Graph Theory

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Lecture No. # 22

Gallai-Roy theorem, Acyclic coloring, Hadwiger's conjecture

Welcome to the twenty second lecture of graph theory. In the last class, we had discussed the coloring of digraphs. We were talking about the theorem called Gallai-Roy theorem which says every digraph we contains a directed path with chi vertices chi being the chromatic number.

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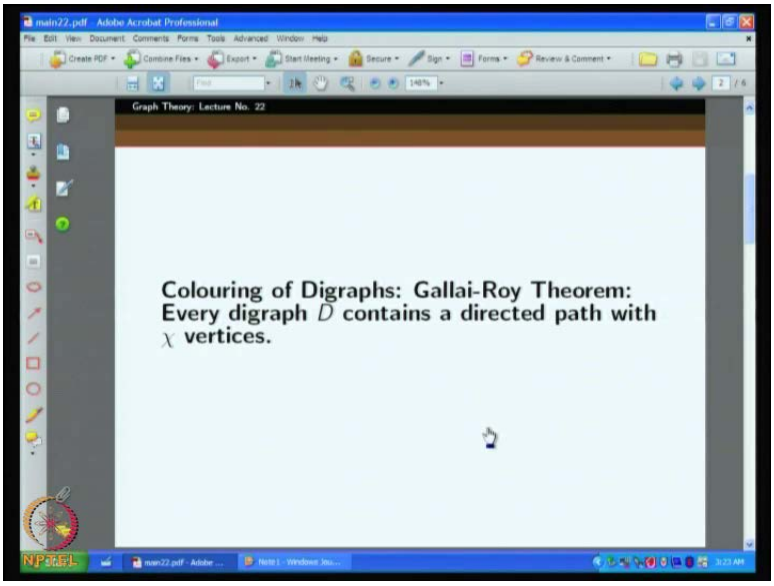

See the chromatic number of a directed graph is not different from the chromatic number of the underlined undirected graph. It is just that whenever we give colors to the vertices, we have to make sure that two adjacent vertices get different colors. So, here it is a directed edge rather than a undirected edge, that is all. So, it is natural to see whether the statement is valid in the undirected case, so we can which we can easily verify as follows. How do we do that? So, we know that there is a critical sub graph of the given graph - color critical; if it is a k chromatic graph then the k color critical sub graph is available, so how do we do that? We we keep deleting vertices and edges until it becomes critical right that means deletion of any more vertex or edge from the current graph will reduce the chromatic number from create to something else k to k minus 1.

So such a structure we know the critical graph has the following property namely, it is it is minimum degree is k minus 1. This we had seen in the last class. So, now if the the minimum degree of a graph is k minus 1, we know that there x is a path of length - path of k vertices, path having k vertices in a graph. How do we know that? It is just we take a we take at the longest path, and because the end points of the longest path should be such that the neighbors of all the neighbors of this end point should be in the path, so k minus 1 neighbors plus that vertex itself will make it k. So, the longest path will be having at least k vertices in it. So, this is the this we had seen in earlier classes that minimum degree implies that there exist a long path of. So…

So, then if the graph chromatic number is chi then definitely there should be a path of length chi. Now, the question is when it comes to the directed graphs does it hold good even after giving directions that means you you are given an undirected graph, we know that there are some paths of length chi here in this undirected case, then now we are allowed to give direction to the edges; is it, but we can always give we can try to give directions to edges in such a way that each path of length chi becomes I mean the directions are not correct in that path right, it becomes it it would not be an directed path though in the undirected graph it is a path. So, is it possible to direct the edges like that? Gallai-Roy theorem says it is not possible; however, you direct the edges, however you orient the edges they will always be a path of length directed path of length chi in the graph. So, today we are going to do the proof of this thing; so, this is the way. So, we consider a directed graph D.

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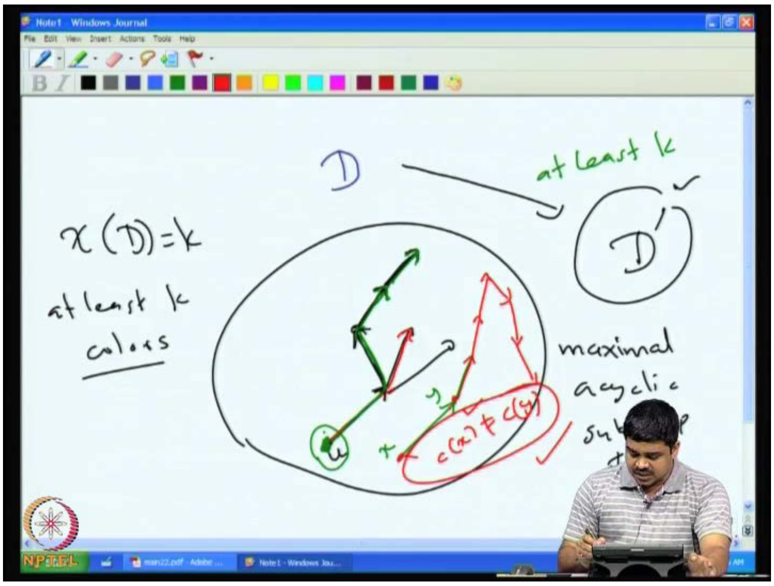

Now, the directions… So, what we first do is suppose this is a directed graph, we start taking some edges from it, in such a way that there are no cycles in it. So, we will keep adding edges as long as we do not introduce. So, we are picking up we are picking up a sub graph D dash of D with the following property. There are no directed directed cycles in D dash, and also this is a maximal structure with respect to their property that means we keep adding as many edges as possible from D, we are taking the edges from D and making D dash, but we keep taking as many edges from D as possible as long as we do not introduce the directed cycle in D dash. This is a maximal acyclic sub graph of D that is D dash. So, this is the maximal acyclic sub graph of…

Now, so the point is we are going to show that, so there is a we know that suppose suppose the chromatic number of D is equal to k. Now, any valid coloring of D should have at least k colors right any valid coloring of D dash should have at least k colors that is what it means. Now, what we are going to do is to show that there exist a path of length k (( )), so this is the way we are going to do. Instead of showing that there exist a directed path in D, we would show that there exist a direct path direct path of length k in D dash in this sub graph D dash - maximal acyclic sub graph of D mainly D dash itself right. This is what my plan is, so we pick up this D dash first.

Now, we will color the vertices of each each vertex as follows; give a color. What is the strategy, starting say for instance if this is u, we will see which is the the longest directed path going out of u, so this is what; suppose so among this is a directed path going out of this, this was one, this was another one right. So… But then the longest one if you look which is this right; this one, this one, this one, this one right, so… And then that how many vertices are there in that path, that we will give to. That will be the color of u that will be the color of u right that will be in the color of u. Now…

Now, what we are going to do is, we will say that so every vertex got a color; we will claim that the colors that we have given to each vertex will define a proper coloring of this graph D - directed graph D which means that the number of colors used should be at least k, if it is a proper coloring the number of colors used should be at least k that mean they should be one vertex which got the value k or more right. So, which essentially means that there if you look at the longest directed path starting from there, it will have at least k vertices in it right that is what it means right, because we our coloring strategy was to consider the longest path starting from there and then give the value of value of the color color has the number of vertices in that path right to that vertex.

Now, the point is, if it is a valid coloring, they should be a one vertex with color k or more therefore, the out the path which starting from here - the longest path is starting from here that vertex should be at least k - that is the path you are looking for. So, now how do you show that this is indeed a valid coloring? So, we can consider each edge of D and show that the n points of each such edge has to be colored differently. Now, suppose there are two cases; so consider an edge of D. Now, in this edge in D, may not may be an edge in D dash or may not be an edge in D dash. Suppose it is an edge in D dash, so we will consider this edge x, y - suppose it is an edge in D dash. Now, what is the color of y? So, the y is, because the color of y will correspond to the longest path which is going from the longest path which is going from starting from y that longest direct path in D dash.

Is it possible that this red one will come and hit this red path that is the longest path which is going out of longest path which is going out of the longest path which is going out of y starting from y, will it come back and hit x once again will it come back and hit x once again is possible, this is the question right. So, this if this can come back and hit x once again, and then we know that so if you start from y this comes here and x y is an edge it will form a directed cycle. We know that in D dash there is no directed cycle. So therefore, this red path cannot hit here, because x y is also an edge of the D dash. So, this will never happen; so it will go somewhere else only, so this will never happen. So, it means that so you can when you consider the longest directed path starting from x, you can first go to y and then follow the y's longest path; so it would not be a cycle at all; it will be longest directed path. So therefore, the value that x gets the color value that x gets will be at least 1 more than the color value that y gets; this is what we can. So, the color of x would not be equal to color of y is what we can infer from this.

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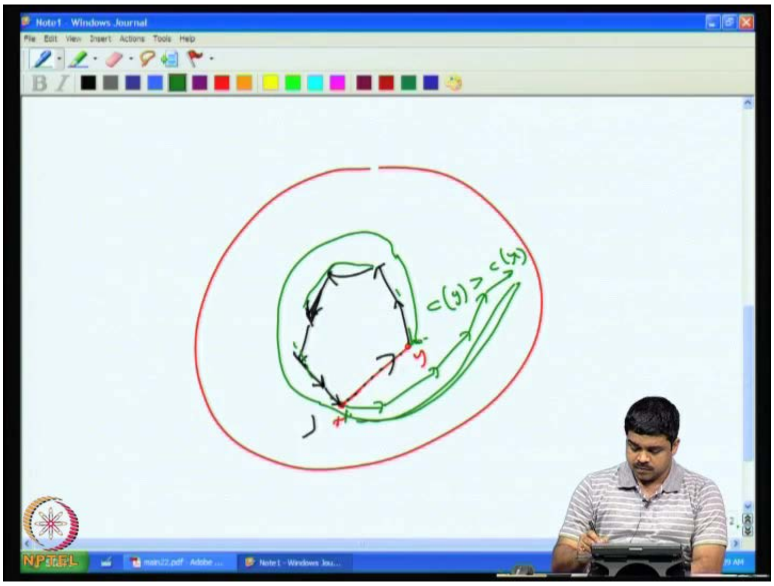

Now, the next situation is suppose this edge x y in D the edge x y in D was not an edge in not an edge in D dash, in D d dash that edge is not there. But what will happen if I add this edge x y to D dash then immediately a cycle will form; why, because D dash was selected as a maximal acyclic sub graph of D; therefore, when you add x y, so they should be a directed cycle in D. So which means that so they should be in D in D dash they should be a in D dash they should be a directed path in D dash they should be a directed path going like this and somehow reaching here right. It should be possible to start from y and reach x like this. This is what we know right.

So, now let us see that so this is this directed path is definitely there, and so such that if x y is added to D dash then this cycle will come that is why x y is outside, and now we see if you look at the the color of x. It will be actually the length of the longest directed path in D dash starting from x is something like this. Now, the question is, is it possible to go and hit y somehow; is it possible to include in this path starting from x - the longest path starting from x can it can we have y. It is not possible, because if it is there suppose then we can always travel like this starting from x through the green edges and through this black edges and then we can come back here to x, forming a directed cycle. So, this will never form never happen. So, this this portion will never happen right. So, this will never happen.

Now, what we see is the color of x corresponds to the length of this path right length of this path; now, color of y will correspond will be more than this, because there is a path going like this and then following this path this right. This path y to x black path and x to the path which is starting from x - the green one which I have the together there will be more than that. So, we see that the color that y gets will be greater than the color that x gets. So, in both cases whether an edge of D is in D dash or not in D dash, the color of x has to be different from the color of y. So therefore, it follows that this coloring strategy correspond to a proper coloring gives take a gives a proper coloring for the vertices of D, and now we can say that because it is a proper coloring, number of colors use this at least k, so they should be a vertex which got the color value k. And now that means that the there exists a directed path in D dash with k vertices in it. So, that is what the Gallai-Roy theorem proofs the chromatic number of the graph is the directed graph is k. Then they should be a directed path of length I means of vertices number of vertices k in it. So, this this we have proved. So so, it means that in directed graph so also the chromatic number can tell as something about the structure of the graph.

Now, in this class, so before leaving this topic of coloring, we would look at some other coloring problems just for a quick over view; so, we look at some other interesting coloring problems and some other some important questions in that. So, here is another version of the coloring problem, so which is called the acyclic edge acyclic coloring problem. What do you mean by acyclic coloring problem? It is indeed a we will seeking a proper coloring of the vertices as usual, but with an extra property namely, we want to color the vertices of the graph such that there are no bi-chromatic cycle in it.

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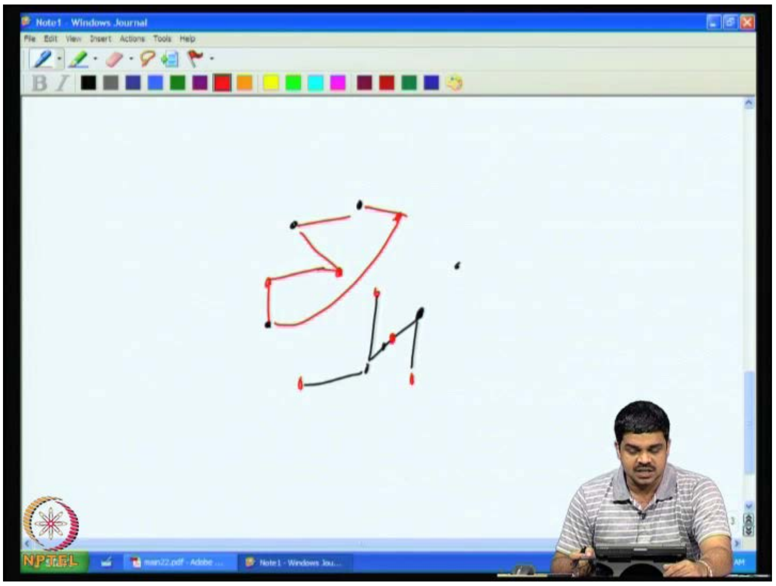

For instance if you look at two colors say black and red, we should not see a cycle of this what, so finally when you pick up the black vertices say and the red vertices along. So, together we should not see any structure of this sort. Say this sort right sorry, so we we should not see any structure of this sort, here black-red, black-red, black-red a cycle. So, in other words, if you look at the vertices black and red vertices along, what the structure should be just forest that means so it can be something like this right. So, forest no cycles should be formed, so this, so of case this connection (( )) there sorry, so forest is (( )). So, now bi-chromatic cycles are allowed or in other words if you examine any cycle there should be at least three colors in it at least the, if you look at the colors of the vertices participating in any cycle they should be at least three different colors. This is vertex says.

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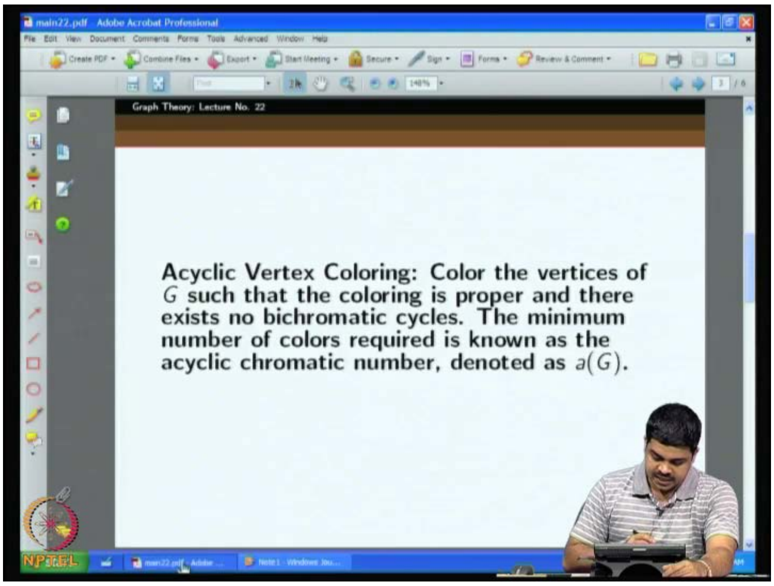

So, now the question is how many colors are required to achieve such a coloring? The minimum number of colors required to achieve such a coloring is called acyclic chromatic number of the graph. So, it is also denoted as a dash of G.

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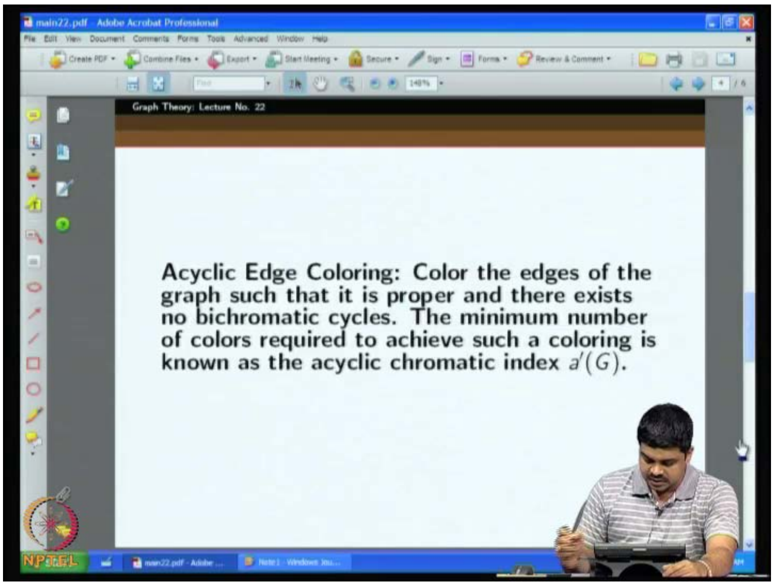

So, let us also discuss different variation of the also an edge (( )) of the problem which is acyclic edge coloring. The question is to color the edges of the graph such that of case we want the coloring to be proper as usual that means two adjacent edges should not get the same color. And we should again we have also this condition that there are no bichromatic cycles. The minimum number of colors required to achieve such a coloring is known as the acyclic chromatic index, minimum number of colors required to achieve such a coloring is known as the acyclic chromatic index a dash of G. This is what acyclic chromatic index a dash of G.

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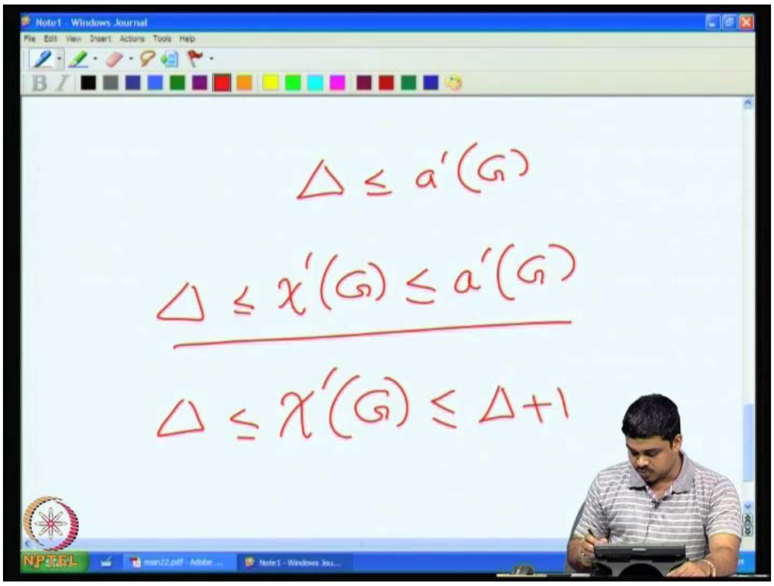

So, the when it comes to a cyclic edge coloring, the so the we know that acyclic edge coloring has to be proper right. So, because it it it has to be proper, so like the proper coloring the lower bond of the delta, maximum degree is valid for the acyclic chromatic index also. See delta we know that delta is less than equal to chi dash of G namely to proper, if if there is a vertex of degree delta then all the edges incident on that vertex has to get different, different colors, so the delta colors are required even to proper color. And acyclic edge coloring has to be definitely it has to be a proper coloring, so it has to be at least more than that right. But how much more which have learned that Vizing's theorem says this number right is at most delta plus 1. We can color the edges of the graph in such a way that there are no bichromatic cycle using at most delta plus 1 colors, but delta plus 1 color is a sufficient. But for acyclic coloring it may not be necessary, sometime may not be enough, sometimes we may need more. So, you can show some examples where it needs more, but we will show that, but see…

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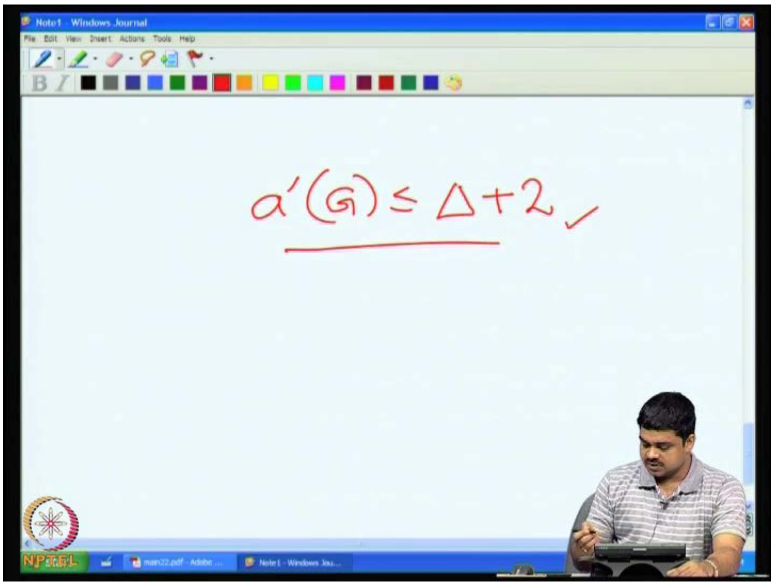

But there is a famous conjecture called acyclic edge coloring conjecture by Allan (( )) also and this conjuncture also also proposed by Russian scientist called (( )) before Allan (( )). And they conjecture that the acyclic chromatic index is at most delta plus 2 is indeed a strong statement that means the requirement of acyclicity that no two colors should form cycles, no or any cycle should not contain less than three colors I mean they should not be a cycle using only two colors that will not increase the requirement on coloring by too much that is only delta plus 2 colors are enough. This is for this thing alright. So, now there is a question of case the delta plus 2 colors are required or not. So, even is it possible that there are some graphs which required delta plus 1 colors of case that we know even in proper proper coloring delta plus 1 colors are required.

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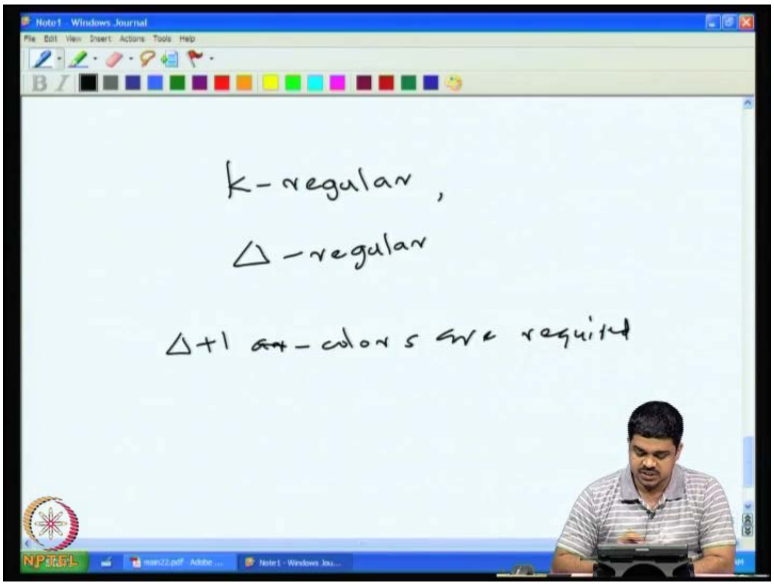

So, now for in a particular we can show that if a graph is k regular k regular, say let us say delta regular, because we will use the same term because the maximum degree is same for same as all equal the degree is equal for all the vertices, therefore we can say delta regular.

Now, we will show that delta plus 1 colors sorry required colors are required. Why is it so? The this properties somewhat different from the proper coloring, because there are we know there are proper there are k regular graph which can be proper colored using just k colors - k regular k colors. It is not that every k regular graph requires k plus 1 colors for its proper coloring. But then when it comes to acyclic coloring we need k plus 1 colors. Why is it so? Because…

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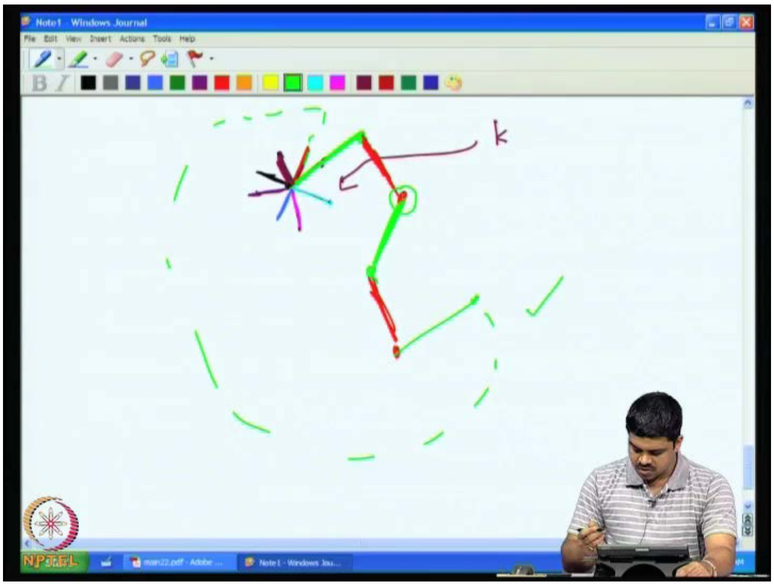

So, you will just start from one vertex, because there are its degrees k or so k is. Now, you know all the… Because only if you are using only delta colors then all the colors should be present here, it should be something like this; all the color should be present here right. So, it should be it should be the edges all the edges should be different and all the… See this we know by proper properness requirement this should be distinct is same correct, but then if you are using only k colors then all the k colors have appeared here, not even one color is missing here. Now, we can pick up two of the colors say as a favorite colors are red and green, so it is that we can we can travel by the green edge say we reach here.

So, now in this vertex also if I search I will find a red color, because you know all the colors are present there right, because only k colors are there, and then all the k edges around them should get different colors, so red color should be present, we can follow the red color. And now here we will found out the green color, green color will be there. Now, (( )) we can find out the red color and so on. So, how long can you go like this? So, always you can go forward by red-green, red-green, red-green. So, now where will you how will you stop, because the graph is finite and it has to stop somewhere. The only way is to form cycles, so you come back here and then and then you have to enter through this red edge here right. So that means there should be a red green cycle; so any two colors if you take, we should form we should a get cycle of using those two colors two type of colors, so right red and green. So, so therefore, it is not possible to have just k colors, because if just k colors of this problem (( )). So, we should have at least k plus 1 color, why because in if there are k plus 1 colors in some point it is possible that we may not see the outgoing green edge or outgoing red edge, so therefore, we may have to stop there this. This sequence will not continue, like at some point it may break even without coming back to the same vertex. So that is why in it is k plus 1 colors.

Now, we claim the next point is that, so the is it possible that so every every graph can be colored with the delta plus 1 colors; why delta plus 2 why the conjecture so then this is the reason, in fact there are lot of graphs which require delta plus 2 colors. In fact there are a large number of them, so the following statement can be made in fact.

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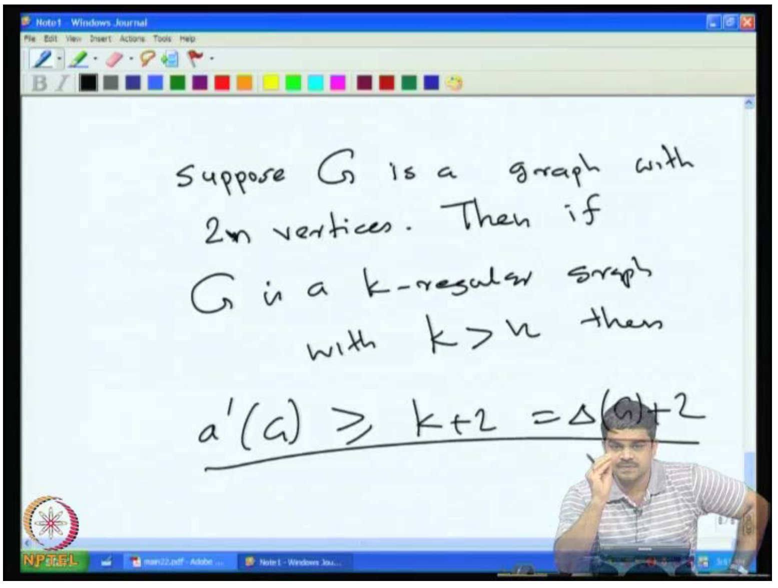

Suppose G is a graph with 2 n vertices 2 n vertices, then if G is a k regular graph with k greater than n then a dash of G is greater than or equal to k plus 2 that is delta of G plus 2 right. This is what we can tell. So so that means the degree is… So, I am considering graphs of even number of vertices, if the degree is strictly more than n if the degree strictly more than n then the acyclic chromatic index has to be greater than or equal to k plus 2, why is it so? So, this is the reason for that.

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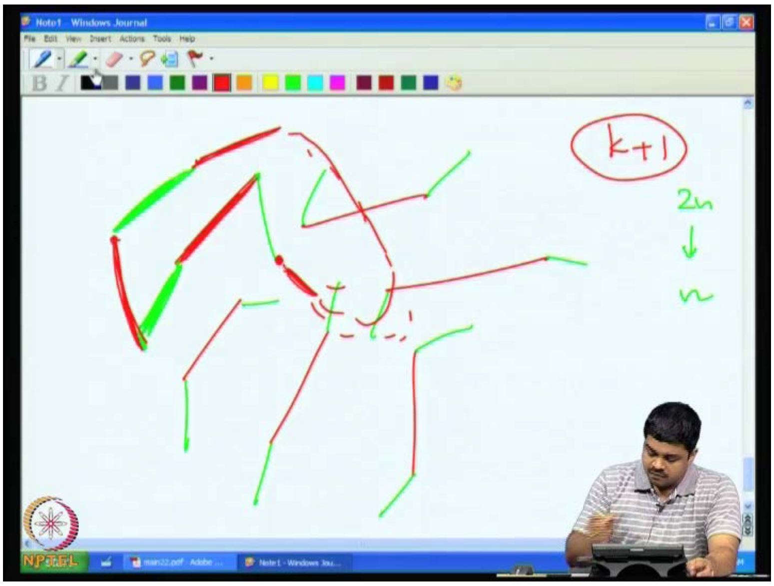

Now, you consider a color glass, color glass glass color glass means you you can consider the edges colored red, say suppose these are the edges colored red right. Suppose I am assuming that suppose there is a coloring using k plus 1 colors suppose there is a coloring using k plus 1 colors, now you pick up the red edges along. Now, you consider suppose this happens to be n edges, so there are 2 n vertices. Can you have more than n edges? No, it is not possible, because so it has to be a matching right, (( )) color glass should give us a, because you do not touch each other because of because it is a proper coloring.

Now, the biggest matching possible for 2 n vertices is only n; suppose you get one of the color glass to be of cardinality n. Now, is it possible to have another color glass say green color glass, also to have be of cardinality n. Suppose it is also if cardinality n that means the green color glass also defines a perfect matching that means every vertex is touched by a green edge also, like the red edge touches every vertex, some green edge also touches every vertex like this right it will you will always be able to find out both red and green at every vertex right. Now, what will you do? So, now we see that like in the earlier argument we see that we can start from this vertex; we follow the red red edge here right follow the red edge here we follow the red edge here the red edge here. Now, we follow the green edge, now here this vertex also will have a red edge, because it is a red is also a perfect matching.

Now, we can follow the green edge from here, because green is a perfect matching and then we can follow the red edge. So, this will where will it stop, this again the question. The… Because it cannot keep going right till, because the graph is a finite, it has to end somewhere. The only way of ending is to come back and join with one of the earlier things like this. So, this will give a red green cycle. So that is a contradiction, because we are saying that in acyclic edge coloring and this should not be a cycle with just two colors. Here is a red-green, red-green cycle that will not happen that cannot happen. So therefore, which means that there can only be one perfect matching if at all, if at all there can only be one perfect matching. The red can be a perfect matching then no green cannot be a perfect matching, blue cannot be a perfect matching, no other color can be find up perfect matching.

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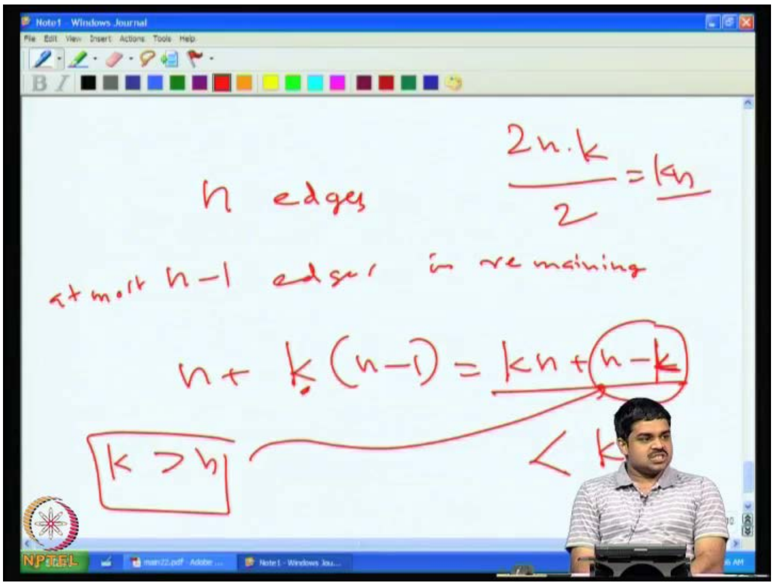

So, one of the color glass can have n vertices, one of the color glass can have n vertices sorry n edges in it, n edges the remaining is at most at most n minus 1 edges suppose to be in remaining in remaining. So, total number of edges can only be n plus… There are k k plus 1 right k into n minus 1 right. How much is this? So, this is k n n plus n minus k, this many edges are the maximum you can get if there are only k plus 1 color glass 1 color glass and k other color glass.

So, now you see that this thing will be suppose we have assume that k is strictly greater than n. This would imply what, so this n minus k will be a negative number. So, this together will be strictly less than k. So, how can it be strictly less than k n? So, if every edge is colored the together if I some of the color glass the cardinality from the color classes, it should give the total number of edges right, and how many edges are there? There are 2 n vertices, it is a k regular graph, so 2 n into k divided by 2 equal to k n edges should be there. Now, we are saying when you summed up the cardinalities of all the color classes we will get something strictly less than k n which is a contradiction which cannot be true. So, we can infer that it is not possible to have a coloring using just k plus 1 colors. They should be k plus 2 colors in it. So that is a interesting (( )), because if the regularity as such tells us that k regularity as such is that we needs k plus 1. But in the case of done(s) regular graph, done(s) regular graph in the sense is the k becoming greater than half of the vertices - half the number of vertices then it is says that we need k plus 2 colors, it is not just about the regular graph; one may thing that it is it is something special about the regular graphs.

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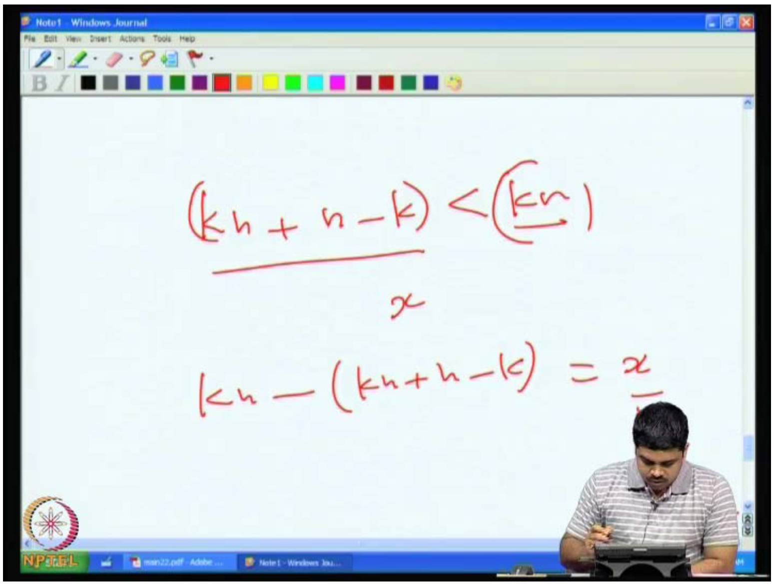

You can here you can see what we have told is because the when the color classes are added together we get k n plus n minus k, and if it is strictly less than k and we have a contradiction. So as long as it is suppose the difference between this side and this side is x that means this total k n minus this quantity that is k n plus n minus k is equal to some x. So that means we can remove less than x edges, so x minus 1 edges we can remove from the current k regular graph we are taking about, and still sorry right, so we… That means we can reduce the side by x minus 1 still we will be we will not be reaching the total number of edges by this some right.

So, regularity can be destroyed, but even then we need we can argue that k plus 1 colors are required k plus 2 colors are required. By making here that still the maximum degree is k right, so that means at the some vertex is undisturbed. So… So therefore, there are graph which required delta plus 1 2, but up to now nobody has found out any graph which required more than delta plus 2; it is possible, but there is this conjecture still open after 20 years. It is its conjecture somewhere in 90s - the beginning of 90s. So, it is still open after almost 20 years. So it is an interesting problem so… So, one can try to discrew the conjecture.

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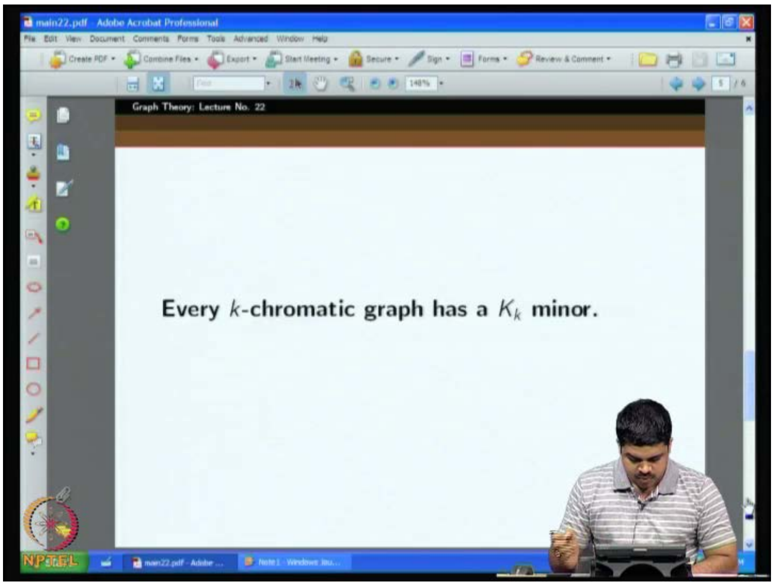

Now, the next idea is to consider another important problem regarding the chromatic number the coloring vertex coloring of graphs. So, this problem this problem one of the most important to open problems in graph theory and this is called the Hadwiger's conjecture. So, this is very important, because it is tries to generalize the famous four color theorem for planar graphs right. And also several people have already worked on this and it this known to be a very tough question, so this is the statement. Every k chromatic graph has a k k minor. So, we can verify it for small cases like we can consider a for instance 1 chromatic graph, (( )) nothing to prove there.

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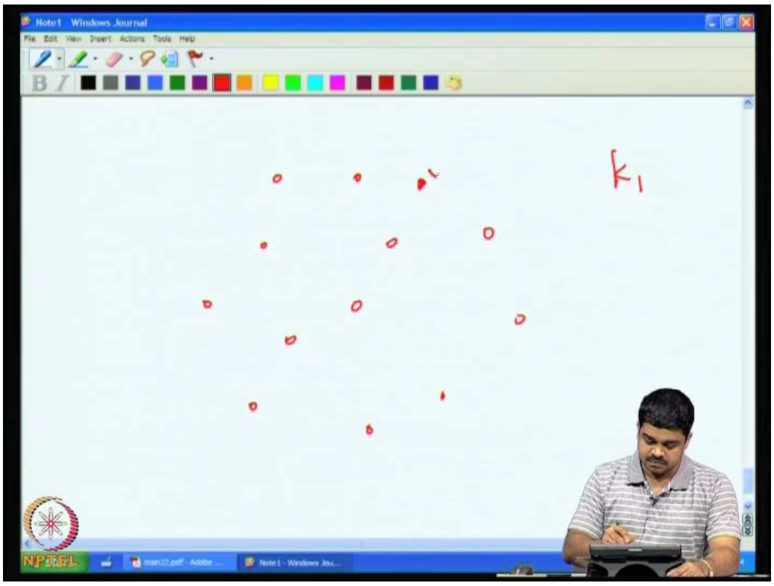

1 chromatic graph means so here we have the independent set right, we have just the independent set. If it is a 1 chromatic graph then there is a k 1 minor which is there of case, this is a k 1 minor. So, there is nothing to prove there.

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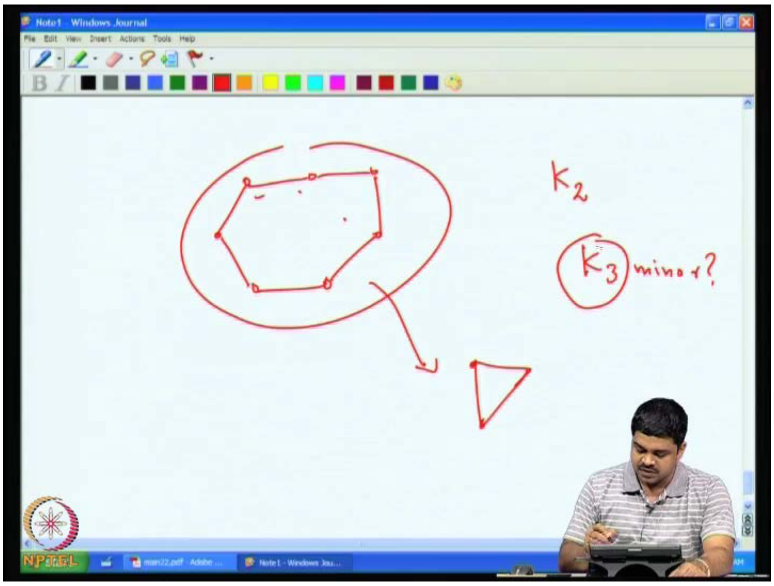

What about 2 chromatic graphs? So, what can we tell about 2 chromatic graphs. So, 2 chromatic graph means bipartite graphs right. So, in bipartite graphs is there a k 2 minor of case there is, so there is a k 2 minor alright. And now what about 3 chromatic graph; so, 3 (( )) graph, it does it have a k 3 minor. So, it is not difficult to see that so of case the k 3 k 3 chromatic means it cannot be colored with 2 colors, so they are not trees or forests. So, there is a cycle in it right there is a cycle in it, somewhere there should be a cycle in the graph, it is not an acyclic graph. If it is acyclic graph then it should be colorable using just how many colors two two colors right. Now, three colors are required.

Now, if there is a cycle then you see that we can contract it and get a k 3 minor right. So, you contract it and you get a k 3 minor. So, how do you contract it? You just contract edge by edge and you can you can convert it to this k 3 minor. So, when we say k 3 minor right so what do you mean by that; you can throw away some edges, throw away some vertices and contract some edges and ultimately get a k 3, complete graph on three vertices or minor.

Now, the next question is, is it possible that there is a 4 chromatic graph, is it sorry, if we if we a given 4 chromatic graph that is always set k 4 minor or not? So, this is also true, because what we can do is if you take a 4 chromatic graph, so we we consider the critical graph of that color critical k 4, critical 4, color critical sub graph of the given graph, there is always one right that we have already discussed. And you know one property of this thing it is already the the minimum degree is already 3. So, other is a we can easily show that it is not a it has a k 4 minor. So, there is a result of Dirac saying that the graph which so so also we because the minimum degree is already 3 that is a k 4 minor.

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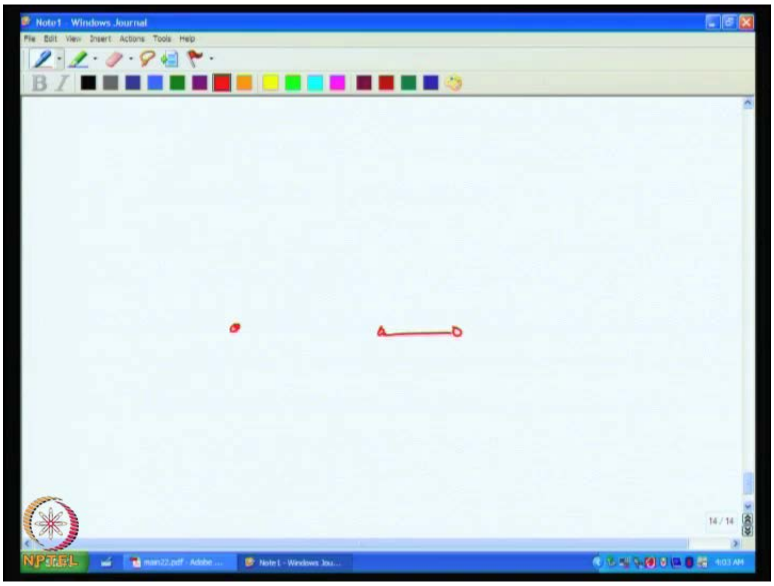

The graph which does not have a k 4 minor or the so called series parallel graphs. So, what is the series parallel graph? So, it is a it is a special kind of planar graph which is constructed like this. So, we can say this is a series parallel graph. These edges also a series parallel graph.

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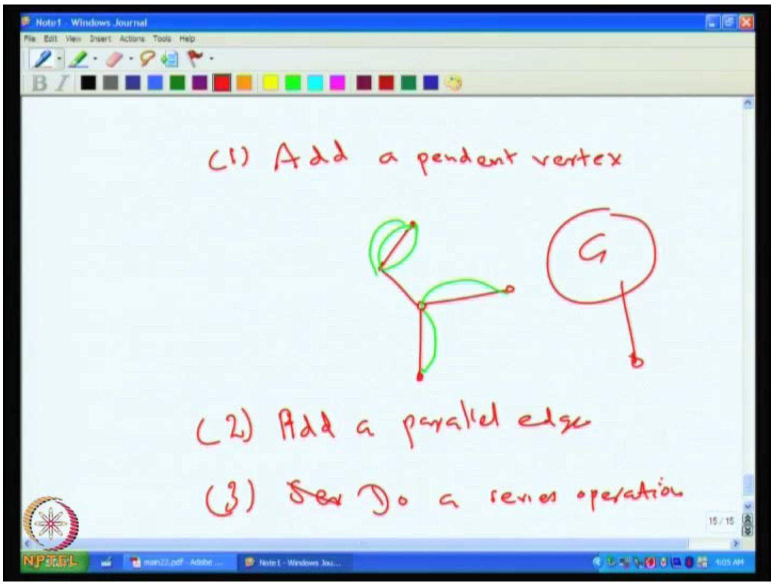

Then the starting from this basic things, we allow 2 three kind of operations; one is add a pendent vertex pendent vertex, this is one operation which is allowed. For instance if this is a series parallel graph, so we can add a this is also, this is also, a series parallel graph this is also, so that trees are all series parallel graphs. You can if you got already got a series parallel graph here, you can add a new vertex like this - pendent vertex (( )).

Then two is that you can… Suppose there is a edge here, so you can add a parallel add a parallel edge which means you can add an edge like this or like this, how many ever you want right. This is the parallel edges. So, if there is an already existing edge you can add a edge parallel to it, so of case one can see that this defines a multi graph not just essentially not necessarily a simple graph, sometimes it can be a multi graph. So, parallel edge addition is allowed. And the third property is that you can do a series operation do a series operation, what do you mean by a series operation?

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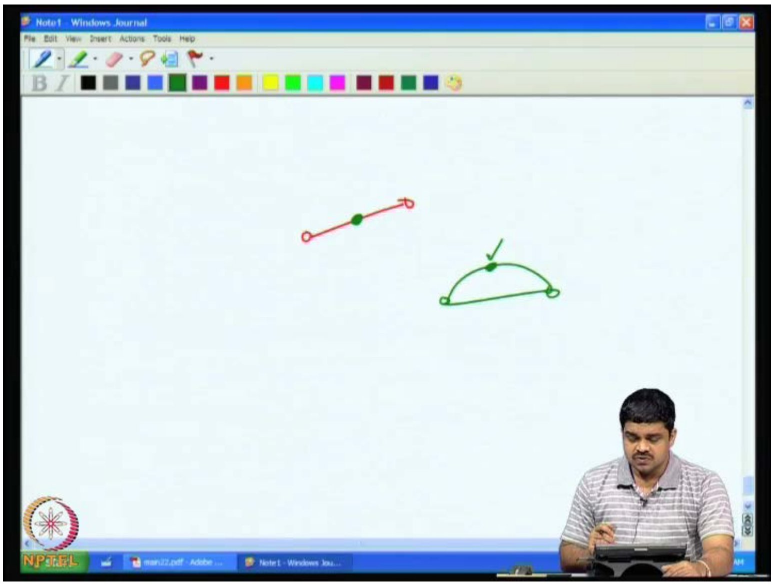

Series operation means suppose you have a edge here, say essentially series operation means it is a sub division operation that means you you can introduce you can break this edge introduce a new vertex here. So, so instead of this one edge you put a path of length two right. You can keep on doing that once. So, this is a series operation. So, you start with an a one vertex and keep on doing any of these k n operation is pendent vertex edition or parallel edge edition or doing a subdivision operation is called series operation.

So, whatever graph you can create like this is called a series parallel graph. We can easily see that right. So… So, this finally the series parallel graphs minimum degree has to be 2, it is a 2 D generate graph means there is a sequence, for instance if we look at the backward sequence, I mean the way you created the reverse sequence if you look, you see that every time you are introducing a new vertex, so that is only single degree. So, when it comes to a simple graph I mean, so the parallel edge is without counting the parallel edges. So, usually typically what we do is we create the parallel edge and then break it by a series operation right. So, therefore this always always going to be a 2 degree vertices, so 2 degree regular graph. So, they can never be 3 regular graph. So, the Dirac proved that these graph does not have k 4 minus and these are the only graph which does not have k 4 minus. And so therefore, anyway so we can use in Dirac's theorem we can show that a 4 chromatic graph does not have sorry has a k 4 minor.

Now, the case of 5 chromatic graphs essentially is much more difficult, because you know 5 chromatic graph means if 5 chromatic graphs have a k 5 minor, if you if you can easily prove that then it means that the planar 4 color theorem can be easily proved why, because no planar graph can have a k 5 minor as we as we have seen; because the operations involving involved in constructing minus preserve planarity, like you can remove an edge remove a vertex without losing planarity and also you can contract an edge without losing planarity therefore, if you if you say that a k 5 is a minor of a graph, because k 5 is not a planar graph, the original graph also cannot be a planar graph. Because when you keep on doing this operations planarity is preserved you cannot reach a non planar graph by doing these things.

So, we get that the any planar graph has I cannot have it a 5 minor. So, if it means that so the suppose there is a planar graph suppose there is a planar graph which requires 5 colors that means it is a 5 chromatic graph. So, if then this theorem if this theorem is true it is like telling that any 5 chromatic graph has a k 5 minor that such a planar graph which is which require 5 colors to color it has a k 5 minor, which will be a contradiction, which would not be true right. So, what can we infer from that? We can infer that no planar graph can be can require 5 colors, whole planar graph can be colored using 4 colors that is the immediate in front right. So, this statement for the chromatic number equal to 5 case will immediately imply the planar 4 4 color theorem. So therefore, proving it would not be so easy. So, because we know that the planar for color theorem was a very, very difficult question and it took almost 100 years to come up with a solution that will not very elegant with lot of computer assistants. So, we have already discussed it before, so this. So, we can only hope probably to use that 4 color theorem to prove that case the Hadwiger's conjecture this that is all 5 colorable graphs have a k 5 minor sorry all graphs whose chromatic number is 5 has a k k 5 minor using the planar graph theorem.

So, of case the (( )) has achieved that (( )) has shown that if you consider the edge maximal graphs. So, the then which are those graphs which requires 5 5 colors, 5 chromatic graphs, they can be either planar graphs or some very special classes of graphs which have shown to be 5 which sorry the the the graphs which does not have a k 5 minor, so and we take the maximal graphs when he have he has shown that there are only two categories and then they are always 5 colors, so they they so, of case they are 4 color. So that this case was consider by Wagner and he has come up with a convincing argument, of case using making use of the fact that planar graphs are 4 color.

Now, the next case was proved by Robertson and Seymour. So that is the 6 color which was also using the planar 4 color theorem for its say say say using 4 color theorem - planar 4 color theorem. They relied on that. And then the later from there on the the problem is still open for instance k 7 on words it is still open, so we that that remains to be one of the major (( )) problems in graph theory in now so. And then we discussed Hadwiger's conjecture it is very natural to remember a historically was another another conjecture called Hajos conjecture which **was** a stronger conjecture, and it told every k chromatic graph has the sub division of k k that means the Hadwiger conjecture only claims that every k chromatic graph has a k k minor, here it is a stronger requirement it says that there is a topological minor locate. So…

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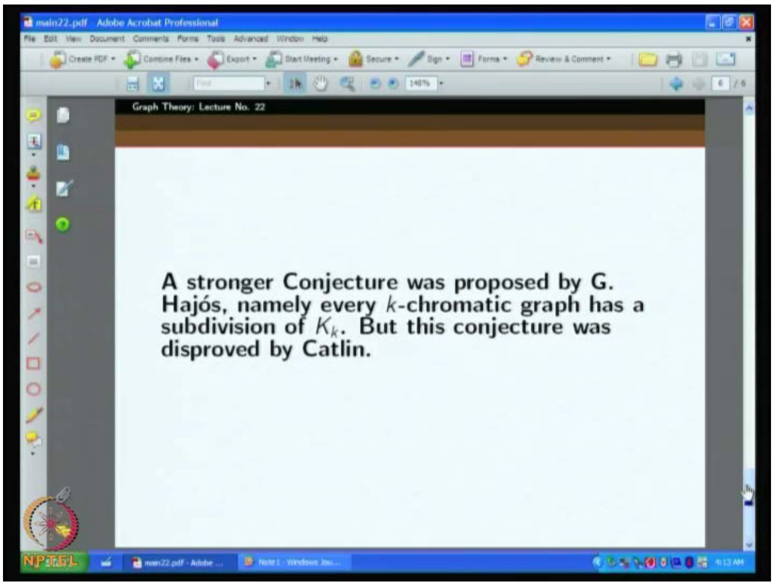

But unfortunately this conduct conjecture was is not true and just this proved by Catlin, he showed a kind of graph which which which is k chromatic, but does not have a sub division of k k. And in fact it can be shown that with some using some other models G a probabilistic models which we have not yet discussed that almost all graphs will be a counter example to Hajos conjecture. So, that is the story about with two important conjectures in the graph coloring theory Hadwiger's conjecture, so that Hajos conjecture was disproved and of case.

So, then now-a-days so many counter examples are now for Hajos conjecture. Now, the the next we are our plan is to discuss some, these are the main conjecture about the vertex coloring. Another thing is of case why this the coloring, how this coloring idea the vertex coloring idea chromatic number, this concept can be generalized. So, for instance one possible way is to define something called a homomorphism. So, there just telling it for the sake of looking at where is space in which people have tried to generalize it, this is one interesting possibility.

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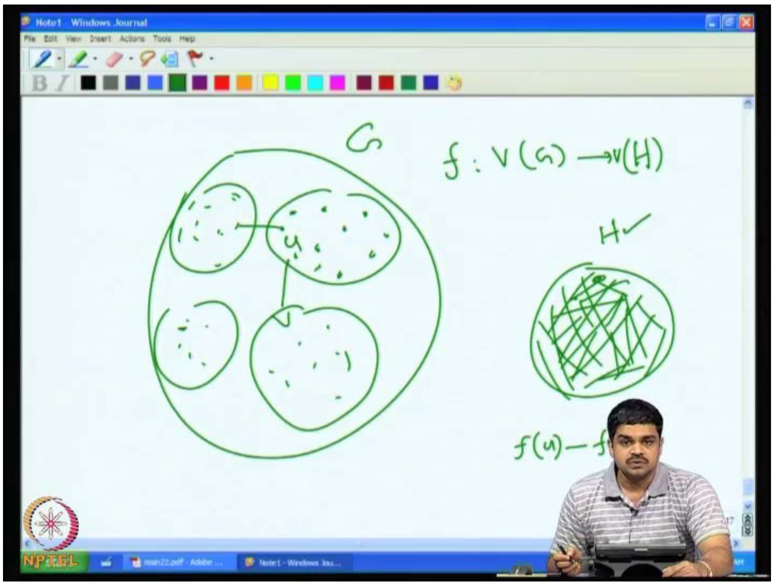

So, here so the here you can you can have a graph G, so here is a graph G and now from the vertex set of G, you can define a function to a smaller graph say H, to H, v of H we can say right. Such that suppose there is an edge here between u and v so right, then the corresponding edge should be there in between f of u and f of v right, so this is homomorphism right. So, see for instance suppose if H is a complete graph suppose H is a complete graph then you see the for each vertex the pre images of each vertex will form a color glass right alright. So, then because between right… So, this is will be a coloring, because between whenever there is an edge here the corresponding whenever there is an edge here then there H be a corresponding edge here also right. So, the therefore, this this will correspond to a coloring, and now suppose H is not a complete graph it will give a slightly different notion of the coloring there. So that is an interesting kind of generalization which one may want to read up.

So, then the next topic we are going to study is the perfect graphs. So, in the next class we will consider this thing, so this is essentially again a topic in graph coloring in some sense, because the definition of the perfect graphs is related to the concept of coloring. But still it essentially it is a very interesting class of (( )) with lot more properties and there are several subclasses of perfect graphs which which is practically useful and theoretically interesting, so we get a chance to study all those things, so that that will be the next topic. So, in the next class we will discuss about that. Thank you.