Graph Theory

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Module No. # 05

Lecture No. # 31

Network Flows: Max Flow Mincut Theorem

Welcome to the thirty-first lecture of graph theory.

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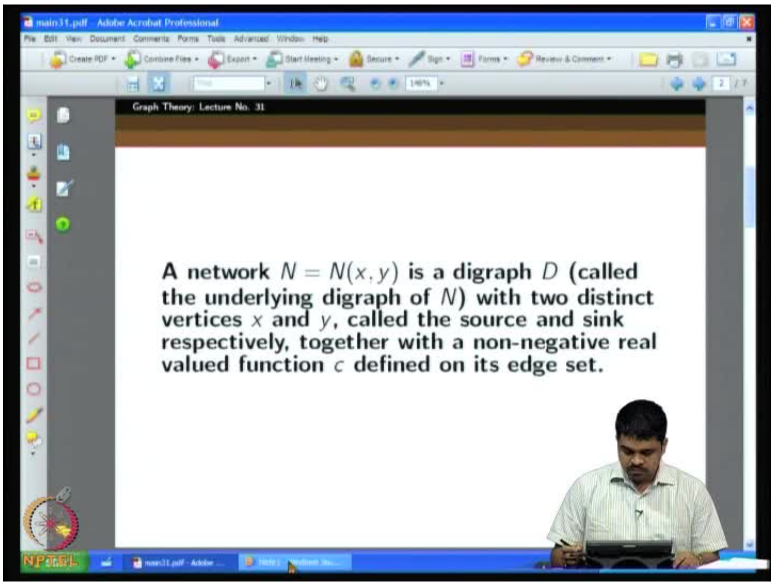

Today, we will look at a new topic namely the network flows. What is a network? It is just a directed graph, I will call it the underlying directed graph of the network n; and there are two special vertices associated with network - one is a source and the other is the sink, say x and y; and also law with each edge or arc of the directed graph underlying directed graph, a non-negative real value will be given.

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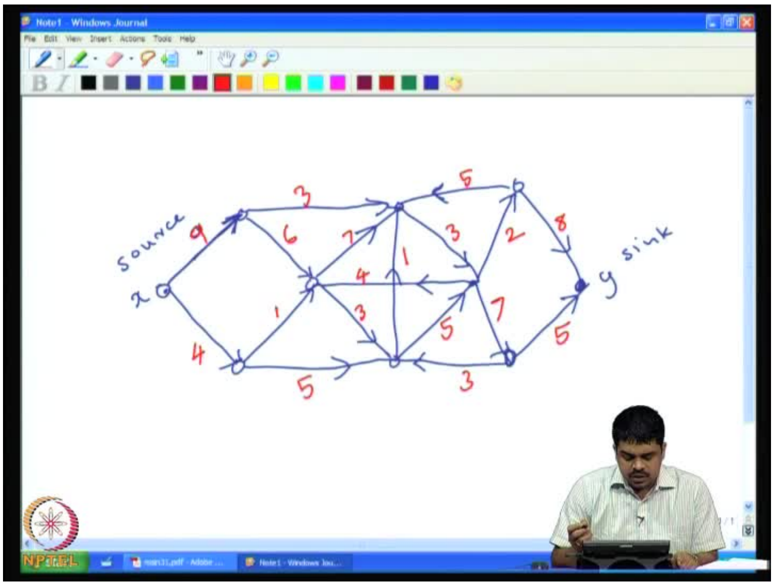

So, that means, a non-negative real valued function c is defined on the edge set; so, this is the capacities, so it can be like for instance; we can draw a networks, let us, we mark one x and so a y, two vertices, special then we drew say two arc's here; and then so let us drew an arc here; so, direction can be like this, and set move vertices strong here so are drawing the underlying directed graph of the network; and here we had here directions being this way direction, opposite direction; and then draw arc's here; so, here is a networks.

Now, the this is the underlying directed graph of the network; so, here is the source; this is the source; and this is the sink; so, you can think that, some commodities produced at x and it has to reach the mark at y; and these are the route through which the commodity has to travel of flow a transported we can say; then you know for each edge, this the routed or road something like that, so there is capacity constrain, you associate a capacitive function on the set of edges, that means, a real non-negative value will be associated with each of the edges - here it is.

So, I will, I am using this red colour for 9 capacities, and 4 so here, 3 here, the 6, 1, 5, and say 1, 4 for this, and 3 for this, and 7 for this, and 3 for this, 5 for this, 5 for this, and 3 for this, 7 for this, and 2 for this, 8 for this, and 5 for this; these are the capacities; so, this this together with the capacity function, this is called a network right; so, what does it means, a capacities means, so you try to send the commodity on a thirteen route say a along this edge, then it says that, the rate at which the maximum rate at which you can transport the commodity through this edges is 3 - 3 units per unit times something like that.

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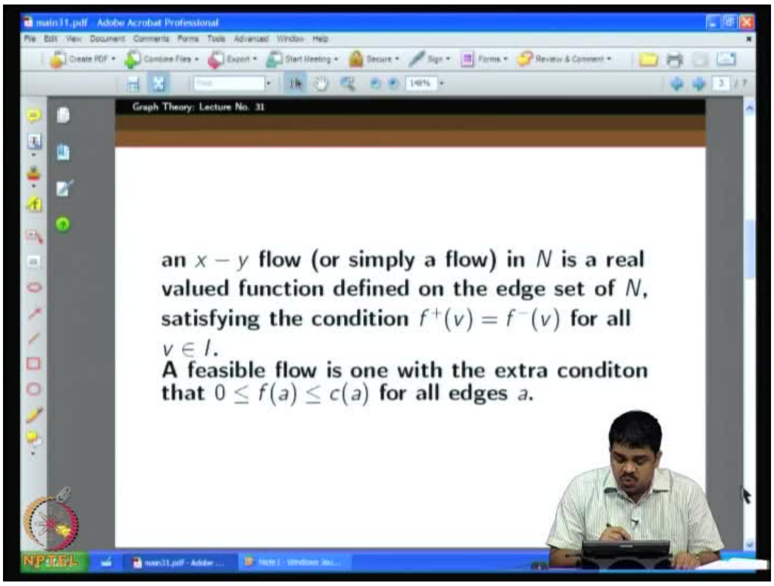

So, but, in practice, what actually goes through that can be less, but we can never be more if it should be feasible; and that is of the meaning of the capacity; that is the capacity of that particular route along which the commodity should be transported not more than say nine unit can be transported in a unit time along this routed, along that edge. Now, let us say, what actually is a flow? A flow is defined on such a network, the flow is again, it is also function defined on the set of arcs; so, it is a real valued function defined on the - set of the network - set of edges of the network, but only thing is, it should satisfied a constrain for each vertex which is not x or y; the outgoing flow has to be equal to the incoming flow, what do you mean by that, so for instance, if I consider a vertex here which is not x are y, x is source, y is sink; other than x and y, the remaining vertices are called the intermediate vertices, we can say I.

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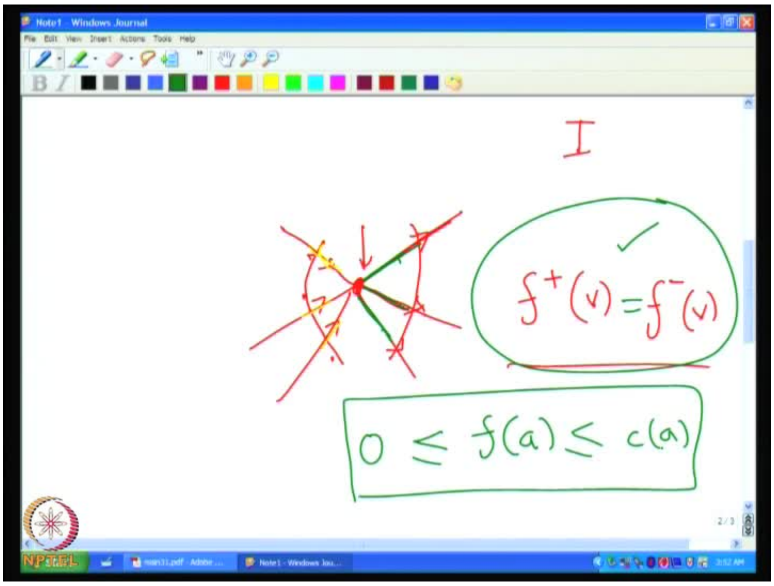

So, the for instance, if the vertex is like this; and these are the outgoing edges; and these are the incoming edges; so, the value the flow should be such that, if you add up these values, it should be equal to the value of the of flow on the outgoing edges; if you add up the values of on this thing, it should be equal to the values on this; so, when we say f, that f is the real valued function which is the flow, it should be satisfied f plus v means, the f plus which indicates the outgoing edges - the some on the flow values on the outgoing edges.

So, whenever we say a function on a set, so that means, you sum up the values of the function for all the member of that set, in this case, edges of the member, and we the here x plus means the member are essentially the outgoing edges of the set, and we sum up the this thing should be equal to these one; so, for instance this these are the - outgoing - incoming edges; so, when you sum up this, that is f minus of we show f minus f f minus of, this is the condition; so, you can see that, this says, whatever comes to the this vertices should go out at the rate of the certain rate unit time, whatever comes into this vertex - should go out - should in unit time.

So, that is conservation constraint, because it concerns right whatever comes in, goes out, means, it is a conservation condition for that this think; so, if these condition is met for a real valued function on the network, then it is called a flow on this network; that f function f defined on the arc set is a flow; so, one should realized, one should not misunderstand or misinterpret the values - the c values and f values - in fact, c values are associated with network, and it is it is not the flow, it is only use to say that the maximum which can flow through an edge is that the capacity of the edge; the flow is another function in fact.

So, for instance, I can I can defined the flow here is, so the also we will say that the flow feasible, if each flow value is such that for for a any arc; if a is an arc edge, if it is non-negative, if that is greater than equal to 0 and less than equal to the capacity of that arc, then it is a feasible flow; it is understandable, because when we say some commodity, so negative flow is not allowed, and also one cannot sent the commodity through an edge more than its capacity; therefore, if the flow value on a particular defined on an edge is at most the capacity of that edge and at least 0, then it is called a feasible flow along with this condition; conservation condition is the important thing along with capacity constrain, this is call capacity constrain; then it is called a feasible flow.

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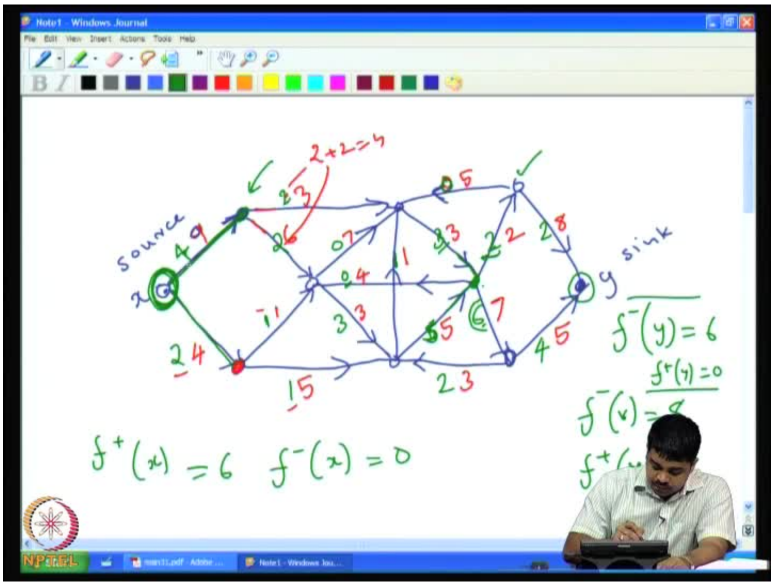

So, you can in this network we can defined a feasible flow, say for instance let me put it using the green colour indicate flow values here, for c this is less than 9 here, we can have set two, so this is less than 4, therefore the capacity constraint and it is also greater than equal to 0 and here I am putting 1 put, therefore a here I put 2, and here I put 1, and these are all satisfy the constraint 3, 3 is less than equal to 3 greater than 0 here, I am putting 0, and here I am putting 0 and here I am putting 2 and so on.

So, you can check whether the conservation condition is satisfied at this vertex; so, this this is the vertex belongs to i; this is an intermediate vertices; so, here what is the flow on the incoming arc's 4 and the flow on the outgoing arc's is essentially 2 plus 2 4 - 2 plus 2 4 - right here, the this is here, this 2 and this 2; therefore, the capacity constrain is satisfied for this; here also we can check for instance, is incoming is 4 outgoing is sorry incoming is 2 and outgoing is 1 plus 1 2; so, here also it met; and I am completing the other values here, 0 here, so it is the green 0 here, and 3 here, and here it is 1, and here it is 6, say here it is 2, and here it is 6, and 2 and 2 and 4.

So, you can check for instance here; this node is the conservation constraint satisfied, which are the incoming arc, this 1 and this 1 totally 3, 3 here, plus 3 here, and 3 here, and 5 here, this is 5 sorry 5, 3 plus 5 is 8, here outgoing on outgoing arc's, we have 2 here is a 2, then here is the 6, already 8 and here we have only 0; therefore, 8 outgoing and outgoing arcs that is f minus of this particular vertex we say is equal to 8 and f plus of this particular vertex is also a 3 plus 5 8.

So, f minus equal to f plus here; and for all the vertices you can verify like that, so for the intermediate vertices; while for x, this is not true, because the only outgoing arc therefore plus 2 f plus of x is equal to a 6; while of minus of x is equal to 0, because therefore no incoming arc at all; what about the y, that is we have f minus of y is equal to 2 plus 4 6; and f minus of f plus of y value that is 0, because there is no outgoing arc here.

So, this is these an example of a flow on this network; see the key points are there, is there two constrain that you have to satisfied; one is the conservation constrained, which is to be satisfied for all internal intermediated vertices not on the source and sink; the other vertices remaining vertices; so, that means, whatever is coming inside into the vertex should go out. So, the sum of flow values on the incoming arc should be equal to the sum of flow values on the outgoing arc; this is one main condition. And the other condition is the capacity constraint namely that all for all values should be non-zero sorry non-negative, that means, greater than equal to 0; and there is a bound upper bound on each arc each which namely the capacity of that arc, we cannot push more than the capacitive, that arc it always; so, in real life, because we cannot in put in push infinite commodity through a capacitive link, while of case if there is an infinite capacity link; and you can find such assigned a path starting directed path starting from x and you can reach y through involving only links of infinite capacity, then you can always push finite flow.

So, but, in practice that may not happened; so, if usually will be some capacitive associated with any network, real life network, so this is the definition of the flow. Now, the next thing is to defined, what is the flow value? What you mean by flow value? So, the flow value means, how much we are able to, the rate at which able to send commodity from x to y; x is the source and y is the sink; from source to sink in what rate we can send the maximum rate at which we can send commodity through the network.

So, this will depend on the network; given a network along with the capacitive, so the edges we can ask this question, what is the maximum flow? Maximum flow; so, the therefore, the value of the flow is essentially how much we can send, what is the rate at which we can send from x to w; and given a flow, there is a value associated with it, for a particular flow, you know, that flow will send certain amount of commodity from x to y, that that amount is supposed to be the value of flow, how do we define it? That is the question.

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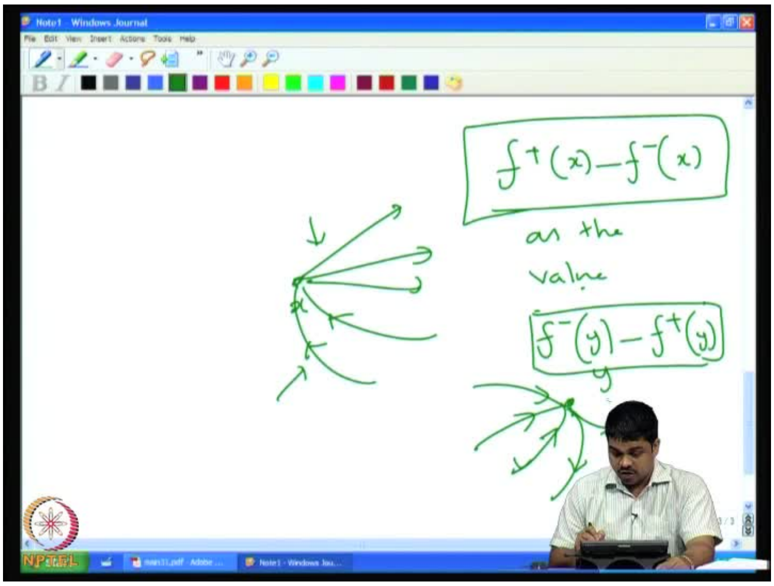

So, the let us say this, this is natural to thing like that, for a very intuitive thing that so the what we send out of x right is the value of flow; here we sending 4 along this thing and 2 along this thing, so they show, therefore it is natural to assume that to think or defined that the value of the flow is 4 plus 4 plus 2 here, for instance, but then the source need not be always like this, you can have a source where see source x may be may be some outgoing edges and to the source some incoming edges also may be there; in that case, we will talk about f plus of x minus f minus of x as the value as the value of the flow, because that is the net outgoing flow from x - net outgoing flow from x; suppose, we defined this is the value of flow, so which essentially is like if you if you consider this x is the source, it is like the outgoing total outgoing, the rate at which the commodity going out of x, that will be the value of flow.

Now, but, then is it consisting; so, it is very an interesting thing is that, so if you see why so for instance sending x from x this much, then is it reaching y, that much, the same amount is reaching, more is reaching or less is reaching, this all this things are interesting for instance in y, what will be interesting, what will be interested, what is entering y; so, for instance, there are certain edges going into y, and there are certain edges going outside of y, you will be interested f minus of y minus f plus of y, this will be what entries y the net inflow into the y.

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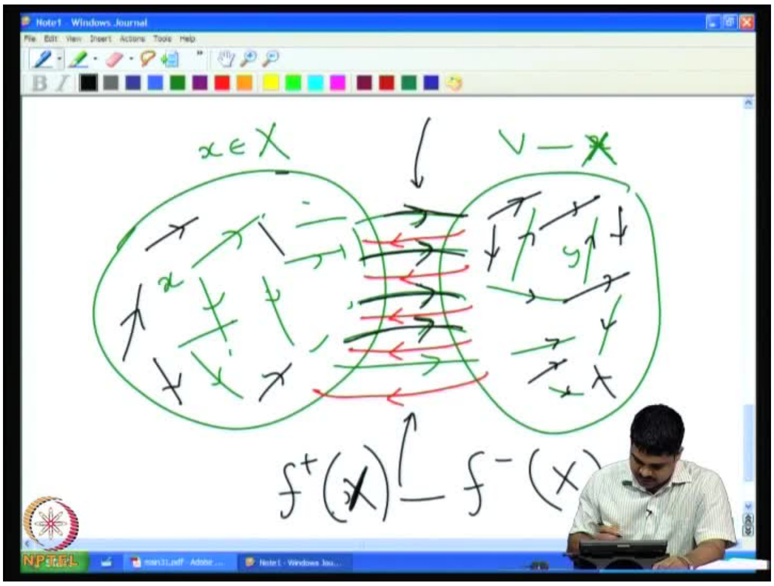

So, therefore, whatever is sent from x should reach y; so, we are very much interested to know whether this quantity the net out flow from x will be equal to the net inflow in y correct; so, we will rather proves a much more stronger statement, we will say that, not only the that, there equal, in fact, if you take any set suppose you have this network with x and y here, and then you identify a collection of vertices in the network; some vertices in the network with some connections some connections and this lets say this is x; so, we have this source is inside it, y is outside it; this is v minus x; so, this is v minus x v minus x.

So, therefore, there will be some arcs going from this side to this side; there will be something inside here; so, will be a some arc going from here here to here; and of course, you should also count the arcs which are there are some arc coming back also, there are some arcs which are directed backward from this x to y.

So, some this side to this side; so, of case, there are several edges inside with various directions, various direction; now, so, what will we expects, suppose from the source the commodity is flowing into y, we would tested in how much is the value of the flow here, what will give the value the flow here, what what is meant by the value of flow at this cut, this point is, how much is the outgoing flow here, namely, f plus of x minus f minus of x; so, what you mean by f plus of x? Essentially, if this is the set x, the edges which are directed, such that, it is going out of x; this collection of edges is called as plus of x.

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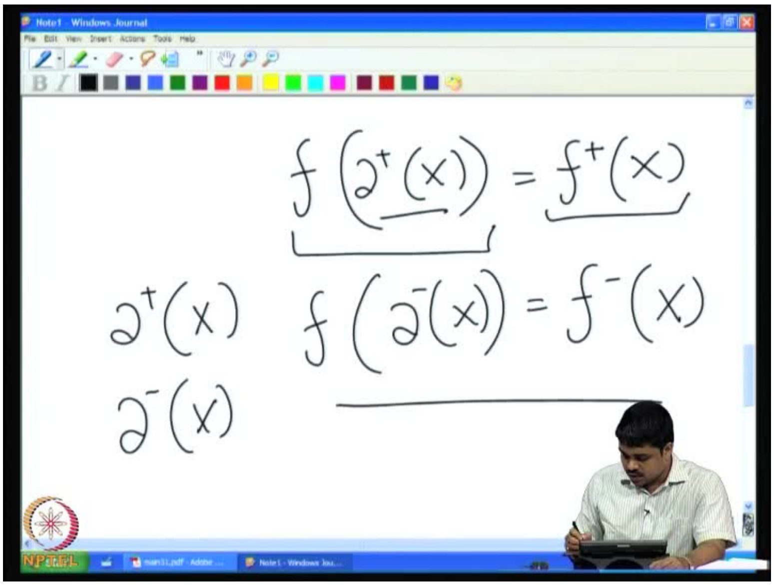

In fact, the notation for is dou plus of x, but then the flow on dou plus of x is as a short notation will use f plus of x; so, usually dou plus means, the set of outgoing edges of x; dou minus of x means, the set of incoming edges of x; essentially, this f means, the some of the flow values on this set and then from other flow values on this set will be this; and this is short notation is given for is f plus and then this is f minus x. Now, we are interested in f plus of x minus x minus of x y, because the total out flow is that, because this much is going out, because and this much is coming in, so the total out flow will be this minus this.

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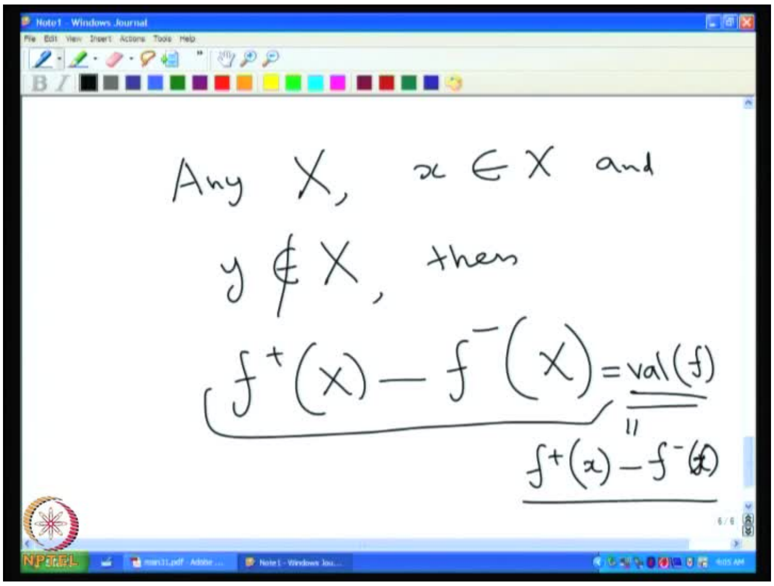

So, what we claim is, now we claim that, if you any take any x, such that our source belongs to that set; and y does not belongs to that set; that means, y is mean v minus x; then if we consider f plus of x minus f minus of x, this will be essentially the value of the flow that we have the defined; this value of the flow we have defined, what was that, it was with respect to the source, means, the net outgoing flow of this source was the value of the flow by definition; that is what we was thought, we should take the value of the flow to be.

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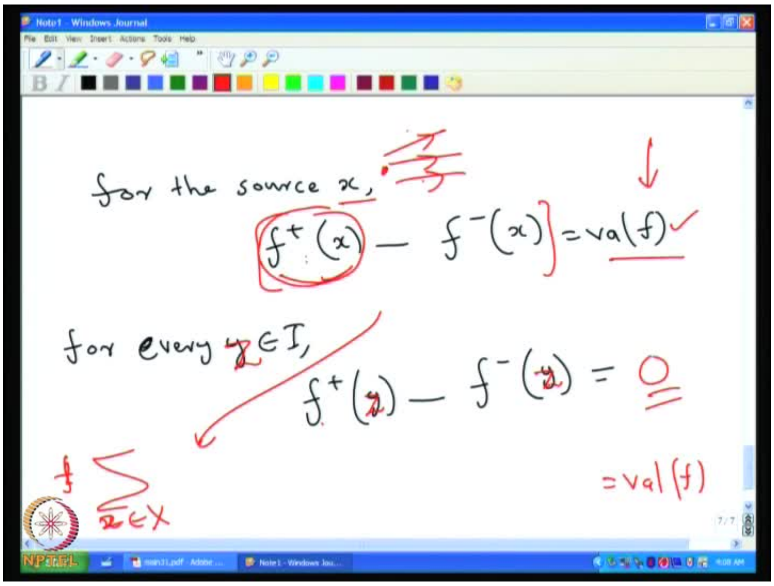

So, it makes sense because not only that it is the net flow going out flow of the source, it also the net flow going out of any subset of vertices containing x and not containing y; so, you can see that, it is something going out of x and towards y, that is the interpretation of that; but why is it is true, so that is because because you can see that, if you take x and right down for for x, for the source x, for the source x we have f plus of x minus f minus of x equal to value of f, this is the definition; this is the way we have the way of x; and then for every y element of I we have, f plus of y minus f minus of y is equal, so maybe we should use a different notation say for every set element of y say set is equal to 0, why because, this is the intermediate node; and this is by definition of flow, this condition should be met, this is the capacity sorry conservation constraints right and we do not have y in this thing, y means, source so sink is not there in this collection because in x.

So, when sum up of this equation for all summing for all x element of y sorry for set element of x, what will happen? These inequalities are summed summed for all the vertices in the given collection x, we will happen, so we see that these side will be, these side this right hand side will sum up to definitely value of f, because only one of them here is nonzero and all others are zero; so, therefore, only value of x, because see for x, this is value of x, all others because there all from intermediate vertices why is not in our collection; therefore, these are going to be zero for all others; therefore, the total here will be value of f, but when we sum up these things, so what will happen? You can see that, this is essentially the collections of values here, in fact, plus of x means, all the edges which are out going from a x; the values we are summing up the flow value on these edges and that is what this is; and this is the sum of the values of the flows on the incoming edges of x, similarly, here this are the values of on the edges, which are going from set and these are the values some other values of the edges which are incoming to set.

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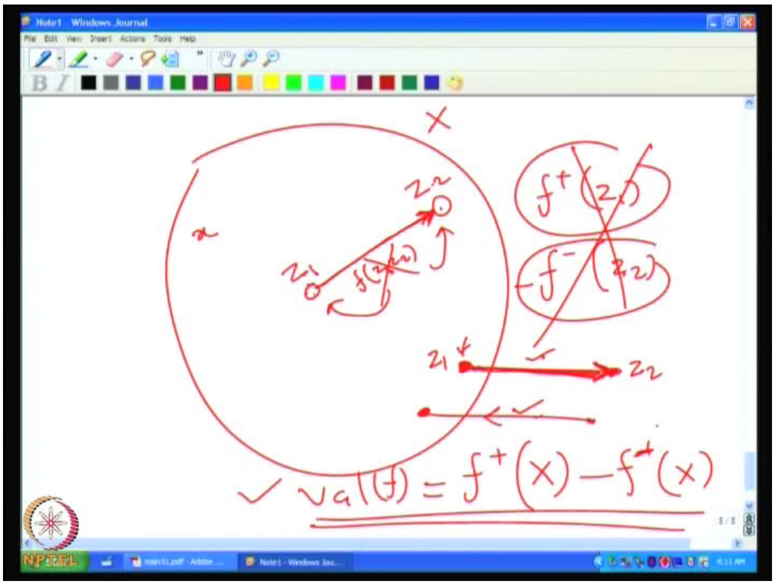

So, it means that, if you consider any edge like this, suppose you consider any edge like this, some z 1 z 2; and suppose, both the end points are intermediate, then what will happen? This z 1, so z 1 z 2, both are in our z not intermediate; suppose, both x 1 and x 2 are in our set x, so this is the x, and this is x, and then both of them are, this edge will the value here; say suppose the value of this edge say, z 1 z 2 say f of z 1 z 2 will appear in the in the some force z 1 x, because f plus of z 1 we have this value; along with many other values this will be included in that.

Similarly, in the f minus of z 2 along with other values f of z 1 z 2 will be included, but this come as minus, this come as plus, so this will get cancelled, so the positive term from this, and positive negative term from this, therefore they will get cancelled; so, what kind of edges will not get cancelled? The only the edges which are of this type, which are one end point here and other end point out side will not get cancelled; so, for instances, there are two type such edges, one is the beginning n point, the start end is one side, and the head of there is on the outside or another type is like this.

So, the it is coming from outside to inside; it these type kind of edges for instance z 0 is here z 2 is here, so there is z 2 components, this component is will not be therefore this plus component will remain; so, that is these value will remain as plus; so, for all the outgoing edges from the entire set x or out, there value will remain as positive and then the on the incoming edges whatever is there there will remain as negative; therefore, we will get, when you sum of these kind of contribution, we will get a f plus of x; and when you sum of the kind of contribution from this, incoming edges will get f minus of x f minus of x, so that is exactly value of f, value of f is the sum on the other side.

So, this is what we get, so we see that value of f that, we have defined as the total outgoing flow from x, just looking at x, that means, f plus of source minus f minus of source, that is essentially the value, the total outgoing flow of any set containing the source and not containing the same so by the simple argument that we have no established.

So, now, the next stages, because the value is defined, value of the flow is defined, and that is consistent in since that any cut which separate x and y, if you take the to flow going through that cut, I mean, from a subset x which containing the source, the net flow which the going out of that set is essentially equal to the net flow which is going to out of a source; therefore, that is definition of the value of the flow; and then we are interested in maximize this value, which is which flow will achieve the maximum, because there are the various possible given a network; we can have a several flow functions on the network, we want to get the flow which will maximize the value of the flow, because that will achieve allow us to send the maximum commodity from form from the source to sink.

So, what flow will achieve the maximum commodity? This the maximum flow maximum value, this is the next question; so, two, we will next show that, this to answer this question, you have to establish a connection between the value of the maximum flow and the value of the some of the capacities on the cuts of the networks; so, given in a networks, we now it the directed graph underlying is the directed graph and then we can we can consider cuts of these underlying directed graph, such that, x is on one side and y is on the other side, source is on one side and sink is on the other side, such a cut will be called a cut of the network; in fact, usually, if it was a cut in the directed graph, we can even consider cuts with both x and y on same side, the source and sink on the same side, but we will not consider such cuts for the in the case of the network; therefore, whenever we say cut in the network, we mean, source is on the one side of the cut the sink is on other side of the cut, so x belongs to one side x and then y belongs to minus x, these is the meaning of that.

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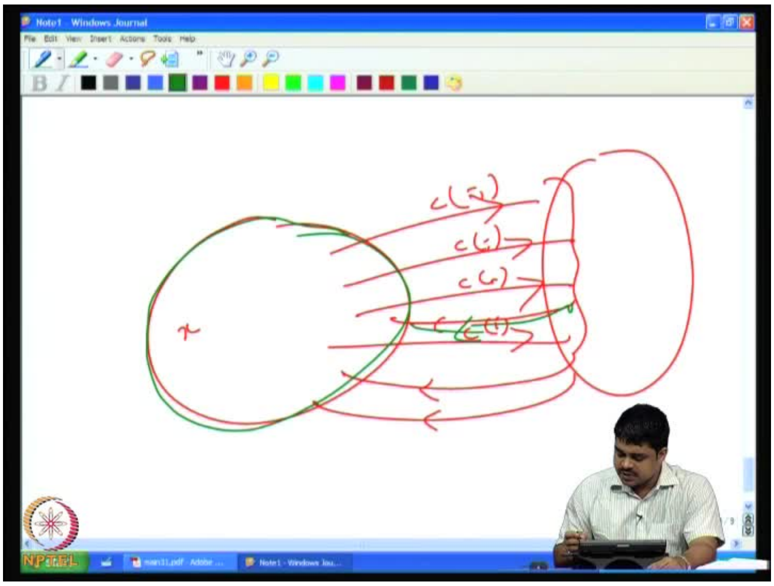

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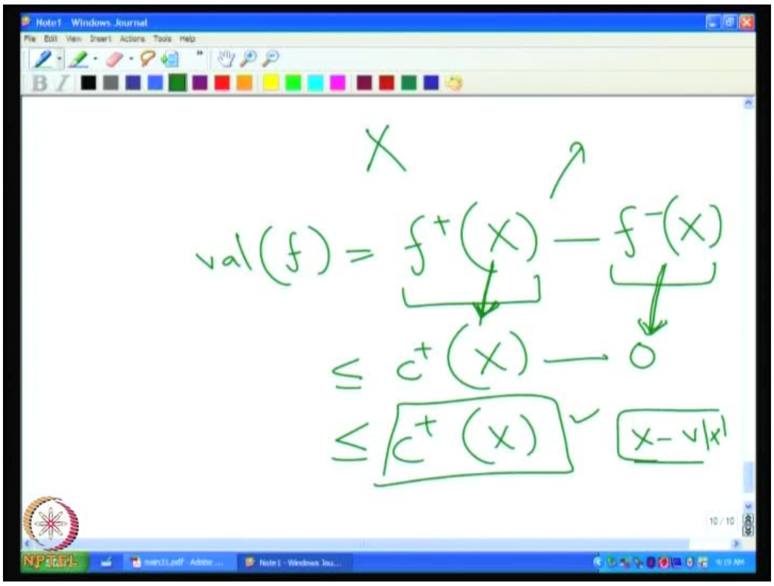

Now, the what is the capacities of such a cut, is essentially, if the capacity of the cut means, if these is the cut, so x is here, and then source is here, and these are the outgoing edges, the capacity of the cut means the sum of the capacity so the outgoing edges, so c of this edge, c of this edge, c of this edge, c of this edge; if the sum of edge capacities, this capacity, this is the some of the capacity, this is the capacity of the cut; there are the incoming edges, they do not, they do not contribute the capacity of the cut; the incoming edges like this will not contribute to the capacity of the cut; when is a capacity of the cut we are just considering the outgoing edges, so remember when we say the outgoing, we are considering the cut is the part containing the source outgoing from that, the source side to the sink site, that is that is the direction; and the first and easy observation that we can make is that, any flow, is it is the feasible flow has to have value less than equal to capacity of the any cut; you can take the any cut in the network with c as source on one side and sink on the other side; of case then the capacity of that cut will be an upper bound for the value of the flow, which our is the flow otherwise cannot be a feasible flow; why is it so, it is easy to see, because already we have already seen that, the value given x suppose x is a cut; than the value of the flow is equal to f plus of x minus f minus of x and this is what this is essentially the flow value on the outgoing edges from x; and you know, each flow value has to be less than equal to the capacity of the outgoing edges, therefore, this is defiantly less than equal to the c plus of x; that means, the sum of the capacity of the outgoing edges x; of case, this is minus, what minus, what of case we can when it is smaller and smaller, these quantity, this entire quantity can became bigger.

But, so, how small these can go in the waste case, they can it can go at 0, because you know the 0 is lower bound for any flow value, therefore, this is minus 0; therefore, this is less than equal c plus of x; and this is exactly the capacity of this cut x v minus x cut, therefore, this is value of the flow can be at most the capacity of the cut; but then one will ask, can this is an inequality value of the flow for any flow is less than equal to capacity of any cut, but can it be ever be equal; first, if you carefully look at this thing, you can easily see that, if it is equal, then the this has to be 0, and this has to be equal to c plus of x, otherwise how can to equal; because in both cases, here also it should achieve the maximum possible here, it should achieve the minimum possible.

That means, if the value of f is actually equal to the capacity of that cut; then we know that, each flow value, flow value on each outgoing arc has to be equal to the capacity that arc, which means, in our terminology will say that, each arc has to be saturated; their saturated mean, it achieve the maximum capacity possible on that arc; the flow that the is flow on that the particular arc edge is exactly equal to the capacity the maximum possible, then it is equal to the capacity, it is so that arc has to be saturated with this flow.

So, otherwise, even one arc is not saturated, it will the sum will not become equal to the capacity of will be totally equal to c plus of x; similarly, for every incoming arc it should have 0 capacity, we should have 0 flow value, because if it is little more than 0, there will be a reduction here and how can it be c plus x; so, the equality is achieved only if all the outgoing arcs are saturated by the flow and the incoming arcs are 0, have zero value, it is a 0 value flow on the arcs that, that is only the way you can achieve the this thing.

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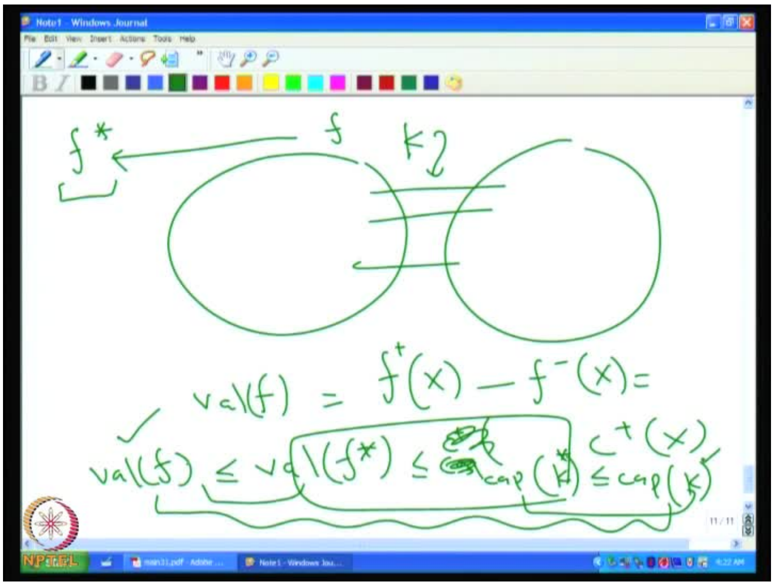

So, now, suppose, a flow is such that, this condition is met, that mean the flow is such that, its value is equal to the capacity of a particular cut x, then we can immediately say that it is a maximum flow; why is it so? And not only that, it is a maximum flow and among all the possible cuts of the network what we located is a minimum cut, why is it so? Because see this is the cut, where we have seen the equality and then say f is the flow such that, f of x write the flow f plus of x minus f minus of x is equal to c plus of x, supposed it happened, this is remember these value of the flow.

Now, anyway, suppose, f star is the flow which achieve the maximum; so, one will ask, will they be a flow which achieve the maximum of case, but see if it is, as I mention before, if there is an infinite final sequence of arcs so directed path which all infinite it can infinite; because, in general it, but general let us assume that, for all the capacities are find it there is a maximum flow that can be achieved; so, there is the maximum flow; suppose, so then, let f star be that maximum flow, then value of f has to be less than equal to value of f star because the definition of the maximum flow; and then we can see that, these f star also being the flow satisfied property that, it is less than equal to capacity of the cut the c plus of that cut, so c plus of that particular cut, say let us say, let us call it the cut; x is the cut, let say, l for this cut along we can give name cut and this let us say the capacity of the cut and then this has to be; so, capacity of minimum cut and this has to be less than equal to this particular cut capacity of the, because the minimum cut any way equal to less than cut.

But, then we know that, suppose, we assume that, this and this is equal, value of the flow is equal to the capacity of these cut, then it means that, this and this also are equal and this is also equal, this is also equal, everything is also equal, that means, this is the maximum flow and this is the minimum cut - the minimum cut; so, our inference is that, some have if we can observe a flow such that, it is value equal to the capacity of the cut, then the particular cut has to be a minimum cut, and flow which have obtain now is a maximum flow we have need located maximum flow.

So, that is a very interesting observation - useful observation - because if you are looking maximum flow and want to get a maximum flow; this is the way to identify that, we have indeed obtain a maximum flow, that means, this identify some such that, the cut saturated by this flow and not only that how do we identify that it is in fact saturated; so, all the outgoing arc has to be saturated also also all the incoming edges has to be of 0 value; this is only the way achieve the maximum value sorry saturated that cut, because you know the, otherwise if there is any non-zero value on the incoming edge, the value of flow will be less than c plus of x, because c plus of x minus something; while, on other hand, any outgoing edges was unsaturated, then how can the value of the flow f plus of x minus of f minus of f is equal to c plus of x.

So, the you cannot achieve that; so every outgoing edge is saturated and also every incoming edge is 0, 0 value of flow on that incoming edges; so, in such a situation, only the minimum that cut will be the value of the flow will be a equal to the cut, and then we are sure that, here what we observed is a maximum flow, and these cut is indeed a minimum cut; this is, this is the key observation; now, the next question is, should should it happen, so we know that the maximum flow has to be less than equal to the minimum cut, that is very clear, because as we have seen the flow value is a f plus of x minus of minus of x and it can never be above c plus of x the value of the cut; so, it is very clear, but can it never achieve for a given network is it guaranteed that you can reach that.

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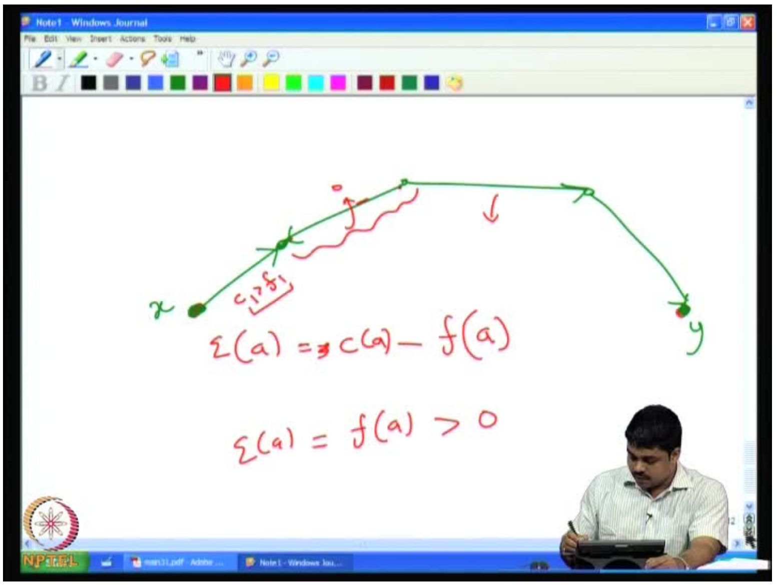

So, the next theorem will assert that it can always be achieved and in fact the their excessive flow, there is always a flow in a given network whose value is equal to the value of the minimum cut - this is the next statement this is called the max flow min cut theorem to how do you prove this think; so, to prove this, think this is the idea, so we have to defined; so, suppose you given a flow, now if it is not the maximum flow, we will we will show how we can increase the value of flow by a small technique, this is simple idea; so, it is essentially, if you give a flow, then we what we going to do, showing a method by which can increase the value of flow; for incidence, see look at, so say I am just taking, this is the source, suppose you have sequence of this arch here, some other arch also can be there, but it just identified this set of arch; this is the path from x to y; and it so happens that, it is so happens that, suppose there is a this arch has capacities c 1, but the flow value is some f 1 less than c 1, that means, strictly less than c 1, that means, not full here; on the other hand and this here the flow value is not 0, it is more than 0 more than 0; and here the flow values is less than the capacity here and here the flow value is again less than the capacity.

So, what we are thinking is to find out a path starting from x to y; sometimes containing forward edges, sometime containing edges backward edge; it is the backward edge, but which gives some possibility of increasing the flow, how does will formally, we will defined like this; so, we will consider, if all the forward edges in this path has to be unsaturated, forward edges has to be unsaturated, so that something more can be pushed through that edge; that means, it is not equal to the capacity, even now there is the some difference; so, this different for an edge for given edge, so this f of c of a minus f of a will be defined as the as the epsilon of that, what is what is left on that edge, what can more to be pushed with edge; on the other hand, if it is a backward arc on the path, then we will take it as the value of flow f of itself, so that is a epsilon of a which essentially mean that, we can cancel that flow and go forward, that is intuitively, that is the meaning of that, but that is why we want to be greater than equal to 0.

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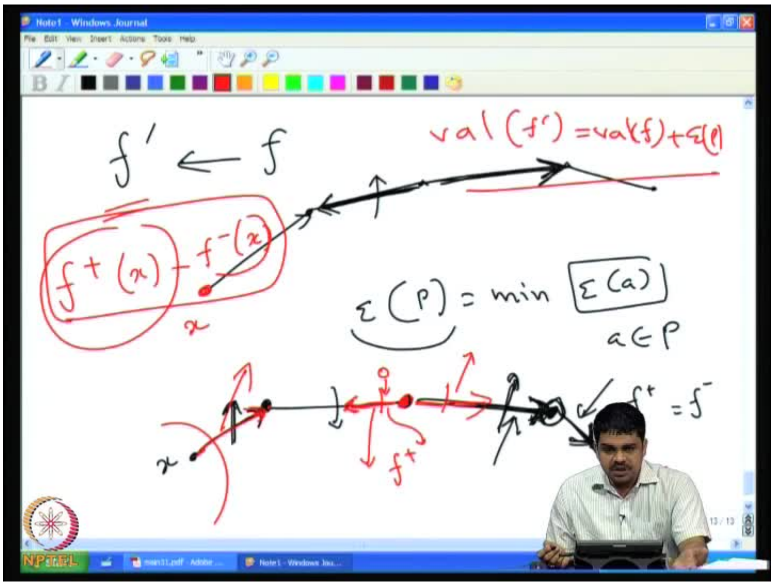

So, it should be, it should not a 0, now we will generally defined x path starting from the source x, so x path is a unsaturated path, if it is composed of edges, if it is consists of edges which is not yet saturated saturated; in the sincere, if it is forward edges, it is it does not achieved capacity fully; and if it is a backward edge the 0, the value of the flow is not 0, it is little more than 0 with respect to the path if it is a backward edge; and then the flow value is to be little more than 0, it should not be, it will not goes the capacity, but it will be more than 0.

So that we can cancel the flow and go forward, and if it is a forward edge, it should be less than capacity, so that we can push further and the epsilon of that path; this particular path will be defined as the minimum of the epsilon of the edges over this path; all for all the edges of the path, we will take the value of that, if it is a forward edge, this value of the epsilon is what is left, that means, capacity of a minus the flow values of a the flow value of a flow value on the a and if it is backward edges, it is just a flow value of a, because it is what we are thinking it to reduce flow there right, so that should be known 0 right.

So, these if epsilon is known 0, then we will say the unsaturated path; and if you have the unsaturated path starting from x and reaching all the way to y, then that is an incrementing path, because we can push some flow through that and at to all the way to y right; so, why do you think take, we can push a flow, what does it mean that, suppose this epsilon quantity we increase so on any forward edges on this path, if increase a flow by these epsilon quantity, the capacity constrain still satisfied, because we can indeed a increment, because that epsilon was the minimum our what was loft the different form the c minus f a for the all arc's, which are in the forward direction path.

So, we can always increase by that quantity in this path and also whenever we had a reverse arc, we rather decrease the value, decrease that flow value, the claim is that flow condition is not so after doing this think whatever new flow we get f dais f dais is obtain from f by doing this operation on the set of arc on the other arc whatever flow values there we will keep it unchanged; then you can see that, for instance, what change have done here in this kind of in say for instance in this kind of a situation, where there is forward arc coming in forward outgoing in; so, when you see f, f minus is increases a little bit, but f plus is also increased a little bit, some incoming edge what an increment, so there an outgoing also edge also got the same increment; therefore, f plus will remain equal to f minus for that particular vertex right; therefore, the conversation constrain will not loosed and also capacity constrain we were careful now too valid.

Similarly, in a place like this, for instance, if you look at vertex like this, this an outgoing edge, and this is an incoming edge; here we are incrementing the value on this, but decreasing the value on this; both are outgoing edges here, so f plus got on increase increment due to this, but then decrement it was decreased due to this edge; so, f plus is unchanged because it the f minus there is no change at all, so is see both are outgoing edges, so there is the we increase, this we decrease because here this a reverse direction; therefore, our flow value was reduce by epsilon right here.

So, therefore, this here also conservation is not lost, because earlier it was it was f satisfies the condition; so, if you constructed a flow f dash like this it will still again be a flow and not only that it has increased value, why is it so? Because if you look at x, if you look at the x, these edges as well as the these edges concern, it as got an increment for increment, if it is outgoing edges, if it is on incoming edge then what will happen? So, that it is reduce that, there the overall f plus of x minus f minus of x will increase this flow value this right; because if it is in the opposite direction, these quantity got a reduce where reduction, and therefore, the overall value in increase forward direction, these quantity will increase; therefore, f plus of x minus f minus of x anyway will increase the flow value the overall flow value I will more.

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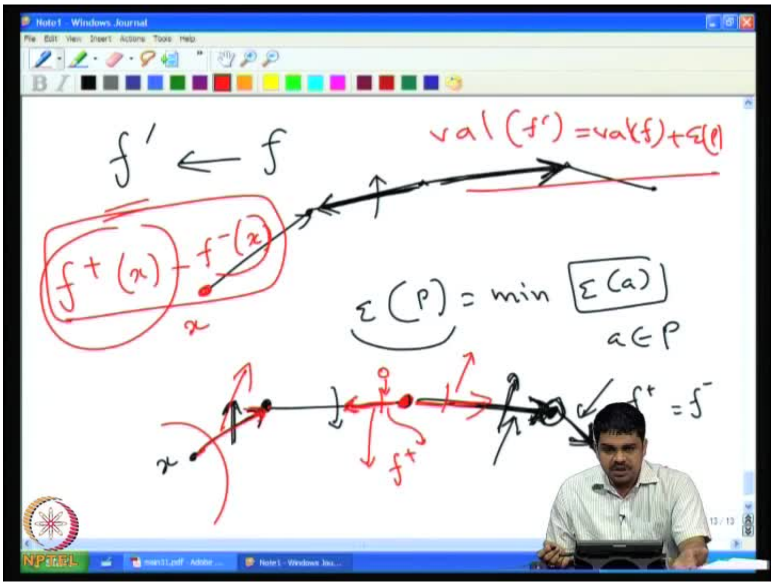

So, value of f dash will be equal to value of f plus the epsilon of p, that we are the, p is the incrementing path that we have selected; so, this is this is how we can increase the flow value; so, we can keep on doing as long as do that, so then can we do these thing if we find can be incrementing path we can do that; or in other words, starting from x we can, if we can search through this saturated path unsaturated path and reach y and then if even if one path is obtain like that, an unsaturated path is an obtain using sometimes using forward and backward edge; sometimes, it does not matter our definition included both kind of because if it is backward edge; we are planning to reduce the flow there, if it is forwarded edge, we are planning to increasing the flow there; so, if it is backward edge then it should be a non-zero value there currently; if forwarded edge, it should be a value which is strictly less than the capacity of that edge; therefore, as long as we can find one incrementing path starting from x and reaching y, then we can increasing the flow value, so we reach we keep on doing this, we reach is situation where we cannot do it further.

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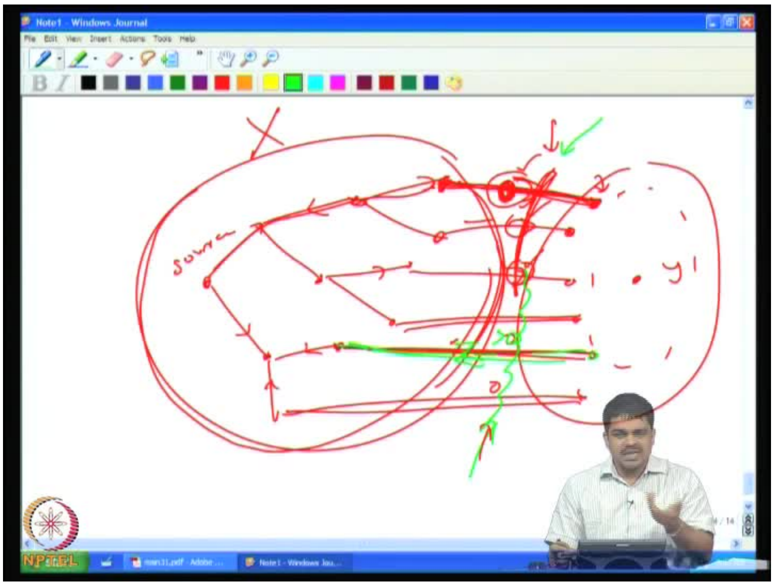

So, suppose, that is, it is stops; so, when does is stop? We call were unable to find any incrementing path starting from x; so what we can do, we can start from x and collect all the vertices which can be a reach by travelling via some paths, which are unsaturated; so, less we collect, all the vertices like that starting from x right of cases, it is very obvious that, we cannot reach y is outside y and collection of vertices may be are outside.

So, we naturally see that, there is there is a cut here, in the cut means, this x means vertices which are reachable from the source via unsaturated parts right up to here, I can reaches due to some reason, I cannot go further in that is why we cannot reached y there; we definitely define a cut, this cut will have following property, so we claiming that, this cut you also already minimum cut or in fact the current flow is actually with respect to flow only; we are doing this thing actually, if there is no incrementing path from x to y, then it is already a maximum flow cut; we have got is actually minimum cut, these will these will give as evidence that, it is a maximum flow why is it so? When you when you inspect the edges - forward edges; why cannot I go forward from somewhere here because this is forward edge and I can reach up to here by an unsaturated part; why cannot you go here capture this, here also that is because the capacity of these edge is equal to the flow value on these edge, thus why we cannot push further flow on that; therefore, this is a saturated edge each of these forward edges has to be saturated edges; otherwise I could have captured one more vertices, that we could that we have captured all possible vertices, we can be which can be reach x via is saturated path is not unsaturated.

So, for all the forward edges are saturated here and also all the reverse edges have 0 flow, because if it is even one was non-zero, then we could capture if the say a flow edge like this with non-zero greater than 0 value on this, then we could of capture on these vertices also as a vertex can be reach from x via an unsaturated path; so, we can we know that, all these backward reverse edges incoming edges of x this set is a non-zero and all the edges which are outgoing from x are saturated; therefore, as we have seen these flow has kind of saturated this cut.

So, the value of the flow equal to the value of the cut; here c plus is equal to that plus of x minus minus of f as we have seen; and therefore, it has to be minimum of cut in the flow maximum flow, so we indeed have reached a maximum flow by these things; so, remember, what we have done is, to start with say x 0 flow something start pushing flow through an implementing path increasing it; so f is made f dash f dash is made as double dash like that we keep on implementing the value of flow until we cannot do it further; why cannot we do it further, because we are unable to find anymore incrementing paths in the networks.

So, in these case, we have shown that; so, if you collect the set of vertices which are reachable from the source via unsaturated path, it will defined a cut, in case y on the other side, otherwise we could have got an incrementing path already; so, these cut will be saturated by the flow in the sense that, all outgoing edges are saturated, and also all incoming edges are of 0 value; therefore, this is the value of the flow is equal to capacity of the cut, and that means, it is the minimum cut and the flow is a maximum flow; and then we have indeed achieved and max flow by that, so this algorithm of pushing through incrementing parts until we cannot do any further will indeed achieve for maximum flow; so, we just will discuss how to implement this algorithm, thus called ford Fulkerson an algorithm in the next class.

Thank you.