Graph Theory

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Lecture No. # 30

Chvatal's Theorem, Toughness, Hamiltonicity and 4-Color Conjecture

Welcome to the 30th lecturer of graph theory.

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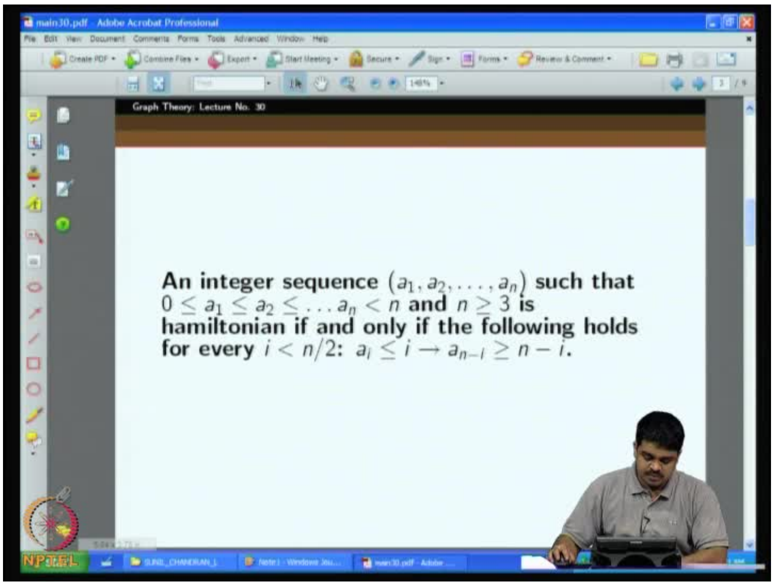

In the last class we were going through the proof of chvatal's theorem about a sufficient condition for the existence of Hamiltonian cycles; the theorem is stated like this, if g is a graph with n vertices and degree sequence d 1 less than equal to d 2 less than equal to d n, then the it is called Hamiltonian; if every sequence, which is point wise greater than or equal to this given sequence it corresponds to graphs which are Hamiltonian, that means, if a graph has this as its degree sequence then it should be a Hamiltonian graph.

So, an integer sequence a 1, a 2, a n, such that 0 less than equal to a 1 less than equal to a 2 less than equal to a n strictly less than n is Hamiltonian if and only if the following holds, this is the condition, for every i strictly less than n by 2, if a i the number in the ith position is less than equal to i then the corresponding number a n minus i, that means, if you count from the other end the i plus 1th number has to be greater than equal to n minus i.

If this condition is met, then it will be Hamiltonian, that sequence will be Hamiltonian, that means, not only this the graph this sequence or and all graphs with sequences which is point wise greater than equal to this will be Hamiltonian; and so, we have shown why, for instance, if this condition is satisfy, then all sequences which is point wise greater than equal to this sequence corresponds to graph which are Hamiltonian.

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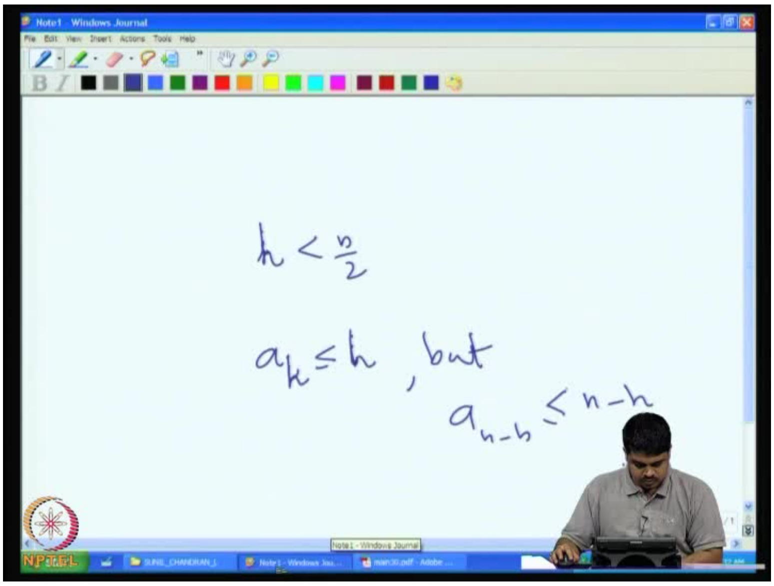

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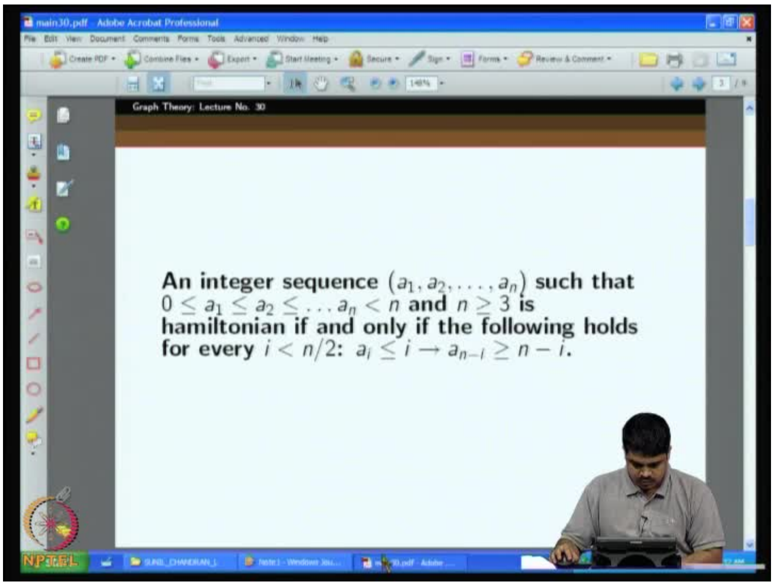

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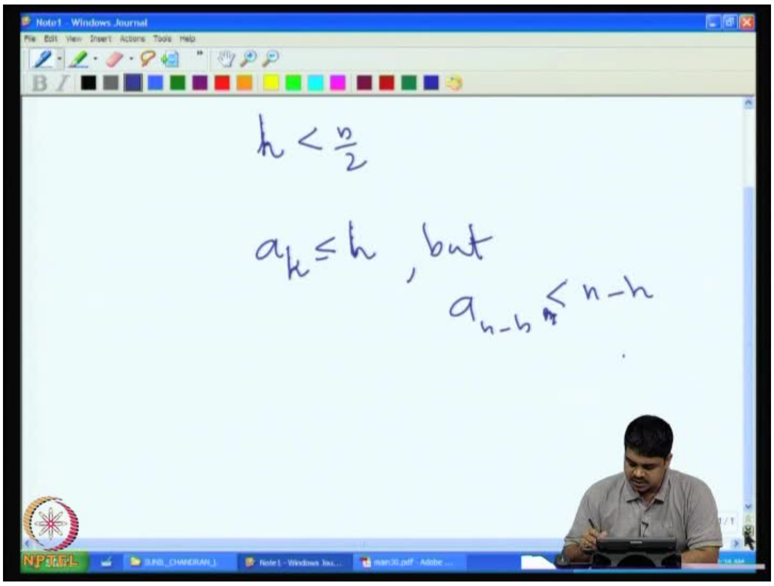

Now, we will look at the other side namely think about the graphs, so so a sequence which does not satisfy this condition; that means, suppose for some i, you are given a sequence, suppose some i, suppose some i less than n by 2, it is so happens that, i is in fact, less than equal to I, so maybe you can some edge, we can just change the notation to h, h is less than equal to h, but the corresponding in equalities not correct a n minus h is not greater, that is also less than or equal to n minus h, it is not greater than or equal to n minus h. So, then so say let us look at this, greater than or equal to n minus h, then suppose this is if we want it to be greater than equal to n minus h, so suppose it is so happens that it not greater than equal to n minus h it is strictly less than n minus h.

So, now, we will construct a graph which is point has a degree sequence, which is point wise greater than this, but this not have a Hamiltonian cycle. So, how do i construct because then we we we want we want to say that, if this any any sequence which does not satisfy this condition, the chvatal condition will be such that there exists at least one graph which has a point wise greater degree sequence compare to the given sequence and which is not Hamiltonian; that means, it is an if and only if statement for characterize characterizes the Hamiltonian sequences, not that if its satisfy is always Hamiltonian, if it does not satisfy at if sometimes Hamiltonian sometimes not Hamiltonian, is not that if does not satisfy some, where we can always find some graph, which has a degree sequence, which is point wise greater than equal to this, but does not have a Hamiltonian cycle.

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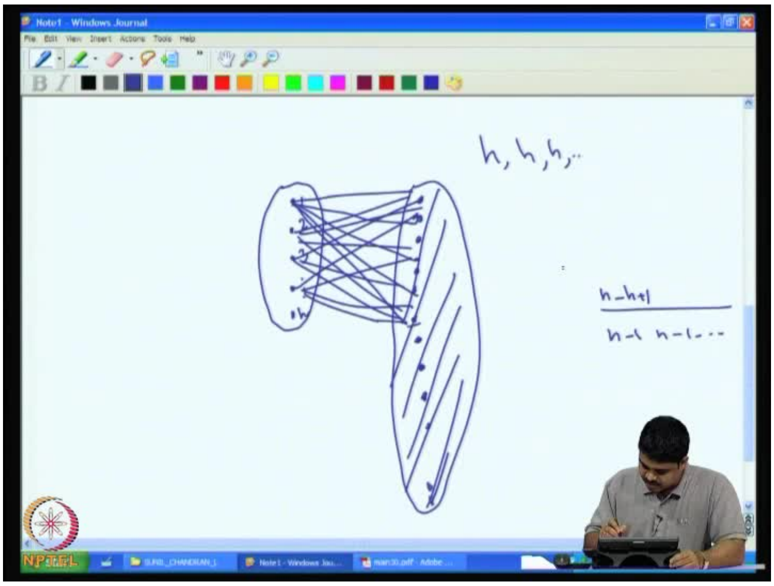

So, how do we see this thing? So, this is the construction. So, let us construct a graph like this; there are can I take edge vertices now, you take edge vertices here, this is 1 2 3 edge; now, here we will take the remaining n minus h vertices, but here this is this will corrupts corresponded to h; h vertices are special and the remaining vertices are different.

Now, the the point is construct to graph with a degree sequence equal to see initial…, what we going to do, connect all these vertices to the corresponding h vertices here; that means, the degree of each of these vertices will be h so that the degree sequence will start with h, h, h, etcetera, how many times? So, h times, so then so that our hth number will be h, that is a h will be less then equal to h.

Now, the these vertices or universal vertices, because they they will connect everything, because we will make it a complete graph here; so, but on the other hand, here these vertices up to this is nth vertex, this is n minus 1th vertex, this is n minus h plus 1th vertex. So, n minus h plus 1th vertex onwards the position onwards we will have n minus 1 degree, n minus 1, n minus 1 degree, and then the remaining vertices will have degree, these these are the remaining vertices, they will have degree n minus 1 minus h; that means, n minus h minus 1.

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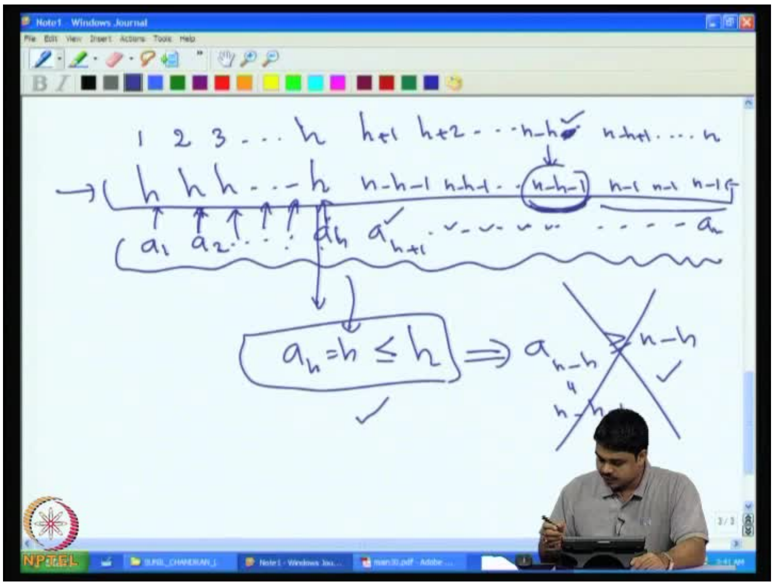

So, the degree sequence will be h. So, I am writing the position numbers here, I have 1, 2, 3, 4 up to h. So, h, h, h to here this is h; and then h plus 1, h plus 2, till n minus h, minus sorry n minus h; we will we will have the numbers n minus h minus 1, n minus h minus 1, up to here n minus h minus 1 and from here onwards n minus h plus 1 onwards to n, we will have n minus 1, n minus 1, n minus 1; the point is that, if we look at the hth number, a h that is equal to h, this is less than equal to h, so the a h is less than equal to h; then we would chvatal's condition says we should have a n minus h should be greater than or equal to h, n minus h should be greater than equal to n minus h, but if you look at the n minus h the number this is only n minus h, so n minus h minus this is equal to n minus h minus 1, so this is not true, this is violated. So, the condition is violated. So, that is the way the graph is constructed.

So, you may ask, so, how is it true that, so, my I my sequence was something else initially a 1, a 2 up to a h, I only knew that a h is less than the equal to h; and here this numbers were anything, which may be greater than or equal to this thing a n, this was the initial sequence, how can I take this sequence instead of this sequence; that is because this sequence is point wise greater than this, because here this number was anyway less than or equal to h. So, all these numbers were less than equal to h, we just increased the values here to h, so and also we knew that here the h minus 1th number was strictly less than n minus 1 is the maximum possible was n minus h, n minus 1 and all these numbers can be at most that much 1.

So, we just increased all the possible numbers here, all the numbers here; it is possible that they were already n minus h minus 1 or it is possible that they were strictly less than n; if they was strictly less than than we increase them to this number; it was it is not possible that they they values for actually greater a n minus n plus 1 2, a h plus 1 2, a n minus h; any of these numbers cannot be strictly greater than n minus h minus 1 initially, because in that case how can this number a n minus h be strictly greater than n minus h, because all these numbers are suppose be smaller than or equal to this number and the later numbers we made it as big as possible namely n minus 1.

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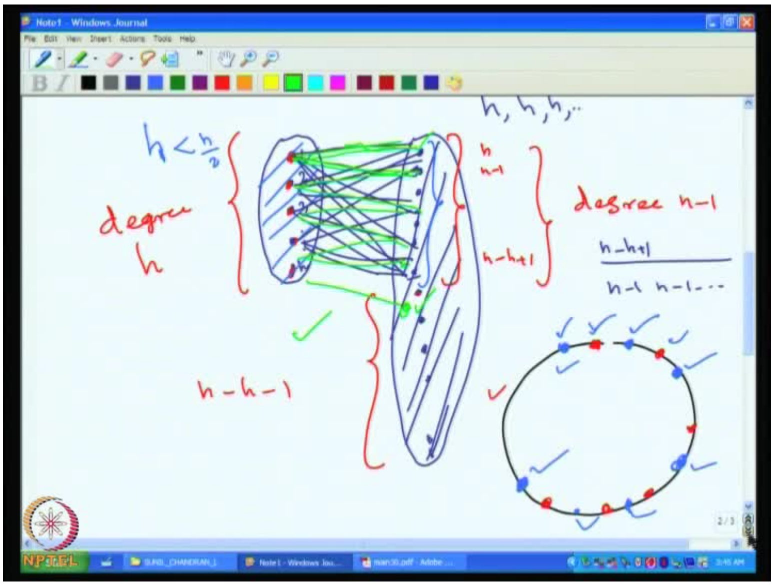

So, therefore, whichever is this initial sequence a 1, a 1, a h, a n, a h plus 1 to a n, we have just increase the corresponding values, every a i has only got increased, therefore this sequence is a point wise greater a degree sequence, greater than or equal to d, the original given sequence. So, therefore, we just have to show that, this sequence corresponds to a non-Hamiltonian graph, some graph exists with this sequence, but not Hamiltonian, that is enough show that; therefore, any given this a sequence which violates the condition, we lead to a point wise greater than equal to a sequence which is point wise greater than or equal to this having a having degree sequence corresponds, which corresponds to a graph which is not Hamiltonian. So, which is that graph, the graph is this, this way have constructed namely, see these are all h, the first smallest values are coming from these vertices, so I will mark it as red vertices see.

So, the and the next, so because these are all degree h, these all degree h, because here there is no…, is an independent set that its connected to all the last h vertices; now, on the other hand, these vertices are the last h vertices namely nth, n minus 1th up to n minus h plus 1th vertex; and they all have degree n minus 1, these are the vertices which coming between with degree and n minus h minus 1, because they are connecting to all vertices in this part except the these h vertices, therefore n minus one minus h is the degree.

Now, we climb this graph cannot have a Hamiltonian cycle; suppose, it has a Hamiltonian cycle, then what is a wrong with it? Suppose if so there is a Hamiltonian cycle, you can identify these red vertices, these red vertices in the Hamiltonian cycle; so, in the, but these vertices red vertices are non-adjacent vertices, they are they are independent set there are independent set. So, in a Hamiltonian cycle, for instance, if we locate them Hamiltonian cycle, so so it may be…, it may come like this, a red vertex, a red vertex, a red vertex, a red vertex, a red vertex, how many neighbors they should have in the Hamiltonian cycle?

So, here if I count this neighbor one, so and this neighbor, this neighbor, so this neighbor, it is possible that every consecutive red k measure this neighbor, so like that, like that, like that, and finally here; unless this number was strictly equal to n by 2, then we can clearly say that the number of neighbors of this set this red set, that means, the number of neighbors such that it is adjacent, so the number of vertices, so that the adjacent at least one red vertex is at least one more than the number of vertices; we know that this number number of vertices is h, that is strictly less than n by 2; therefore, these last two vertices cannot be the same. So, therefore, though even if you say that these are all shared neighbors, these two cannot be shared neighbors, we should have one more neighbor than h for this set in the Hamiltonian set.

But on the other hand if you look here, this is only exactly h neighbors, we do not we do not have enough neighbors, if there is a Hamiltonian path it is not possible to have just h neighbors because there are more vertices here . So, in other words, if you want to see it in a different way the Hamiltonian path will be go here, then it has to come back here, then again it has to go here, it has to come back here. So, every time it goes here and come back, then you have taken for one more vertex and finally because the the cycle has to close it has come out of it. So, therefore, it need to once it enters through one vertex and once it comes out through one vertex, so therefore this should be two one more vertex to which in the in the neighborhood of h exactly h neighbors are not enough. So, we see that this is not a Hamiltonian graph.

So, therefore, we have shown that, if the chvatal condition is violated then whichever is the sequences, if it is violated even in some i some a h less than equal to h, the corresponding a n minus h is not greater than or equal to n minus 1; if it happens to be less than equal to n minus h minus 1, then we know that there is a graph, there is a graph which is whose Hamiltonian whose degree sequence is point wise greater than or equal to this sequence and is not Hamiltonian. So, therefore, chvatal's condition characterizes the Hamiltonian sequences that, so that is what the theorem says.

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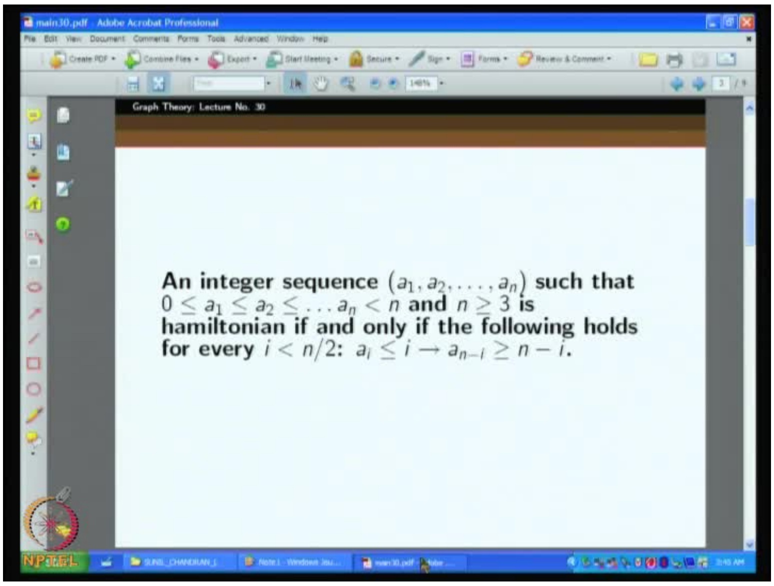

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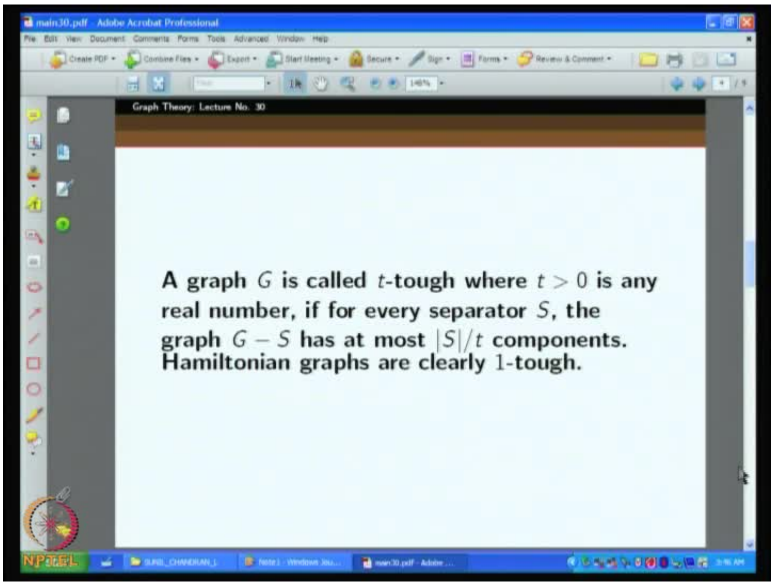

Now, we will so now, our aim is to look at some of the related theorem in Hamiltonian regarding Hamiltonian circuit; in this class we will spend some time to understand some other related results in regarding Hamiltonian cycles; and from the next class onwards we will start on a different topic that is our aim.

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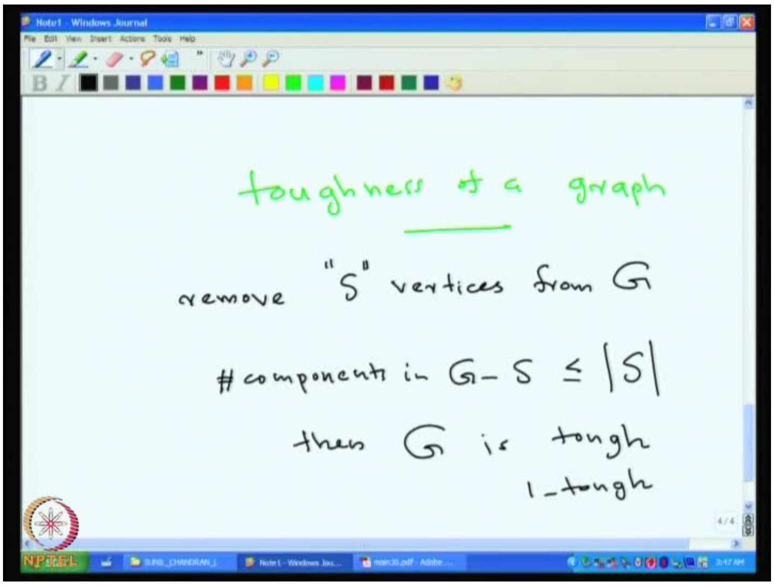

So, here is another concept. So, this is called toughness, toughness of a graph. So, suppose, you removes some s vertices from the graph remove s vertices from g then and then count the number of components number of components in g minus s; if it so happens that the number of components in g minus s is less than equal to cardinality of s, then we will say that g is tough tough, g is a tough graph. So, g is one tough is another because you can generalize this concept further g is one tough.

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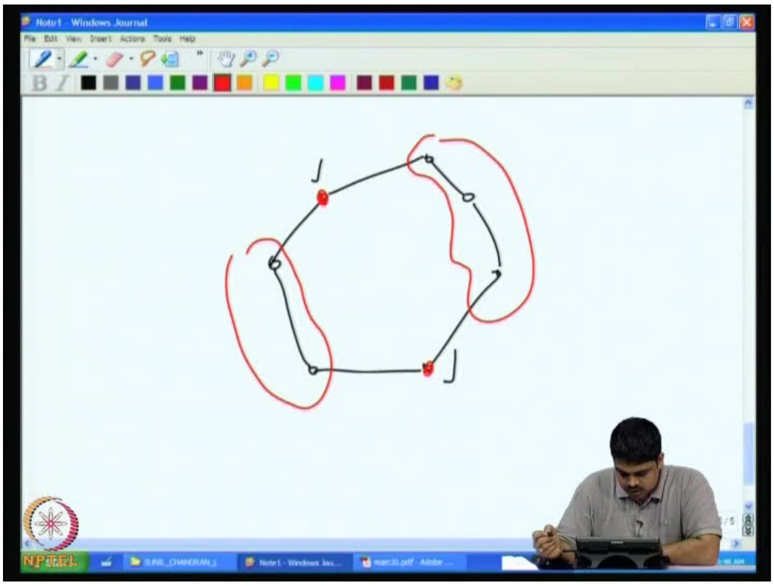

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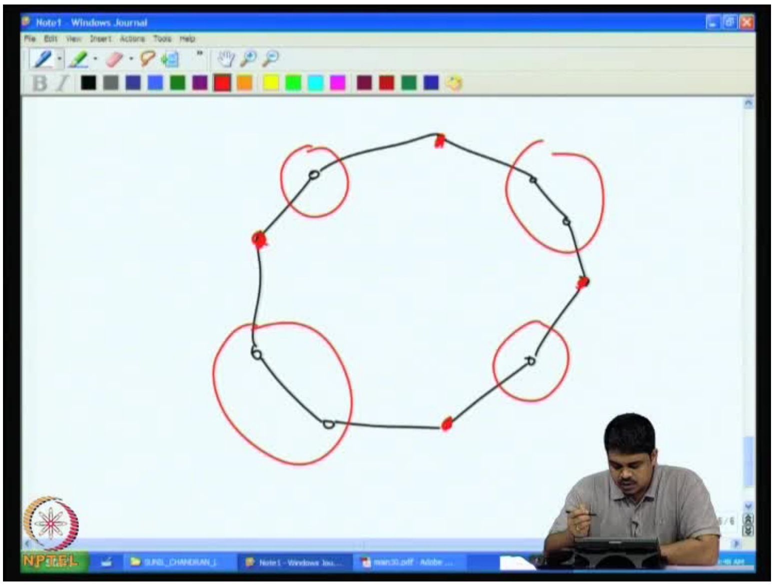

So, for example, if you say in example, so suppose if you take a cycle and you try to removes…, you removes two vertices here, so maybe you can remove these two vertices then how many components results, only two components results. So, number of components which result it is only two; now, we can try removing more vertices may be.

So, so big cycle; now, let us a try to remove some more vertices, see how many vertices are removed, here we try to remove four vertices; the number of components which results is one two three four, four vertices removed and four components result it.

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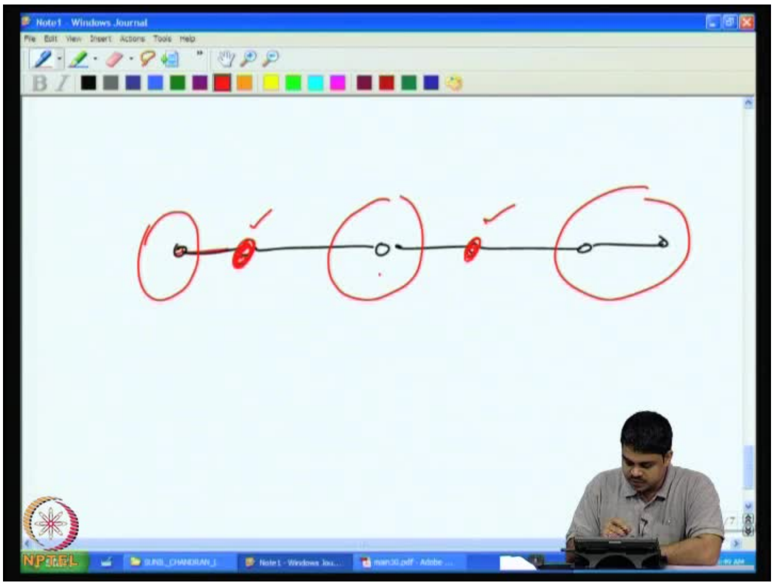

So, in some cases this may not be true; for instance, you take a path, so in a path let us try to remove some vertices, here one two, two vertices you get see one two; the number of components is more than the number of vertices removed, one two three, three components are result it from that it is not. So, the toughness essentially means that, if whichever number of vertices if you removed the the the remaining graph, the number of components is less than equal to the number of vertices that you have removed and see of case this concept can be generalized.

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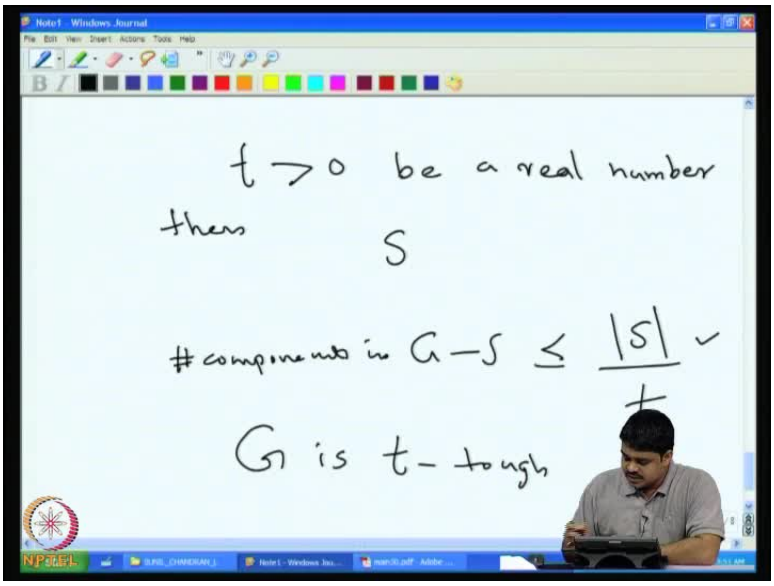

So, see here you show a graph, which is not tough which is and also so a graph which is tough and you see the the we have generalized, it is to give say…, you can you can give arbitrary values of t, for instance, let t greater than 0 be a real number; then you can probably ask, suppose, I have remove s the set of s, s be the set of vertices, I have removed from the graph any set which we have removed; then can I guaranty that the number of components in g minus s is less than equal to carnality of s by t, for any s it should happen, whichever collections of s vertices we remove the number of components in g minus s has to be less than equal to s by t.

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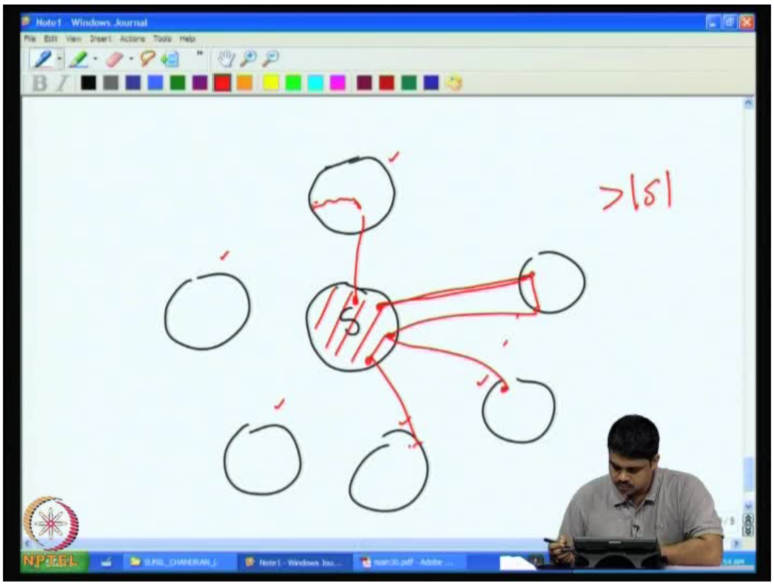

Then we will say that g is t tough; put t equal to one we get the original definition of tough. So, toughness means one toughness. So, why is toughness interesting in the case of in the in the context of Hamiltonian circuits; so, the toughness is interesting, because you see that, if a graph is Hamiltonian, it has to be tough, it has to be at least one tough; why is it so, it is a because, it is because, it is because you you take a set s vertex, suppose you have a Hamiltonian cycle in the graph, now you take some any arbitrary set s of vertices let us say this is s, now it is so happens that the number of components that resulted is more than s; so, that means, the number of components, we got in the result in a graph and you remove this s, then the number of components one two three four five, how many of them? This is greater than cardinality of s.

So, it is not a tough graph, that it is possible that there is a Hamiltonian cycles in it, but if it so happens that, there is a Hamiltonian cycle. So, it should be that, so we should have the portion of the the Hamiltonian cycle here. So, whenever the Hamiltonian set cycle goes out of these, it should enter here; similarly, for each of this thing so at least one vertex, through one vertex it should going, and then through one vertex it should come out may be the same vertex can be used go out of these thing.

So, it should count at least one vertex for entering each component. So, you if you are if you are thinking that a starting from a vertex in s, you can enter through that vertex to one one other components, finish of that component, and come back and then you can enter the another component from that every time to enter a component, you need one new vertex and finally you can come back to the original vertex it is.

So, therefore, we need at least as many vertices as there are components; for instance, if there are more components then there are number of vertices, you will not even one be able to enter a component without repeating a vertex, without re visiting a vertex of s; so, that is it is an obvious thing. in fact, the number of vertices in s has to be at least as many as much as the number of components, otherwise we cannot have a Hamiltonian cycle, it is very clear.

So, therefore, toughness is a necessary condition for Hamiltonicity, but on the other hand toughness need not to be a sufficient condition for Hamiltonicity; is it possible for you to draw a graph which is Hamiltonian, but sorry which is which is not tough, but still Hamiltonian, is it possible to draw? So, that is an interesting exercise, you can try to draw some graph, which is Hamiltonian, which is not tough. So, the I leave you to try that. So, the, but it is not very difficult to find out some graph were Hamiltonicity is available even without toughness.

So, the point here is that, toughness is not enough to insure the Hamiltonicity, but toughness is indeed a necessary condition, sufficient c is not there, how is not sufficient. So, therefore, what we can see is that, but on the other hand if we increase the toughness of a graph a little bit, if you increase the toughness of a graph little bit, is it possible that the graph may become Hamiltonian.

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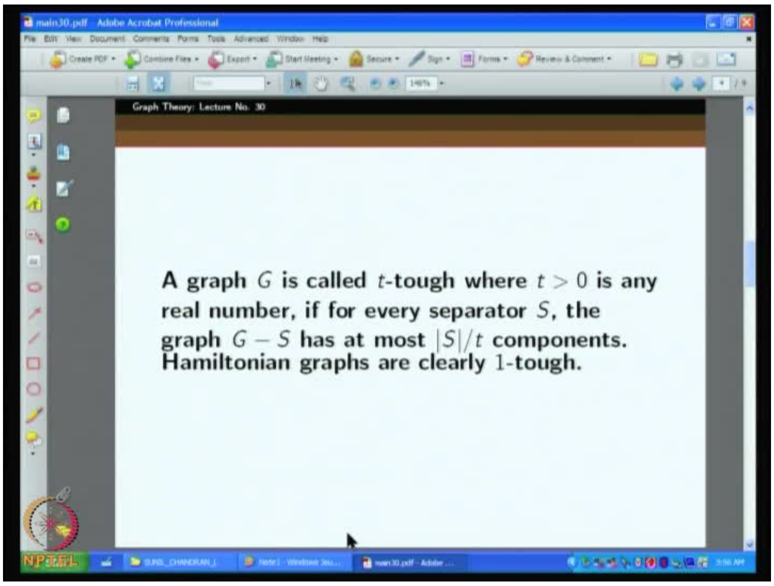

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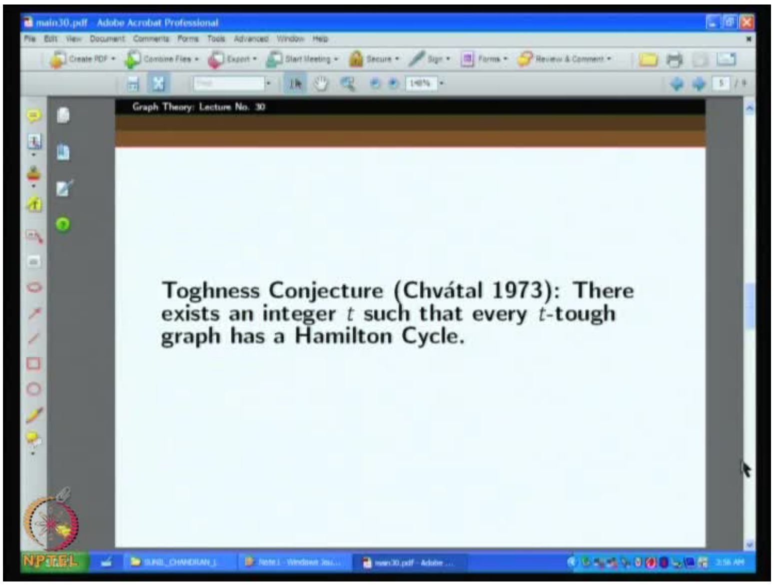

For instance, look at this conjecture, a graph is called…, so suppose the graph is some t tough for a greater value; so so, this toughness conjecture by chvatal’s, it says there exists an integer t such that every t tough graph has a Hamiltonian cycle; there exists an integer t such that every t tough graph has a Hamiltonian cycle. So, how big t should be for this to happen; so, this is an interesting question; so, one can this is still operate; and now this is about the relation between toughness and Hamiltonicity.

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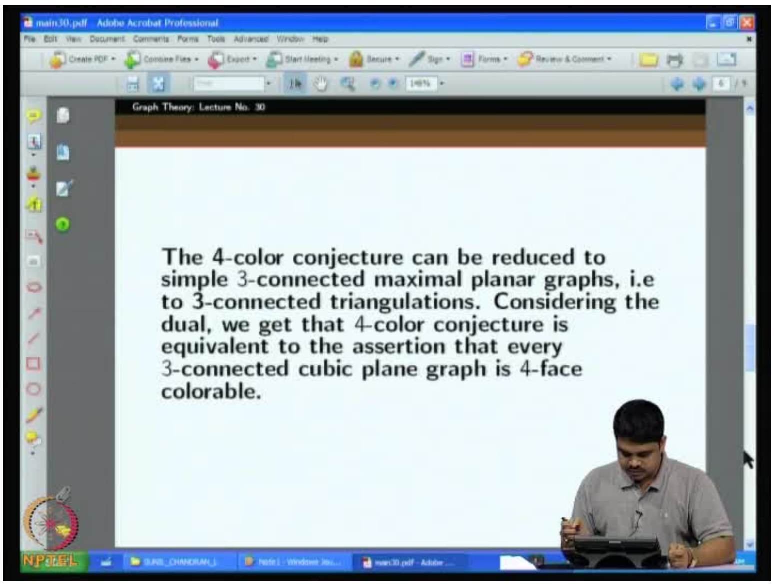

So, well, some minimum toughness of oneness cannot guarantee Hamiltonicity, but it is definitely necessary, but if you increase the toughness to some high value some constant, which is reasonably high; it may be possible that the that may be enough to guaranty Hamiltonicity, so is it some interesting question to try; and now, we will getting to another related results. So, it is somehow the Hamiltonicity is related to the four color conjecture; the how is related to the four color conjecture. So, it is a just briefly explain the the sequence of arguments; this is essentially due to tait and when tait try to proof the four color conjecture, he observed first…, first observation here to make is the four color conjecture can be reduce to simple three connected maximal planar graphs; see the what does four color conjecture say, four color conjecture says we have seen it earlier that every planar graph can be vertex colored using properly vertex colored using four colors at most four colors.

So, of case we can concentrate always we can concentrate on maximal planar graphs, because if a every maximal planar graph can be colored using four colors, any planar graph also can be four colored; what do we mean by maximal planar graphs to remind you, it means that you keep, if you give a planar graph, you keep adding edges until you cannot add any more edge without losing the planar planarity.

So, we also know that all maximal planar graphs are triangulations. So, it is just easy to see that the problem of asking the question of asking whether a planar graph can be all planar graph can be colored using four colors is equivalent to asking whether triangulations; that means, the triangulated planar graph, the maximal planar graphs can be colored using four colors; it is also not very difficult to get rid of cases where the connectivity is two or one or disconnected graphs anyway easy to handle, because we have to only color each component one connected case also is not very difficult to show that.

So, that if each component is can be colored, then each then together it can be colored using four colors; similarly, two connected case can be handled; let I will leave you to work out. So, the point is so by some little effect we can show that, if you want to prove four color, there conjecture we just have to concentrate on three connected maximal planar graph, three connected planar graphs, and we can always maximal by adding more edges; because see that chromatic number can only increase by adding more edges; therefore, if you show it for a maximal case itself and that is enough.

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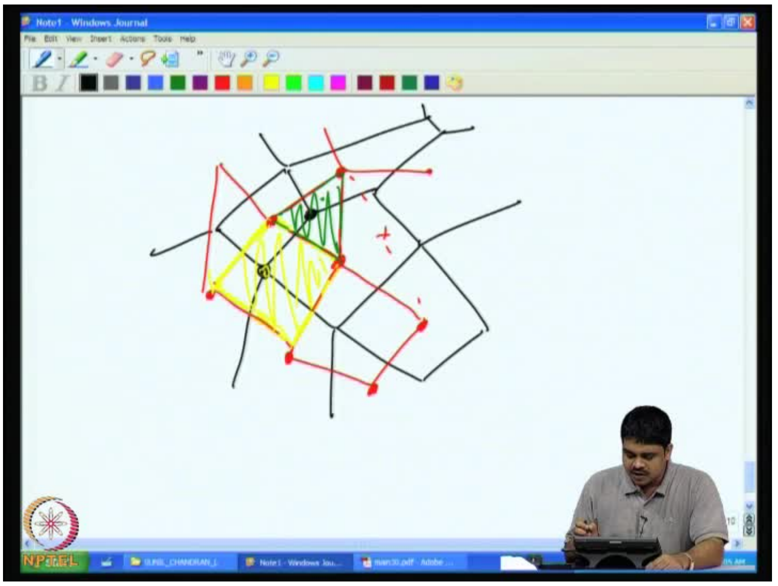

So, now, what we will do is, instead of looking at these three connected triangulations which have the maximal planar graphs, we will rather look at its dual, so that dual is a concept which we did not discuss during…, the when we studied planarity, but it is very simple concept. So, what so suppose so we have a drawing something like this. So, we can consider each face as a vertex says here I can associate the vertex corresponding to each face, so we can constructing a different graph; the original graph is this one, the black one; and now, I will construct red graph from the original graph by this rule corresponding to each face, we will we will assign a vertex and whenever two faces are adjacent, that means, there is one edge between them then I will put a an edge connecting them like this, like this, so because this are separated by just one border. So, it is not that I can put an edge between this and this will not come because this is true for.

So, this is why I can construct a planar graph. So, this is also its very easy to see that, this is also planar graph this is the dual of the given planar graph, black graph, dual is the red graph; now, the point is that, so if you want to vertex color the given planar that the black graph, you can in fact concentrate on the dual and color the faces of it, isn't it? The color the faces, for instance, if you want color this vertex then we could have asked for a face coloring, because so the the reason is that, if you look at the dual the in that dual. So, this will become a face say for instance this is a face of the dual and this vertex have the original correspond to a face of the dual. So, look.

So, therefore, if I want to color the vertices of the original, it is equivalent to asking can I the color faces of the dual in such a way that, whenever two faces are adjacent they should get different colors; for instance, this face is adjacent to say this face this face. So, this face and this face should be colored differently from this face, this is what we are saying.

So, it is very much equivalent to the original question, because you see this face correspond to those vertex which is sitting in the middle; and this face, the yellow face correspond to the vertex, which is sitting in the middle, where essentially is saying assigning the color, one color to this, one color to this; one therefore, we have just asking the that two adjacent vertices of the original should get the different colors. So, when we are saying in the dual faces, which are adjacent should get the different colors.

If you remember the original problem, the map coloring problem, which from which we define the four color theorem is very much like this, because these are the countries; and then we just decided to assign one vertex for each country whenever two countries, where sharing a border, then we put an edge between the two countries, two vertices corresponds; so, we going to the back to the map picture, that is all, that is all we have to see.

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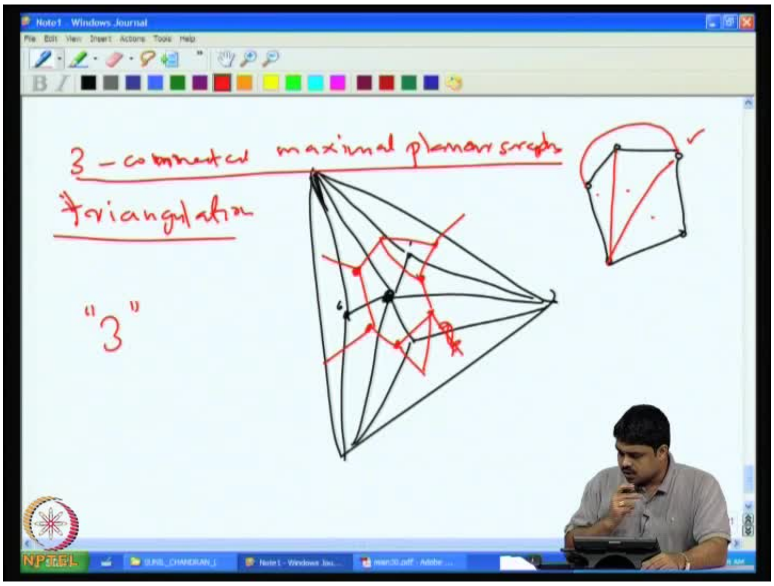

So, therefore, we can see that, it is a question of face coloring coloring the faces of the dual, then what can I tell about the dual, what is the good about dual; the good thing about the dual is that, it is a it is a cubic graph; why is it cubic graph, because in our original graph we have a triangulation, it is a we have a triangulation, though I drew the other picture as our picture will be something like this, as a it will be a triangulation; it will be a triangulation, because because we made it maximal, we just added as many edges as possible.

For example, if you if you do not remember it, if you had a face in the planar graph in the planar drawing like this, then you can always add one more edge like this until it becomes triangulated; you can add one more edge, so this is now all triangles, because otherwise once you get to this picture were all are triangle, then you cannot add anymore edges anymore edges. So, then, because if you if you want add anymore edges, you have to cross, so that is there was the picture.

So, therefore, when you considered three connected…, if you remember we consider three connected get and three triangulate three connected triangulation, that means, three connected maximal planar graphs in the original, in the dual we will get three connected cubic graph three connected cubic graphs. So, this is little bit of cubic graph, because it is very easy to see, because if it takes any vertex here and also you put a vertex for a face, because there are three, it is a triangle there are three edges going out of it. So, that is why every every vertex is going to have degree three now here like this, degree three now, degree three now.

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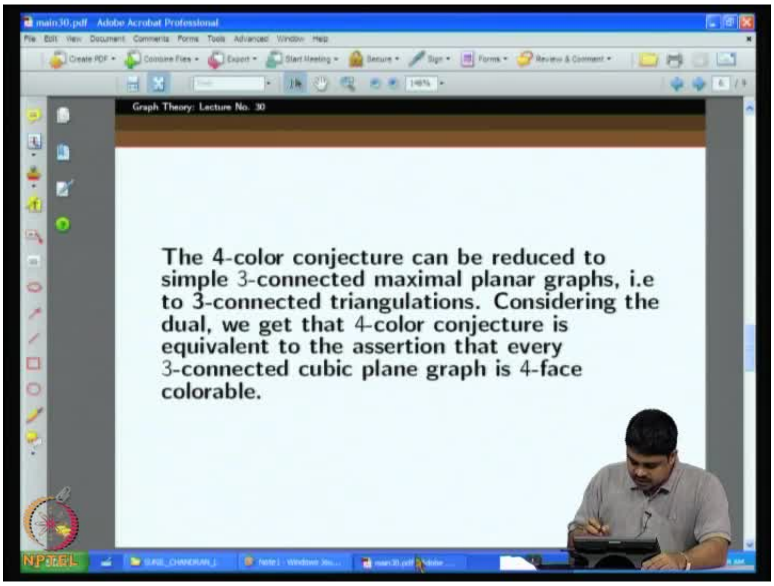

So, it is a dual going to be a cubic graph; cubic graph means, the degree of each vertex, so the dual is going to be three; and it will be a three connected, because original is three connected; and So, the question will become like this, considering the dual we get that four color conjecture is equivalent to the assertion that every three connected cubic plane graph is four face colorable, so we can color the faces of the dual with four colors is what we want to proof; and the dual what is the description of the dual, the dual is just a three connected cubic plane graph.

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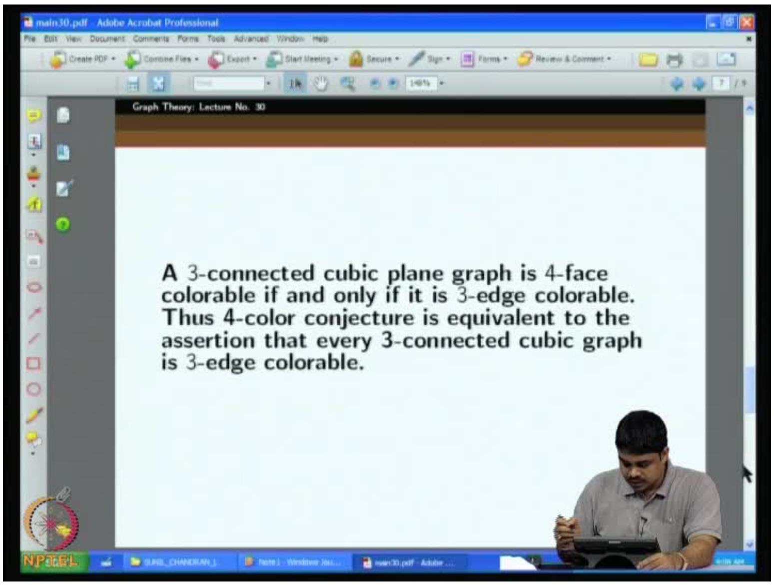

So, if you can show that, every three connected cubic plane graph, the cubic because of the triangulation three connected, because the original is three connected. So, so we just have to proof that three connected cubic plane graphs are four face colorable; now, the next statement is a interesting. So, tait observed that a three connected cubic plane graph is four face colorable if and only if it is three edge colorable. So, instead of looking at the face coloring of the planar graph, we can rather look at the edge coloring of it; we wanted what is three edge colorable means, it means we can color the edges of…, if we can color the edges of that three connected cubic planar graph using three colors, it should be proper edge coloring, proper edge coloring color means two adjacent edges to get different coloring then it can be face colored with four colors.

So, every three connected cubic planar graph will be four face colorable if and only if it is three edge colorable; thus the what is the conclusion? The four edge color then that means, the four color conjecture will be equivalent to the assertion that every three connected cubic graph is three edge colorable instead of go attacking the four color conjecture; we can rather try to solve that every three connected cubic plane planar graph is three edge colorable not cubic graph, cubic planar graph is three edge colorable, this is what will have to show.

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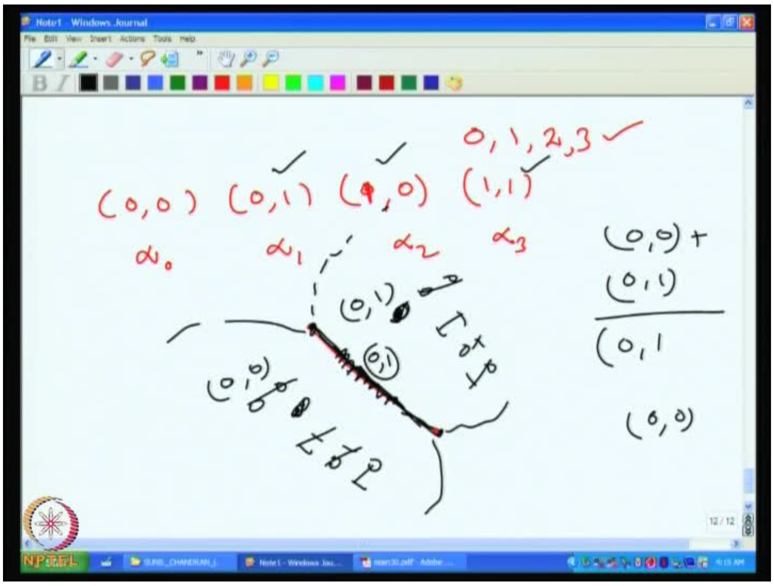

Now, but how do you establish this thing, is it easy? So, to show that, every three connected cubic planar graph is four face colorable if and only if it is three edge colorable. So, tait this nice argument, lets a quickly go through that. So, what it is says is, so, we can we can define colors, the suppose you got a face coloring of the given planar graph, so let us say there are four colors, instead of calling this usually we color the call the colors as zero one two three, this this may be the or may be one two three four; these four colors have the…, but rather than calling it like that, we will use the different notation, we will call the colors as (0, 0) (0, 1) (1, 0) and (1, 1); these are the four colors this can be alpha 0, alpha 1, alpha 2, we can call these these are the colors.

Now, suppose, we somehow managed to color the faces of the given three connected cubic graph using these four colors; now, his argument is, if you just look at any edge, any edge, now this edge is the border boundary of is on the boundary of two different faces, one face here, one face here, two different faces; now, these two faces are adjacent faces, they definitely got different colors, so you may ask is it possible that they are the same faces face; that is not possible, because we are dealing with three connected graphs, because if it is a three connected graph it is not possible to have this face and this face be same.

So, therefore, the color here and the color here are different; for it is not possible to have 0 0 0 0 or 1 1 1 1 or 0 1 0 1 1 0 1 0 combination, because therefore we see that the two colors will when you add the two colors the point is add the two colors; for instance, if this as 0 0, and this is 0 1, and we will add the two colors say 0 plus 0 1 will be added, and then we will write down 0 plus 0 is 0, it is on the modulo arithmetic will not go beyond zeros mode to calculate addition, we will do 0 plus 1 is 0.

So, we will give it 0, one color the after adding these two, we will give the color the corresponding color whatever results from the addition, that color will be given to this edge write this is what; and there are two questions which comes now, is it possible that we only we may get more than three colors if we do this thing. So, first point in notice that, we will get only three colors though there are four colors here, when any pair of them if you add as long as there different pairs, we will never get 0 0 as an answer .

So, therefore, we will end up with either this, this, or this and so therefore, we we we will get only three colors for on the edges; what we have doing is, we are we looking at this edge two faces are they are on two different and they are different faces, they are not the same faces; and now, the face coloring is proper, they colors are different, when you add them up we will get a either 0 1 or 1 0 or 1 1 addition mode two and component wise addition and so we have colored the edges using just three colors.

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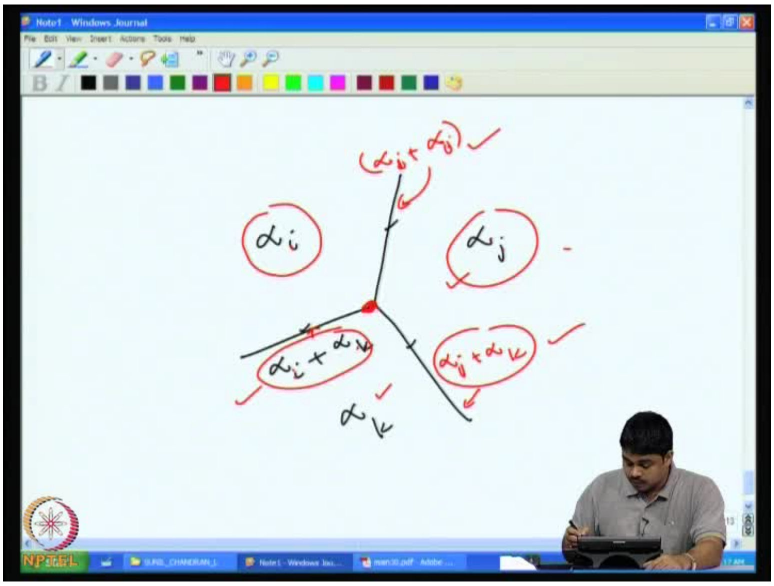

Now, is it a proper coloring? Of case, it is a proper coloring, because if we look at this vertex, so we want to make sure that these three colors and this edge, this edge, this edge is different; now, this face let us say, this is alpha i, this is alpha j, and this is alpha k, what will be the color on this thing, this is the alpha i plus alpha k, this edge will get; this is the color it, this edge will get and the color here is alpha j plus alpha k here this color and here its alpha i plus alpha j this is the color.

So, now you can easily see that these are all different colors, because is it possible to that alpha i alpha k same as alpha i plus alpha j, for that your alpha k and alpha j has to be same, but it is not same, because here this alpha j is given to this face and this face they are different, whether they are adjacent faces they they should get different colors; similarly, if we look at this alpha i plus alpha j plus alpha k, if they are same alpha i has to the same as alpha j, but alpha i cannot be the same as alpha j, because alpha is the color given to this face alpha j is given to this face; and they being faces which share a border cannot be the same. So, they are different colors from the vertex point of view all these three edges got different colors. So, it is a proper three edge coloring.

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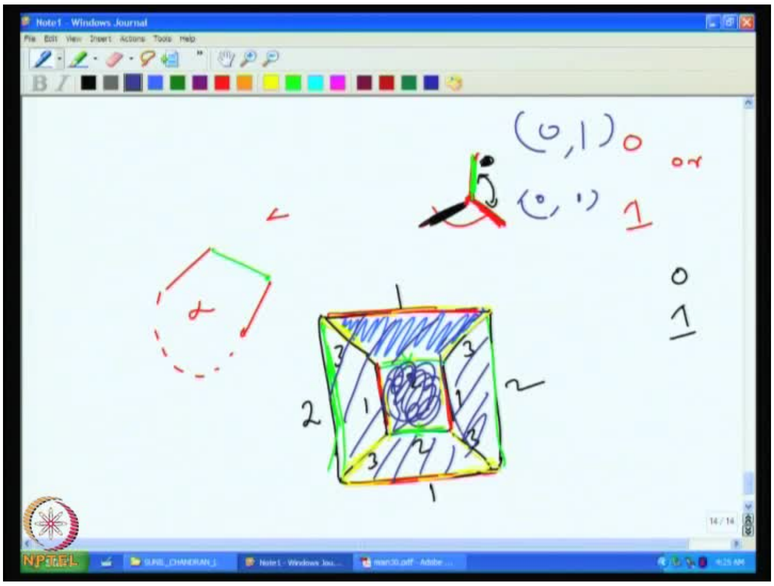

So, what we are now seen is that, if you have four face coloring, we can get a proper three edge coloring of the corresponding graph; now, on the other hand, if you have proper face coloring, so sorry proper three edge coloring we can also get four face coloring the considering it the reverse thing. So, I quickly explained how it is what we do is we because it is the four three edge coloring of cubic graph proper; each color class is the perfect match, isn't it? Because, we know every vertex as exactly three edges incident it and if it is proper three coloring, then you should get different colors on each of this thing; so, it should be like this; so, it is it corresponds to each color class correspond to a perfect match, because everything will get one exactly one edge incident from that one color from one color.

So, so, now, what you can do is, take two of these colors may be green and red is in the green and red; we can call this green and red, if you take then what you can do is, you can consider the edges, which are green and red along they should form cycles; they should form green and red see red green cycles, red green. So, as we have seen it several times that when just two perfect matching is put together, they should form cycles why because, if you if you follow red then green then red finally, it should have to come back here. So, it should be a collection of cycles, that means, as for as the face coloring is concerned, if you just considered this thing there will be collection of faces; these are definitely two colorable, one color here, one color outside like that, because this just a collection of cycles.

Now, for each of the cycles, inner side you can put one color and outside we will have the other color; now, what you can see is, if you had considered say this black and red, there also you will get the same kind of cycle structure and two color two colors are possible; now, based so when I consider red and green you will get collection of faces each face is colored with say colors 0 or 1 0 or 1, because its two colorable when I consider the red and green edges, those edges will partition the plane in to several faces each faces given either a color 0 or 1.

Similarly, when I consider the green black sorry red black edges red and black edges, there also the plane will be partitioned into several faces those faces; suppose, I again coloring using the 0 and 1; now, you can easily see that, if you consider the all graph, then this graph will also partition the plane into several faces and each face of this thing can be thought of as obtained by taking the intersection of two different two faces of the earlier two coloring; for instance, I consider the red and green coloring, we might have got one face and then when we considered the black and red edges we get another partitioning. So, you take one face from that and one face it their intersection will define a face of this thing.

So, an example can be seen like this for instance if you consider this graph, suppose this is colored one 1 3 2 2, and here it is 1 1, and here it is 2 2, and 3 3 and 1. So, let us look at this graph, this when when I say, one we can say all these one's are reds and this all these two's can be greens and two's can be greens and the three's let it be a black.

Now, if you consider a one three face if suppose it discard this. So, how will you construct this face? For instance, from suppose, I am mark it as this face. So, when I consider one three edges alone I will see that say this is a face, this is a face red, and you see a you concentrate on red and black edges alone you see that, this is the face I marking it like this; this is a face, because I am discarding that two edges, this this edges i am discard. So, therefore, this is a face on the other hand when i consider the green and green and red edges, then what will happen? The green and red edges, the one face will be so like this; for instance, so, now, I am say for instance, when I consider the green and red edges one face will be like this; it will be like this, and other face will be inside this inside this.

Now, if you see the outer this face intersecting with a one of this earlier face is given this; in other words, what we should do is, when I consider the red and green face then there are some edges inside the face which are black. So, you imagine that, they are all removed, you will see the entire face; and now, you put back those black edges definitely each of them will be coming from some face of the black red corresponding to the black and red and then so that a particular face will be an intersection of those two faces.

Now, you take the color, because either zero or one is the color of both these faces you pair them up 0 1 or 0 0 depending on where it is coming from that, so this will give you face coloring of the; so, it is very easy to argue that, now it is a proper face coloring, because both the faces cannot have sorry for instance when they are two adjacent thing how can they have two equal colors; if they are I these faces are separated either by one of these edges, which is defining and then based on the two sides; if you look you get two different faces for that particular color therefore, it follows that it it gives a four face coloring.

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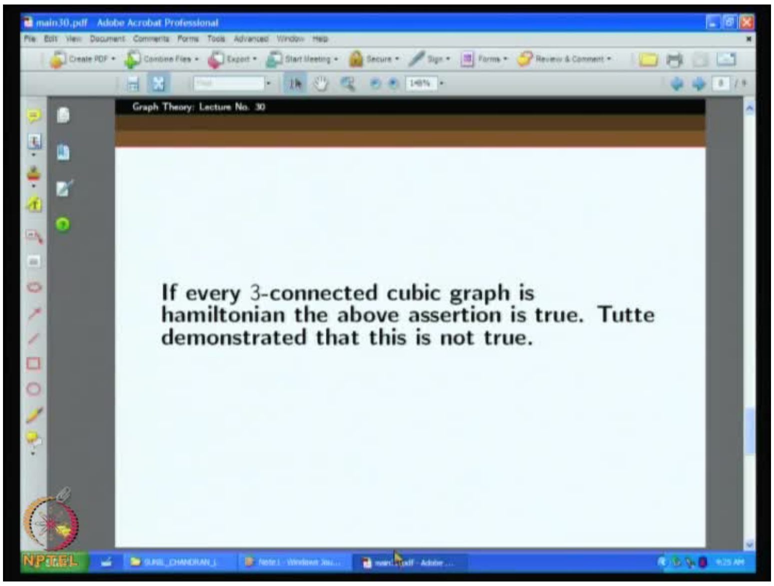

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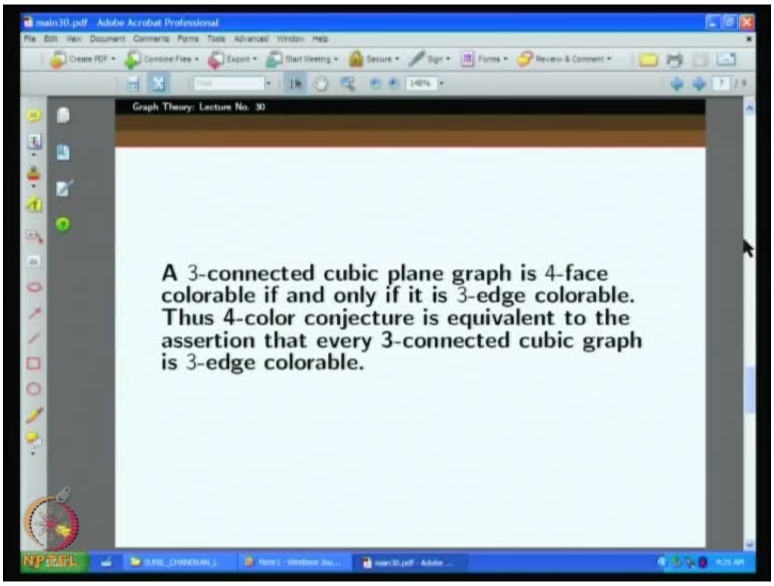

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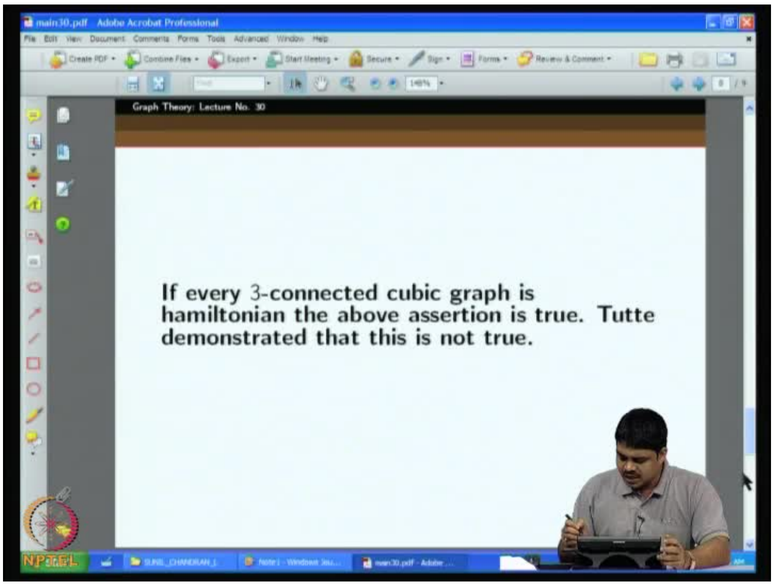

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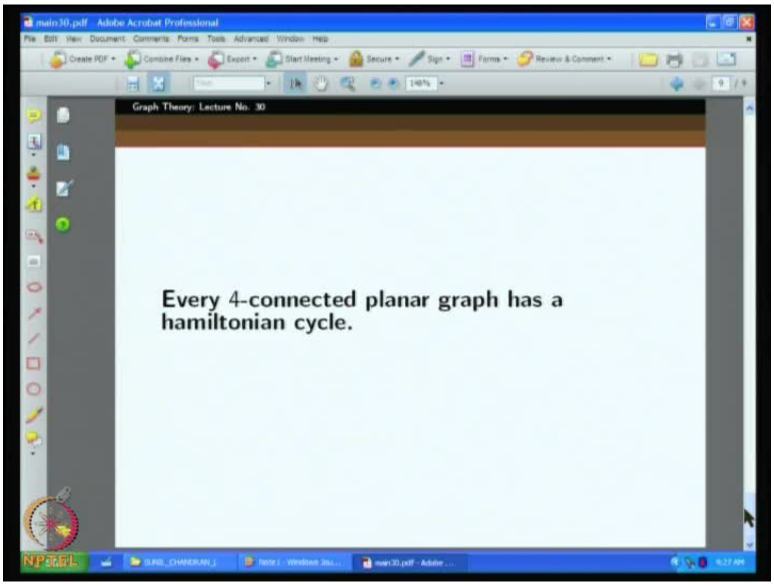

So, now, quickly we what we have done is, we have a proved, we have proven that every three connected here, we have proved that a four color conjecture is equivalent to the assertion that every the cubic graph is three edge colorable; if it is because, if you can three edge color the three connected cubic graph the corresponding face coloring is four face coloring is observed; and then the dual it correspond to the vertex coloring and then that is enough that is that is that is why and now it is a relation between the Hamiltonian cycle tait thought that every three connected cubic graph is Hamiltonian, that is what he thought it is not correct later tait at disproved it; but if every three connected cubic graph is Hamiltonian, then it is very obvious there it can be three edge colored, because the Hamiltonian cycle get gives us two perfect matching's, which have given two different colors and then the remaining is a matching and that we will get the third color; therefore, the assertion will be this will assertion this will be three colorable, if three every three connected cubic graph is Hamiltonian; and immediately, it will be three edge colorable properly and that will mean four color conjecture, but unfortunately that statement is not correct every three connected cubic graph is not Hamiltonian, that it came up with a counter-example later after some time; they were several more examples for this thing and therefore that was but what finally that could proof was that every four connected planar graph has a Hamiltonian cycle.

So, in the next class, we will consider another topic and one more theorem we may want to i i want to mention is that a Fleischer theorem that the square of every a graph is

Hamiltanian every to graph is two connected graph is Hamiltonian, because but I will not touch that interested student may read from the later book distance book.

Thank you.