g without directed odd cycles, then g is d plus 1 least colorable, where d is the outdegree sequence of g. d is the outdegree sequence of g, right. So, this is what we want it to show.

Ah…now, to do this thing, so how are we going to relate this with the the polynomial and combinatorial nullstellensatz nullstellensatz? You can see that the essentially this lists l i corresponds to the lists of lists coloring problem. So suppose, so in the lists coloring problem we are given a graph and each vertex is given a lists of colors; we can say the lists of values. So you know that each vertex is associated with a variable so it is like asking that variable to pick up a value from the list namely the list of colors which happens to be a list of numbers, that list and in such a way that the polynomial does not evaluate to be zero why if if the polynomial does not evaluate to zero, it means that none of the x i minus x j terms have becomes zero. Remember the original the polynomial was the product of x i minus x j, where i and j are… edge i j are, i comma j is an edge, right? v i comma v j is an edge.

So, none of the edges standout be… turned out to be zero. That means… turned out to give a term which evaluate to zero. The x i is always not equal to x j. That means it is a coloring of the graph, right? So so if you can assign values to x I's from the given list, then that means it is least colorable, it is least colorable. Now, the combinatorial nullstellensatz says you can associate value to these variables from the given lists.

So, if certain conditions are known, for instance so the cardinality of the l i. So, let us say that is d i plus one. Now we have to make sure that the degree, the total degree of the corresponding polynomial, the corresponding polynomial is at most the sum of d i's at most the sum of d i's. So, which essentially is true because if we take any monomial, right, it correspond to some orientations and these… sum of degrees essentially will be the number of edges, right, number of directed edges. Because each directed edge will essentially contribute only one term, right, because the corresponds to the degree of contributes only one term. So therefore, that essentially is a total number of directed edges.

So therefore, the total is essentially for any directed degree to sigma d i's, right? So ones one… then the only thing is if you can find out an orientation in such a way that, so the the the second condition that means the x 1 raise to d 1, x 2 raise to d 2, x 3 raise to d, that term, the corresponding term is… has a nonzero coefficient: that is the only thing we will need.

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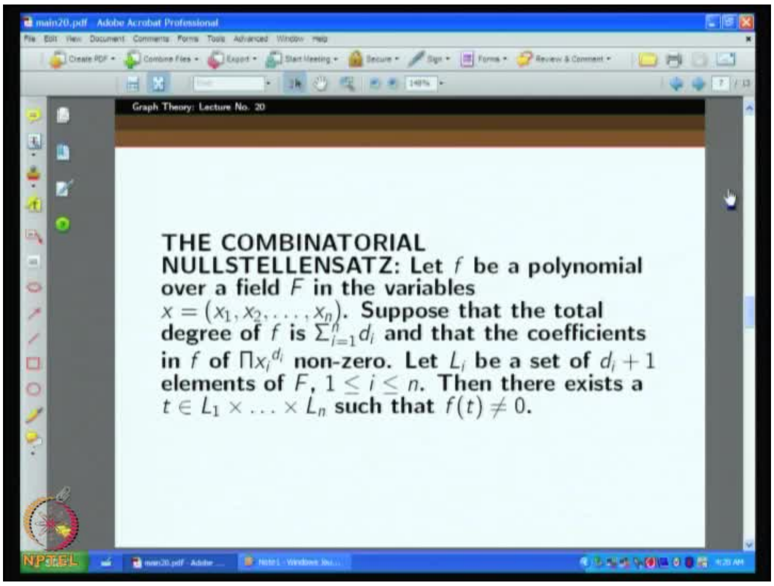

So you can see that in the combinatorial nullstellensatz, we have these two conditions, right? The… one is the total degree has to be sigma i equal to one to n. That mean outcome, so that is going to be the same for all orientations. So therefore, we cannot so we want to get the convenient split up, so we want to get a degree sequence in such a way that the corresponding lists, right, l i has degree strictly greater than d I; some degree sequence like that.

Suppose we can find a degree sequence for an orientation, if we can get an orientation in such a degree sequence that the corresponding outdegree is strictly less than the d I; that is all that is all we need there. And second this one, that means that coefficient of x i is to d i has to be nonzero. If we can show that, then we have a selection from the lists. So which will which will allow us to lists color right. So which will allow us to make the polynomial evaluated nonzero value. That means, which will allow us to a least color the graph, right?

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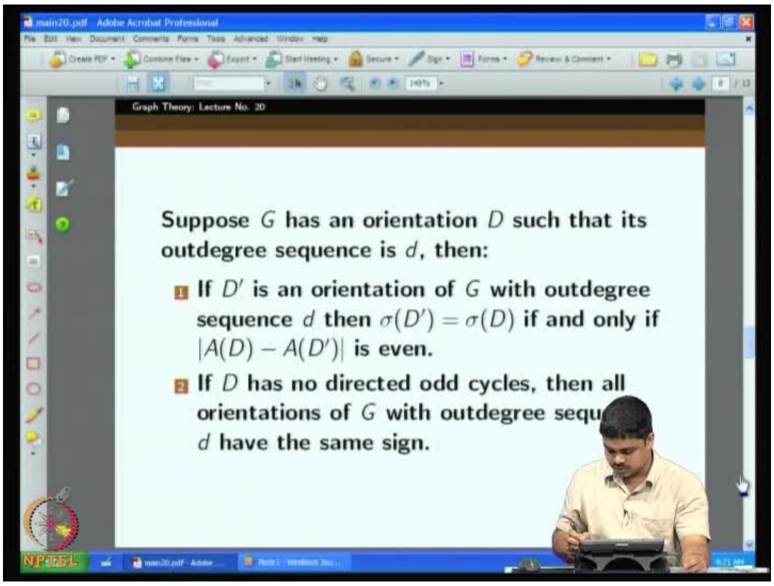

Ah therefore, we essentially we only have to prove that. So, suppose there is a degree sequence, right? So this is a statement we want to show, so suppose this degree sequence that the orientation, the corresponding term monomial x the the or the x 1 raise to d 1, x 2 raise to d 2, x 3…. x n raise to d n has a nonzero coefficient (( )). The weight of that w of d that, that weight of that degree sequence happens to be nonzero. Such an orientation if we can find, then we can say that this lists color right lists colorable, as long as the lists are just at least one more than the corresponding outdegree of each vertex with respect to the degree sequence.

So now how do we say that this term will evaluate to a nonzero term? So here is the reason because we are saying that if so this is the conditions one one which will ensure suppose there are no odd cycle with respect to that orientation, then corresponding monomial monomials for each for instance for that particular degree sequence any any of such orientation any such orientation with the same degree sequence has to add up because they want be anything to cancel. Because the will always have the same sequence; this is what this statement says.

So how do we show show this thing? So suppose D dash is an orientation of G with outdegree sequence d, and the the sign of two orientations will happen to be same if and only if the difference means A D minus A D minus (( )). There are some common edges between the two of them but, the remaining edges if we consider… so in A of D, in one of them in fact, because the total number of directed edges is same, this is the total number of edges. So, either you consider A D minus A D bar or A D minus A D, it is going to be the same value. So, if this is an even, then then we can we can say that the signs are going to be same because because the edges in the intersection of v, edges belonging to both the orientations, they will any way contribute similarly. The other things contribute like, if we see consider an edge D the contribution to D if it is plus the contribution to D plus D dash will be minus, right? If it contributes plus to D and we will contribute to minus.

So, if you keep on considering, say if we write down the edges even then, e 1, e 2, e1, e 2 like that and the first contribution makes it the first time when I consider it is plus and minus, then the next time it is again plus and minus. It will come back to the equal sign then the next time again multiplication will make it two different signs, again next multiplication will make it same sign. So if you keep doing this then you can easily see that if it is even number of edges how which are not in intersection, that means A D minus A D bar. Even number of edges direct edges, then definitely the signs have to be the same, right?

So, so that is what we need. Now, if D has no directed odd cycles, then all orientations of g with outdegree sequence D has to be the same sign, why? It is so because you consider any other degree sequence and you remove the intersection part and now because they have for each when you look at each vertex, the they have different signs and therefore, incoming degree and outcoming has degree has to be equal. We will we will explain this part, it requires some certainties. So we will explain this in the next class. So we will continue with the polynomials are related to coloring in the next class. So, thank you.