Graph Theory

Prof. L. Sunil Chandran
Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture No. # 32
More on Network Flows: Circulations

Welcome to the thirty second lecture of graph theory. In the last class, we had discussed the max flow min cut theorem, which stated that in a network, the value of the maximum flow is less than equal to the… Sorry, value of the maximum flow is equal to the capacity of the minimum cut. The value of the flow we defined as if you you can consider any cut and the kind of the flow which a which goes across the cut is the value of the flow, essentially the some other flows on the forward arcs minus the some other flows on the reverse arcs. While it is very clear that for any cut, this will be less than equal to capacity of the cut, namely the some other capacities of the forward arcs of the cut.

Now, the question is only that is it possible that in the maximum flow will achieve the capacity of the minimum capacity cut. So, the max flow min cut theorem says yes, the maximum flow will be always like that, that means it will be equal to the capacity of the minimum cut.

Now the question is to find out the maximum flow. So, we mentioned that the method is to use a increment the the increment so, given a flow how to increase the value of the flow, that was what we discussed in the last class. So the method was to find f incrementing path starting from x to y; x being the source and y being the y being the sync, right.

So the, what was the f incrementing path? If you start, if you consider a path starting from x so and going towards y, so we will say that if the the edges you can consider the edges on this path and if it is a forward edge, with respect to the direct the the traversal the from x to y, if the if it is a forward edge and capacity of the arc is still greater than the value the flow on the the current flow on that arc, then there is something more which can flow on that arc, therefore, it is an unsaturated edge with respect to this flow.

And similarly, suppose it is a reverse arc on the path, then if the flow value is not zero there, it means we can cancel some flow and make a forward moment, right? That is… If there is a non-zero flow on the reverse arc then then that is also that is and f positive arc, there also we can make it. And f incrementing paths will consist of only f positive arcs and if it is a reverse arc, it should be f positive and if it is a forward arc it should be unsaturated arcs.

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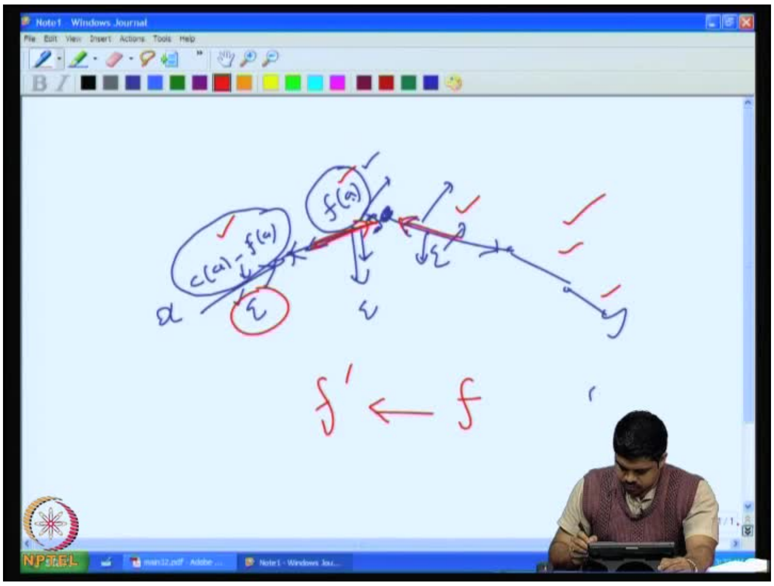

So then, if you can find such a path from f x to y, then you can push a little more flow from x to y, this is the idea. You can increment the flow. What will, what you do is you inspect the edges on the on this x to y path, say x to y path, so it can be something like this, and now you consider each path- if it is a forward arc, then see how much more is there, what is the difference between the capacity of this arc - c of a minus f of a here. And then so if it is on the other way and if it is a reverse arc, then you see how much is the value of the flow here? This is and then the the minimum of these values, right depending on the, if it is a reverse arc, then we we take the value of that edge- the f of a itself, and then if the reverse arc we consider this.

Among all these values you take the minimum. This is the, we can say that this is amount, the extra, which you can push through the path. And then we increment the value of the flow by incrementing by that epsilon on all the forward arcs and decreasing that epsilon on the reverse arcs. And that will be clearly a flow, why because if you consider any any vertex, what if it is two forward arcs with respect to the path, then what does it mean? There is an increment here; there is an increment here also. If it is two reverse arcs with respect to that, then there is a decrement here there is a decrement here also.

On the other hand, if it is one forward arc and one reverse arc, then we see that the flow will, because these are in opposite directions: one is in, both are - sorry, both are incoming arcs, right? So then, therefore, one is an increment, one is a decrement, right? The same situation, when both are outgoing arcs, right, with respect to the path.

So therefore, the and then the other other values are kept as such, so we see that the after this increment is made it is still a flow, right? Therefore, we get a new flow f dash from f by doing this incrementing and the new flow will have you value, have value a little extra, extra by this much, right, this being the minimum of all these these values that we calculated, this from the reverse based on whether it is reverse or forward arc, right.

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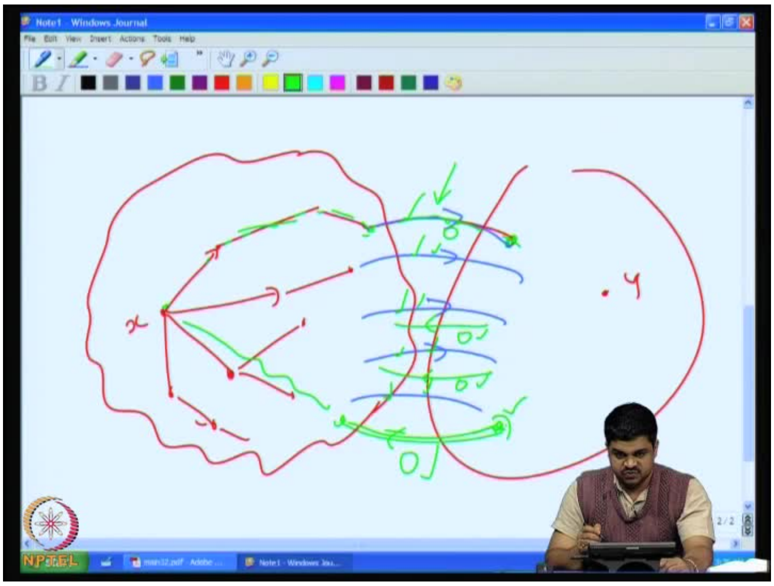

So now, the next point is to see right you keep doing this. At some point, if you can find out one such forward one such f incrementing path starting from x and reaching y, we can increase the value of the flow this way and keep doing this thing and until we cannot find any more f incrementing path. The our claim is that at this stage, our claim is that at this stage, we have indeed got the maximum flow. How do we prove that? It was by so suppose you search for in f incrementing path from x to y, so then you find out all the vertices which can be reached from x via some unsaturated paths, that means paths composed of forward and reverse arcs, where forward arcs have some are not saturated and reverse are not f or f positive, and then, so collect all those vertices from which from x those vertices which can be reach from x.

Say let say these are vertices set of vertices, right? Then now y will be somewhere here, outside. Now, we consider all the arcs which are going from this side to this side, right? So can some of the arcs will be forward arcs, some of the arcs will be reverse arcs. Some can be like this, reverse arcs, is incoming.

So now, the claim is that, if this arc was not if this arc was unsaturated, then even this vertex could have been reached from x, because this was a unsaturated path starting from x and reaching here, and if this an unsaturated edge you can reach here also from x. So therefore, this has to be a saturated edge, that means, the capacity the flow value here has to be equal to the capacity. Same for this all these blue arcs- the forward arcs. As far as the reverse arcs are concerned, they should have values zero on them. Why? Because if they have non zero values, they are f positive arcs, we could have reached up to here from x somehow and then use this arc- this f positive arc to reach here also.

So therefore, this will be an unsaturated path from x to here. So we will we can reach from x to this vertex also, via an unsaturated path. So that will be contradiction. Therefore, therefore all the reverse arcs will have zero in this case, all the forward arcs will have will be saturated, that is equal to the capacity- the flow value will be equal to the capacity. Now, the total here will be equal to the capacity on these edges. Right, equal to the capacity on these edges.

So now, which means that the flow value has equal the capacity of the cut; we know that for any cut, the flow value has to be less than equal to the capacity of the cut. Here, here is a flow which has equaled the capacity of the cut, so that should be a maximum flow. That is the argument. So f of what we get- the flow that we get at that stage- when we cannot find any more f incrementing path from x to y, should be there maximum flow. In this class our intension is to efficiently implement this algorithm, how we are going to search for a f incrementing path, right? So, the suppose our algorithm takes the network as an input and the a feasible flow also has an input, that is all right. So the the feasible flow can be zero, for say zero flow for instance. All the arcs may have zero initially.

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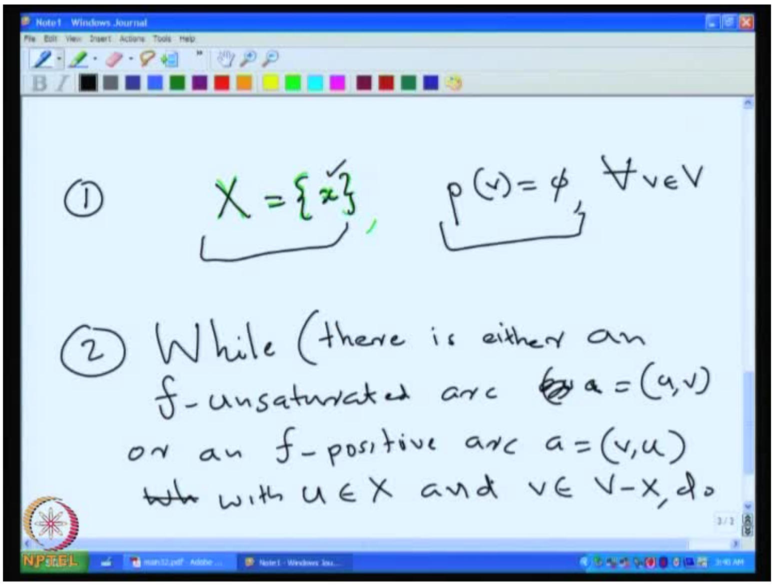

So, first we will define a set x in which the initially only the source will be there in that, right? And of case so this x equal to, then we will define a function p- the predecessor function, for each vertex we will just put it as five in the beginning. v element of v - this is the first step. In the second step our intension is to grow x. what is x? x will be a collection of when a collection of vertices which can be reached from the source x via an unsaturated path.

So, as of now, all the vertices which are there in x can be reached from the source via an unsaturated path. And what is the p of v? This will represent a tree structure on x, in fact, so, a parent function in fact, predecessor function. It is for a given vertex, it will indicate the parent. As of now, the tree is not from that, x contains only one vertex, it is only the root does not have a parent so you can put an empty set for everything. For each stage when we capture one more vertex, we will associate its parent to it, right.

Now, the the key thing is to have to do this thing, while there is either an f unsaturated arc f unsaturated arc a equal to u v or an f positive arc a equal to v u, with u element of x an and v element of v minus x, do. So what we doing is, we will we will do this check. so given the current with current x, we will see whether we can find some forward arc from x. x the with that tail in x and the head in v minus x - outside x, such that this arc is f unsaturated, that means the value of the flow on this arc is strictly less than the capacity of the arc. If such an arc is there then we will continue. We will we will we we are going to capture the head of the arc also; we will capture v also.

Another situation where we can capture v is when there is an arc v u, the from the reverse arc from v minus x to u x there is an arc, coming into the set x, there is an arc v u, which is f positive, which is not f unsaturated: we do not want zero value on that; if it has a non-zero value of flow on it, positive value of flow on it, and then, we will we can we can capture that v also. That means it is, we can still go use that arc to go forward and capture v, right? This is what will do.

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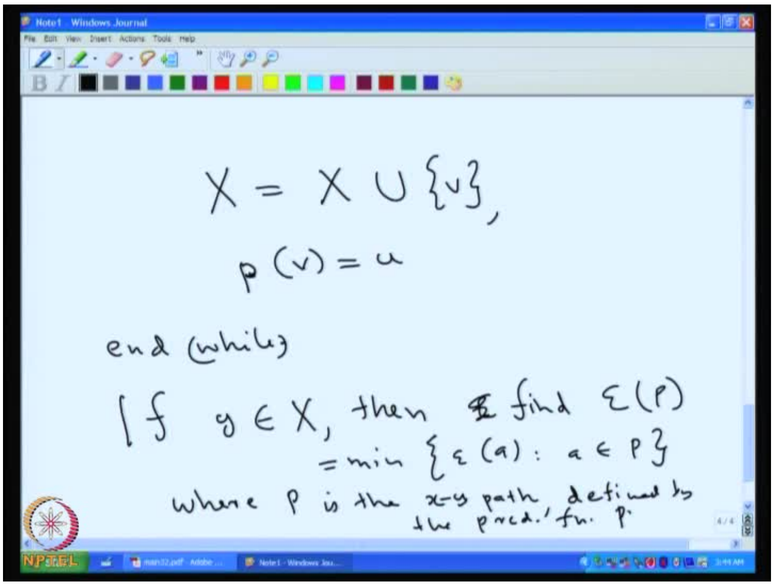

What will we do, we are going to capture v. So, we will increase first, we will add x equal to x union v; v will be added to this, right. And of case, we will we have to define the parent of v as u, the predecessor of u is as v, because we should be able to trace a path from v to x. So currently, we can reach v from x via u, from u of from how to reach x from x to u will be can be found out by following the predecessors function of u. So, for instance, u who will be use parent and who will be use parent's parent like that we will keep tracking, then you will finally reach back to x.

So therefore, it is enough to store v's parent, should be use from where we reached captured v, right. This is what we are going to do; so until until this is end while, right. So as long as we can so we increase the x now, now with this new x, we will again check whether there is a forward arc which is unsaturated from new x to outside or whether there is a f positive arc, which is incoming to the, which is coming into the new x from outside, right? So then we will capture the, in the if it is a forward arc, we will capture the head of the forward arc; if it is a reverse arc is coming into the x and then we will captured the tail of that, which is outside x, right? And then and we add it to the x and make it make a new x and then we do the feature until that.

At some point two things can happen - one is we may capture all of them or we may capture are min at least, we may capture y, right. So if or we may we may not be able to do this thing, right. Now that so we check whether y is captured or not; if y is element of x then, so we will compute epsilon p. Find e of p equal to minimum of epsilon of a, we have defined all these things, epsilon of a equal -a element of p, right, where p is the x y path defined by the predecessors function, right, defined by. So what we are doing now is we will just check whether our sink y - the destination y is already element of x. In that case what we do is, we find out first we have to find out a path p, from y to x. How do you do? y who is y's parent and then who is that y's parents parent, like that we follow that predecessors function p and then we will definitely reach x because we started with just one node on x, so we will have to reach x. And then this path will be used, right?

Now we can find out, which is the smallest epsilon of a for a element of p, for this path. So this we have already seen, what we do is we just look at the forward arcs. For forward arcs the epsilon value is the the difference between its capacity and current flow value; for the back reverse arc it is the value of flow itself and among them you make the take the minimum and this minimum is epsilon of p, right. And then using this epsilon of p what we do is for each of, in this path will make this change. For a forward arc we will change the flow to be, see we will we can change, f of a equal to f of a plus epsilon of p will increase the flow there.

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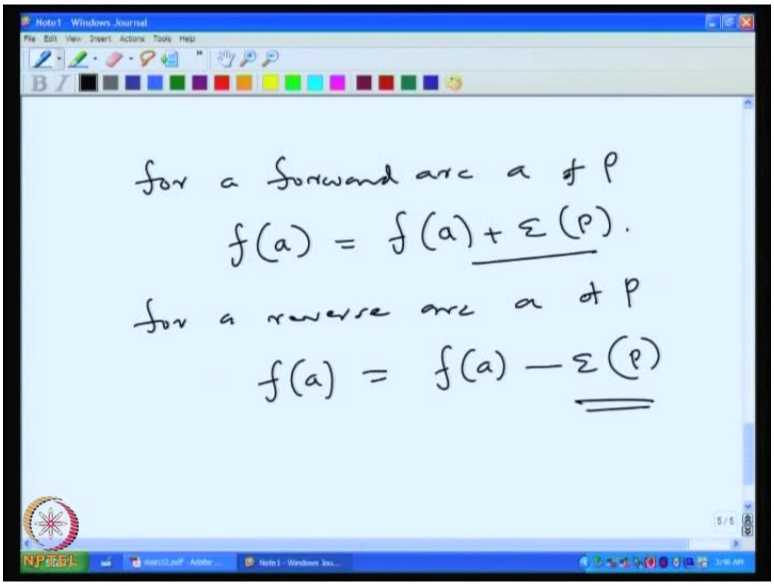

If for a reversed arc reverse arc, p, it will make it f of a equal to f of a minus epsilon of p, right? still be, because see we have taken the minimum here over all the arcs, even if you will not go below zero, even if you add this thing this will not go above the capacity of the particular arc, right. So it will still be a feasible flow. And as we have already seen, if you keep if you do this thing this is the resulting values will still define a flow, because if you look at one vertex, it is, there are several cases, right, both the, with respect to the path, both maybe incoming arcs; the two arcs are so the path which are touching that vertex maybe both incoming, in which case we are, in both case, sorry, in one case it is a increased and in one case it is decreased; it can be just entering and going out, then it is in both case it is in increased, right.

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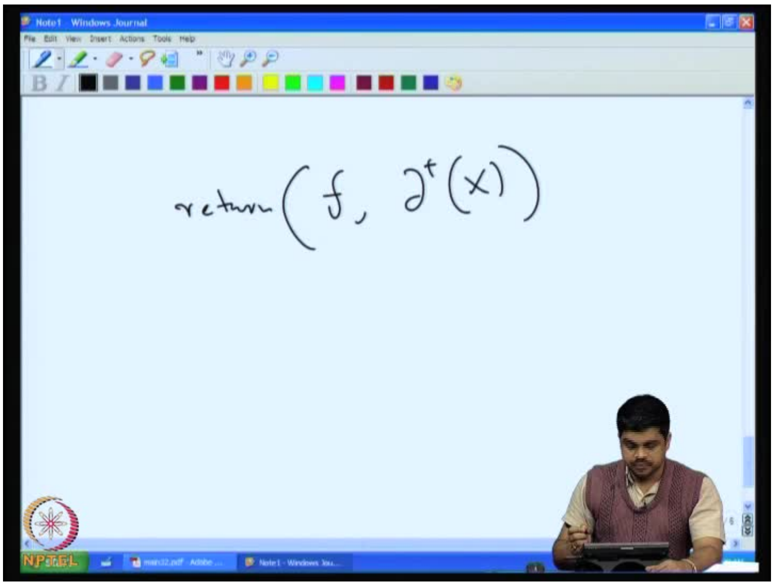

So all those conditions we have considered before, therefore it will remain a valid flow. And then, here we can say that so we got a new flow, right? But on the other hand if y is not element of x, then what will you do? Then we we will collect x, right? We we return that flow, that is the maximum flow, and the cut dou of, dou plus of x, sorry, right, because this is the the, as we have explained, this the maximum flow.

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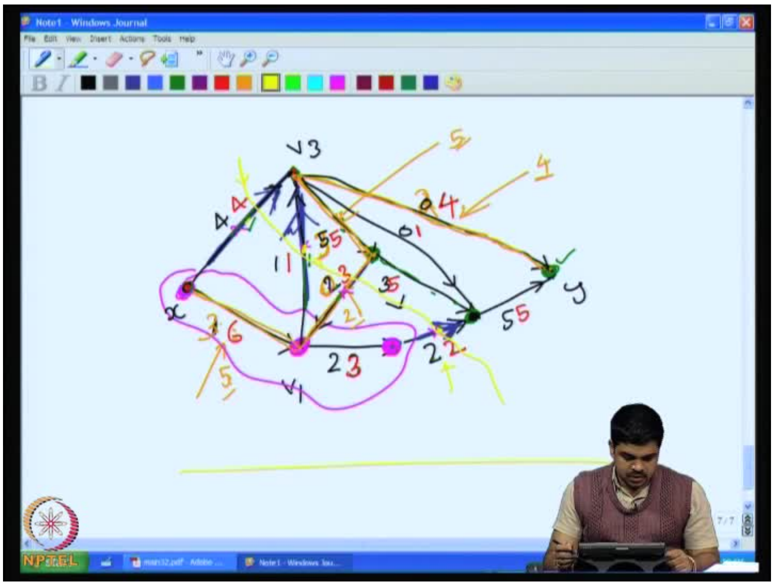

So, for instance, we can we can illustrate this algorithm by considering one small example. Suppose this is x and this is y, and then here, let say this is v 3 this is v 1. (No audio 23:18 to 23:57) Now, it is, look at this network and and will will mark a flow here-so this is four and this is the capacity is red - four and this is one and this is one. And here this will be one and six right. Here, we will take it as two and three and here will take it as two and three, so, two and three. And here, we will take it has two and two, and this be five and five. And here, this be zero and one and here this be three and five. And this is five and five. And this is zero and four.

So we can check the flow conditions here, for instance, this vertex, four plus one is five, this outgoing is five plus zero plus zero, so this is correct. Here, this is, five is the incoming and then two plus three is the outgoing. Here it is one plus two is the incoming and one plus two is the outgoing. And here is two and two and here, it is, five is the outgoing the incoming is three plus two, five.

So this is the, yeah this is let let this flow. Now, for instance, we can we can, with respect to this flow, we can find out enough unsaturated tree. So what we are doing is, we start with an x - I will use the green color to mark it - we start, initially this is x is only this one. And then we will we can capture, because looking at this thing you see that this vertex is unsaturated. Now we can capture this also. Now, this becomes the x. Now, with respect to this x, if you look, this is a saturated this edge this is a saturated edge, this edge is also saturated edge, but this edge is not saturated because the reverse arc but is f positive, see - here this 2 is greater than 0 therefore, we can capture this vertex, right?

So now this new things, this is saturated, this is saturated, what about this one? This is also saturated, now this is a not saturated edge because this is in the reverse direction 5 5. It is capacity is equal to flow but the direction is reverse. So we are only looking for f positive arc, so this this edge is got you you can be used therefore, this vertex can be captured. Now, I have captured this, this and this, right. Now, here look this can be captured, because this is an unsaturated edge - two is the flow value three is the capacity, so this can be captured.

Now, let us look at this vertex. Can it be captured? So, now this this is a saturated edge because two is the flow value, two is the capacity. This one, this is one is three this can be captured, because 3 is unsaturated - flow value is three, five is the capacity, so this can be captured. This and this edge okay, so this can be captured.

So now see, you should understand every time you are capturing, we are remembering the path. So, for instance, here, this was capture due to this, this was captured due to this, so p of this will be this, p of this will be this, and so on, p of this will be this and so on. Now, can this be captured? This is a saturated edge, this can be can cannot be used but then this can be used, right? Because this is ah 0 4, this can be used, this edge can be used to capture.

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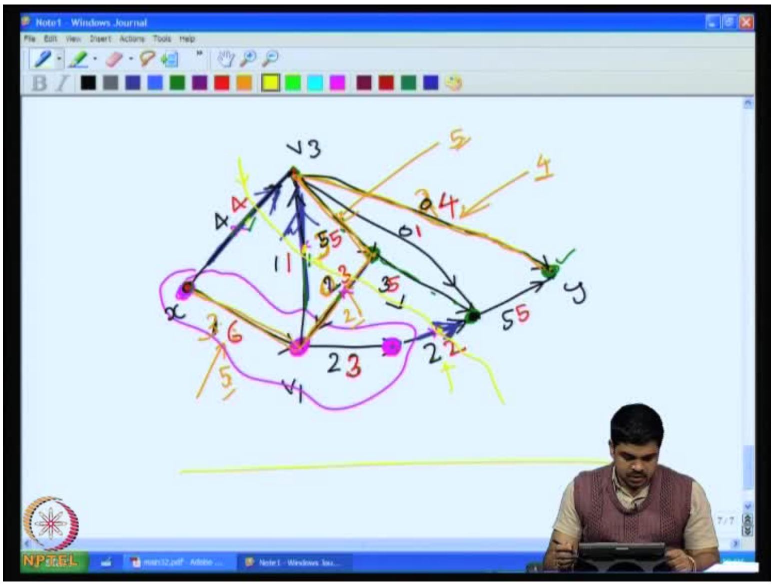

Now, you look to, now y is already there in x, right? y has become green, so that x we are increasing, improve, adding more and more vertices to the set x. Now, this y has gone into the set x. Now we have to track the path by which we were reached here. See y proceeded predecessors, y was reach from here. And now this was reach from here, you if you remember, you captured this from this. And then this was reached from here and then this was reached from this. This is a path that we are considering from y to x or x to y, right? This is the path, right, that, if you remember the predecessors, and this is the path will come.

And now we can calculate, what is the value of the epsilon? Here it is five, because the gap between the capacity and flow value is five here. Here, it is reverse arc right therefore, the flow value is epsilon, so two is the value here. Here, it is five itself, because a reverse arc so, the it is a reverse arc we have to consider the flow value itself, not the difference between capacity values. So, the five is the value here. And here it is a forward arc, zero and four- the difference is four right. So, here its forward arc, that is why we took six minus one, five. So among five, among these numbers, right, one various arcs five, two, five, this is five and four, the smallest two; therefore, we can push a flow of two through this thing. So, that means, we will change one to three and this will become two plus. So here when you, in the reverse arc, when you are pushing the flow in this, this will become zero, that we are decreasing. Here it will decrease to three, right, because again through the reverse arc when you are pushing the flow it will become 3. And then when you are going through this thing, it will become two, instead of two zero it will become two here, right, it will become two here.

Now, if so we have increased now, we can with respect to the new flow this will be a flow as we have already explained, because we have made the changes only on along the paths and the changes are such that the flow the conservation condition is still valid for one valid on each edge. Now, if you just consider it, we can we can, with respect to this flow, we can again calculate the the set of vertices which can be reach from here. Now if you want to do that, so let us say we will use a color, this color. So, start with x, right. This can be captured again because is an unsaturated edge, right. But this edge is not useful; this edge is not useful. Now, from this thing can you capture this? No, this cannot be used, so can I can only use, this edge is not useful, this edge is not useful. This edge is not useful because the reverse arc, the capacity the flow value zero then we cannot use it. So, this is a forward arc, we can use this so this can be captured. This, now from this thing only this is the possible edge but this is also not useful because is now these are the only set which are reachable from x.

Now you can see that y can y is not there in the, now we cannot make progress. At this, we do not have any f incrementing we cannot get any more vertices to the set x, so we stop here but y is not there in it. So we see these edges, see these edges this edge, this edge, this edge, this edge, these are the cut edges, but again this is the reverse edge. Actually the forward edges are the cut are only this, this and this, right. So this creates the, these three create the cut. So this, this and this - these three edges creates the, the capacity of the cut is initially four plus one plus two, so that is total six plus two, eight - well on the reverse only only reverse edge its flow value zero; the flow is also equal to the capacity of the class eight, right. So this is a maximum flow.

So that that is, so the flow we have obtained now is a maximum flow and it has this is a minimum cut also. This is the hit is done therefore, the so to demonstrate how the flow algorithm works. So with this thing we have completely described the flow algorithm of a flow algorithm of a Ford and Fulkerson.

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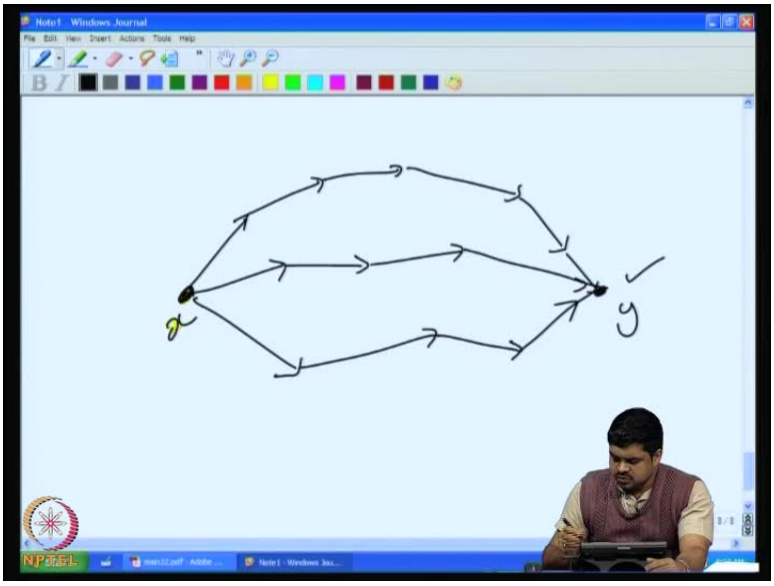

Now the point is to use the flow theory to answer another question. So we have, in when we studied the connectivity, we had consider this question for undirected graph, namely, from x to, suppose you have this undirected graph undirected network and here this is x and here this is x and here this is y and now this and. How many directed paths are there from x to y? How many arc disjointed directed paths are there from x to y? These kind of paths are hold or disjoint paths.

So why are you interested in the maximum number of arc disjoint paths directed arc disjoint paths from x to y? If possible it is a communication to work. And each directed edge we mean that we can communicate in the forward direction from x to y, from sorry, from a vertex u to v if there is a directed edge from u to v, right. So if you want to communicate from x to y they should be a directed path from x to y x to y. Suppose somebody destroys an edge, communication link between two vertices and then naturally we should seek another directed path, right.

So if you want to be show that we are safe, we want lots of directed paths from x to y. That should be arc disjoint because if there is a same arc is present in all the directed paths then the enemy can destroy that particular arc and then we can we can destroy all the directed paths from x to y in which which case, our communication link from x to y will be broken completely: we will not be able to communicate from x to y. Therefore, see from the point of view of the network designer it is important to have several arc disjoint directed paths from x to y in the network.

So discussed in, given a network and two vertices in x and special vertices in y - source and sink - how many arcs disjoint paths are there? What is the maximum number of arc disjoint paths in the network x and y? On the other hand, if you are trying to break the communication between x and y, what we are interested what would be interesting to us? It would be the minimum number of arcs that we have to destroy, such that all the directed paths from x to y are destroyed.

In other words, the we will be interested in the min cut - minimum directed cut, right? So in the directed graph when we say cut, you only mean the edges of the cut in the forward direction, we are not can discard the arcs which are coming backward, right. It is from one side to the other. You only have to worry about from, we are we are want to go from x to y so, one side of the one subset contains x, the remaining subset contains y and we are only interested in the edges going from the x side to the other side, right? The coming back is not very interesting.

So, the as far as the enemy is concerned, he will be interested in figuring out the minimum cut; that means the minimum number of arcs that we can that by removing which we can disconnect x from y. So disconnect x and y means, there will not be any more directed paths from path from x to y.

So these are interesting question like in like the Menger's theorem. Here also, we can say that these two quantities are going to be same; the maximum number of arc disjoint path is going to be equal to the minimum number of arcs that we have to remove, so that we can disconnect x from y the the so that there is no more directed x y path from x y path from the network, right.

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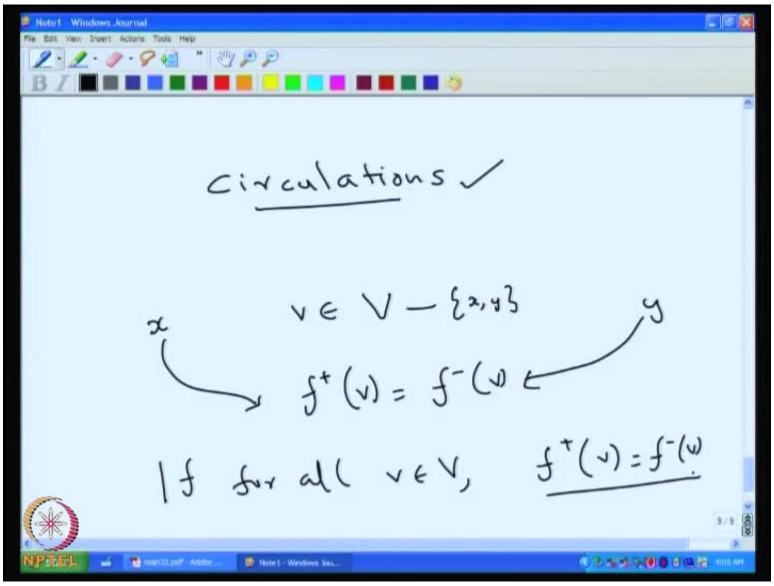

To prove this thing we will use the idea of the flow as flow follows. So, for this purpose we defined a notion called circulations. So, though we are saying for this purpose it is it is much more it has much more wider application than that but in now context we will say that we defining for that purpose. What is this circulation? It is just like a flow; in the case of flow we wanted the conservation condition only on the intermediate nodes. We left out the two vertices, which are which we calls as source and sink. x and y it did not require the conservation condition to work.

For the all vertices, other than x and y, we wanted f plus of v is equal to f minus of v, right? But suppose, we want this conservation condition to be valid for y and x also, then we say that this is a circulation. That is a circulation. That is, if for all v element of v we want f plus of v equal to f minus of v. This is not a flow: this is different because circulation, the all the that conservation condition is uniform.

Now and also we do not have capacities. We are not worried about capacities when you are talking about the circulation. We can always defined capacities and talk about feasible circulations later. But as of now, we are not bothered about capacities. So, we are only requiring that, the conservation condition should be met in every vertex. While in the flows, we wanted it only for all vertices except y and x, here we want it for x and y also.

So such a such a thing is called a circulation. And again we are not bothered about source and sink here. Now the circulation, the point here is that once you talk about circulation, circulations in flows are very much related. Why are they related? So if you have a particular circulation, then what we can do is, suppose you can identifying the in the network there is a circulation - f, then what we can do is we can identify an x and an y and then, suppose there is an edge between them, right, x to y edge is there, right.

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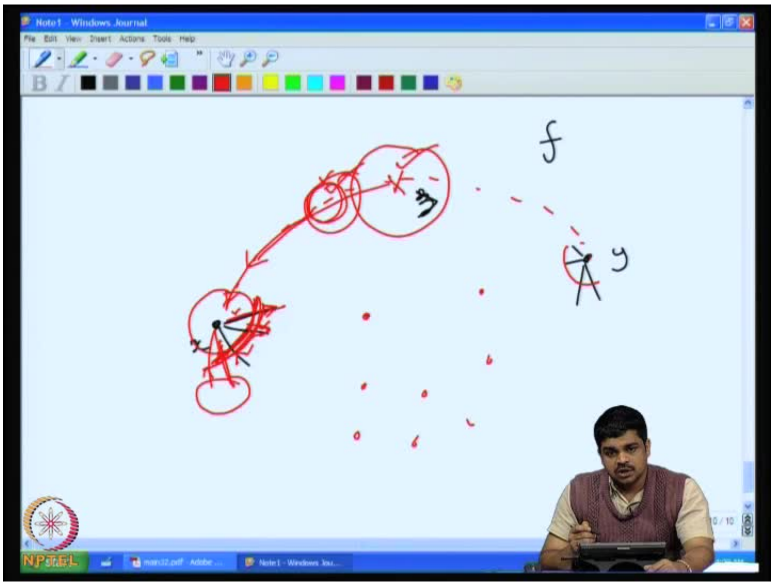

Now, what if, sorry, suppose there is a y to x edge is there, right. Now what if I remove this edge from the network, suppose I remove this remove this edge from the network, right, sorry. So this edge is removed now. Now, you can see that we have destroy the circulation but then this is become a flow. Why? Because except in the x and y are the only two vertices, which affected by this removal of the edge; there the conservation condition will be violated but in all other vertices the conservation condition will still hold, because this edge is not affecting them.

And now, the value of the flow, this will become a flow now, the value of the flow will be equal to the value of the flow on the if you that will be equal to the some of the flows and outgoing edges of x minus the some of the flows on the incoming of x. That will essentially be equal to what was here right, because that is out got balanced. Because you know, this was incoming edge. The total flow, if you take up the sum on the outgoing edges here and minus the flows on the incoming edges, this total will be exactly equal to what is incoming here, so that in the original case when it was a circulation that could add up to zero, right: incoming flow of equal to outgoing flow, that is why the sum of incoming flow is equal to sum of outgoing flow. So if you just remove one edge, then here the sum of incoming outgoing edges minus the sum of remaining incoming edges going to be the flow value on this thing. So, the with respect to the circulation whatever value was there.

So therefore, the value of the flow is going to be the value of the circulation on this removed edge, right. And now so that means, that is how we can convert a circulation to a flow. You can identify two vertices which are adjacent source, especially where if there is a y x directed edge we can removed and immediately it becomes a flow with value equal to the value of circulation of that edge.

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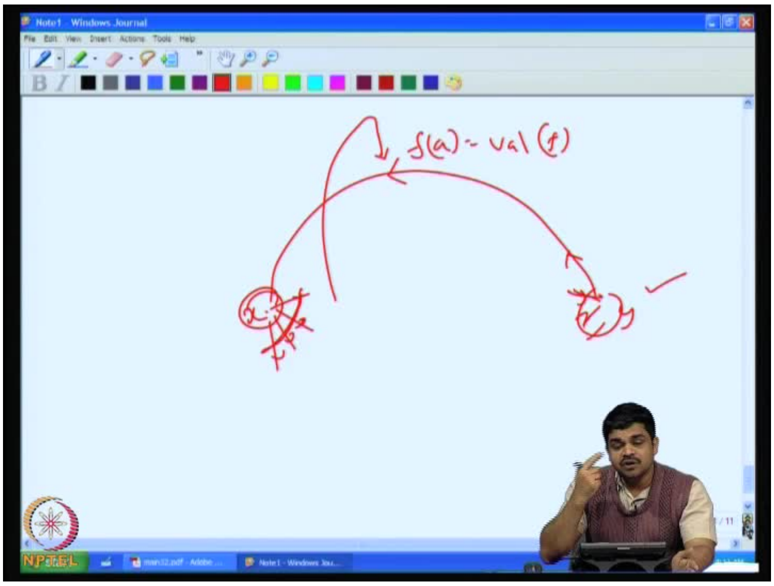

Now, the other way, suppose if you have a flow from x to y with a particular value, how will you convert it to a circulation? So there is an x y flow here, you know the value of flow is this much, right, and we can always put an edge from y to x and make the value f of this arc equal to value of the flow. And then what will happen? So then, for this x also this will incoming flow is equal to outgoing flow. And for y also, because this is incoming flow and this will be outgoing flow. That will be equal to, incoming flow will be equal to the outgoing flow, and therefore, all vertices uniformly will satisfy the condition. And therefore, it will become circulation. This is a way to convert a flow to a circulation.

So therefore, see the point here is that you can rather study circulation, rather than flows. Because if you get some results, some get the result regarding the using the circulation then it would not be very difficult to convert the result to a corresponding result to about the flows, because this is difference is very little. Therefore, but on the other hands studying circulations is much more convenient. Why it is much more convenient? Because of the uniformity for instance, in the flows there are two special nodes which has to be taken care of separately. While in circulation, every vertex satisfies the conservation condition, it makes it little more easier to study.

Therefore we would rather study circulation rather than flow. Finally, if you want to make a statement about the flow we will first make a statement about the circulation and then convert it to the flow using this idea, right. In most cases it will work out. That is why. That is all we wanted to say. And again so, for instance, because the uniformity of this conservation condition, we can formulate the circulation like this. That system of conditions on each vertex can be return using a matrix like this.

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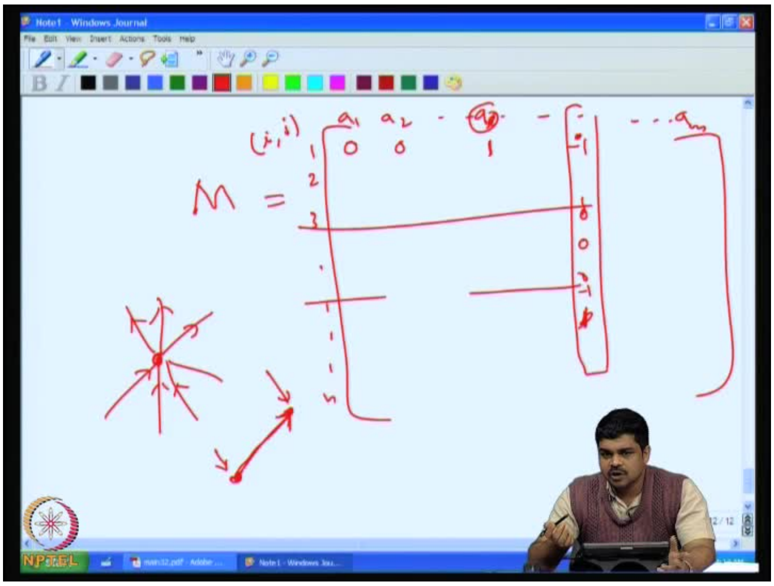

So let us define this matrix m. m is equal to this matrix, so where the matrix these are all arcs. This is called incident matrix of the directed graph network. Here we will put the arc set, arc set meaning the edges -the directed edges, so m. and here the number of vertices one two three. So this is n n by n this thing. For each vertex what will you do? So we will see for this arc, if it is one of the end points of the arc, then only we need the put anything, otherwise it will put zero. If it is not an end point this arc we will put a zero.

So suppose this is i j, suppose it is the tailor arc it put one, right a i it put sorry here the particular arc we will put one if it is, on the other hand, if it is the head of an arc we will put a minus one, right in corresponding to the arc. In other words for every vertex the how many entries will be there, equal to the number of arcs which is incident on it, right. For all the incoming arcs we will corresponding columns, corresponding places will have negative ones; all the outgoing arcs, in the corresponding places will have positive ones.

So, like that we can construct each column of each row of this thing. In a column what will you see, so that is means for a particular arc which all vertices will be affected. Only those vertices will have many of them will be zeros, only two entries will be non-zero: one will be minus one, one will positive. Because if it is an arc one will be saying it has a tail and then one will be saying, this vertex will say that for this arc I am the tail this vertex will say that for this arc I am the head. Therefore, in one row we will we would have entered minus one and the other row we would have entered plus one, all others will be zeros will be zeros. So every column will have one minus one, one plus one all other zeros, right, because it is an arc: it will work as the tail for one vertex it work as a head for another vertex and will not affect any other vertices, that is it.

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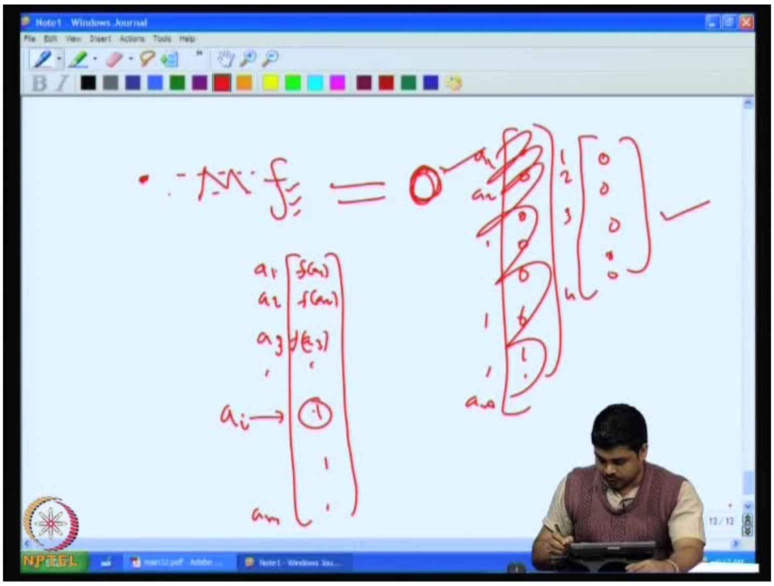

So this is the incidence matrix. And then, this circulation can be expressed as m into, this circulation matrix into, f- f being the flow of flow function. So the what you mean by that? So this f can be seen as a column vector, where this are again a one a two -the arcs- a three, so a m, so there are these are the arcs corresponding to each arc we we have this precision. And then in that whatever is the flow value, this f of a 1 will be written here, f of 2 will be written here, f of 3 will be written here. This is the vector corresponding to this function.

With flow is the function, on a particular value, that means an arc right, if this function is defined on the arc set, if on a particular arc a i it is taking a particular value f of a i, and that f of a i will come in the position corresponding to the arc a i, that is all. So we have just written it as a see. What will this be equal to? This has to be f is a circulation see we are not talking about flows we are talking about a circulations and then m of has to equal to zero, why is it so? Because if you consider any row of this incidence matrix, when it multiplies this thing, you know is a row corresponding vertex here, that vertex has non-zero entries only for the arcs which are incident on it, some other arcs are incoming they have one entry one some other arc arc, sorry, incoming arcs have entry minus one and outgoing arcs have entry plus one.

So, it will sum of the values flow values on the outgoing arcs minus the sum of the flow values on the incoming arcs. So, naturally, because of the conservation condition that will be zero. So every row this will happen, therefore, we will get a zero vector. This is zero vector, this is vector of this matrix zero, zero, zero, zero, zero, zero, zero, zero, zero on any dimension a 1, a 2, a n, right, in corresponding to s.

So, sorry, this not one dimension, this is a n dimension vector, because a number of for each row zero, zero, one, two, three and being the number of vertices, right. Now, so this conservation can condition can be easily expressed using this incidence matrix as m f equal to zero.

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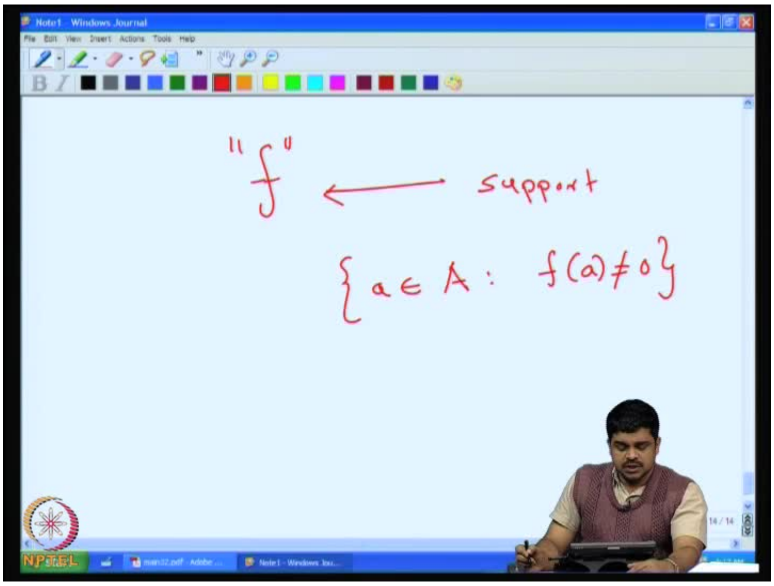

Now, now we will let us look at this vector of f, the vector corresponding to the circulation, right, the again whichever I have discussed. So the the each component position correspond to an arc of the graph and the circulation, the value the circulation at that arc will be written on that particular position, right that is a affect. And then, now what is the support of this thing? The arcs corresponding to which circulation values non-zero is called the support of that, usual word, support.

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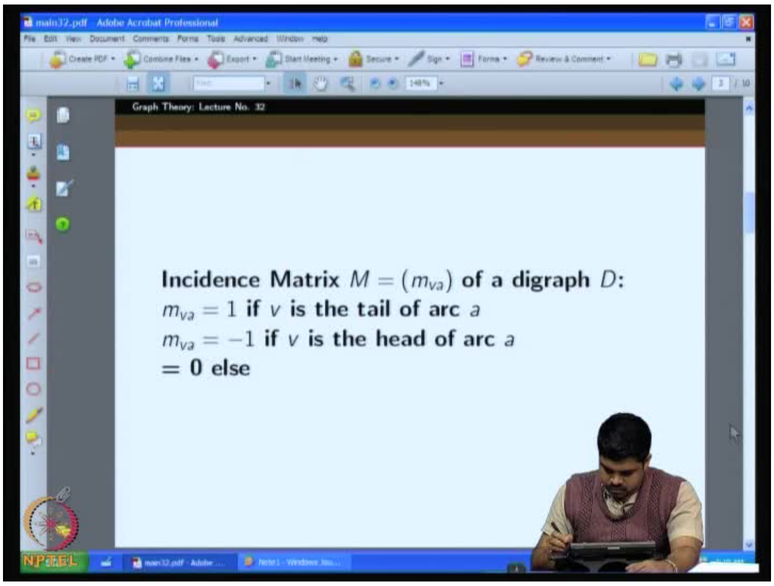

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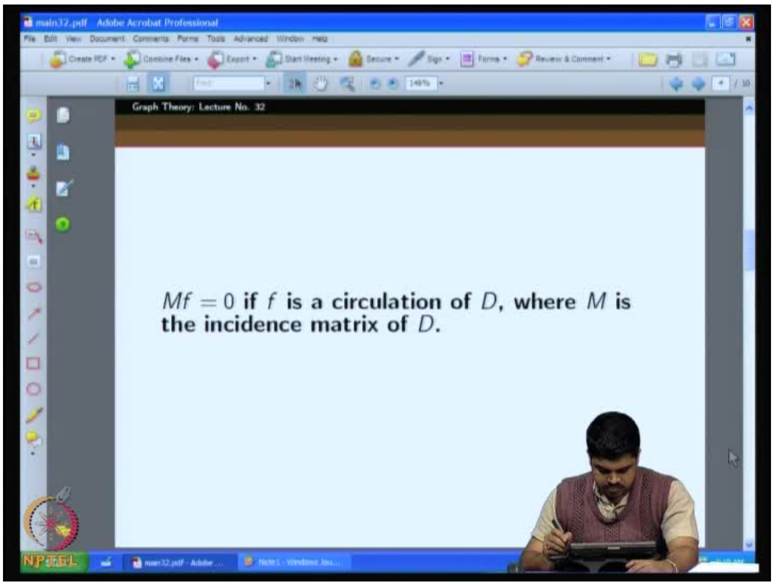

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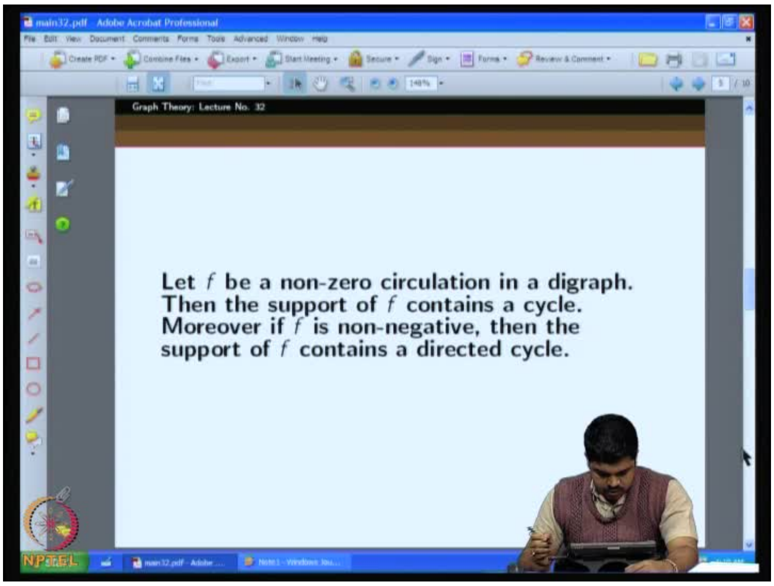

Support means the set of arcs a element of a, such that f of a not equal to zero, right. Now, the next statement we want to make is that, yeah this incidence matrix, is the incidence matrix of D, and so if there be a non-zero circulation in a diagraph, non zero circulation means at least one of the circulation value is non-zero not that all are zeros, then the support of f contains a cycle. Moreover, if f is non-negative then the support of f contains directed cycle.

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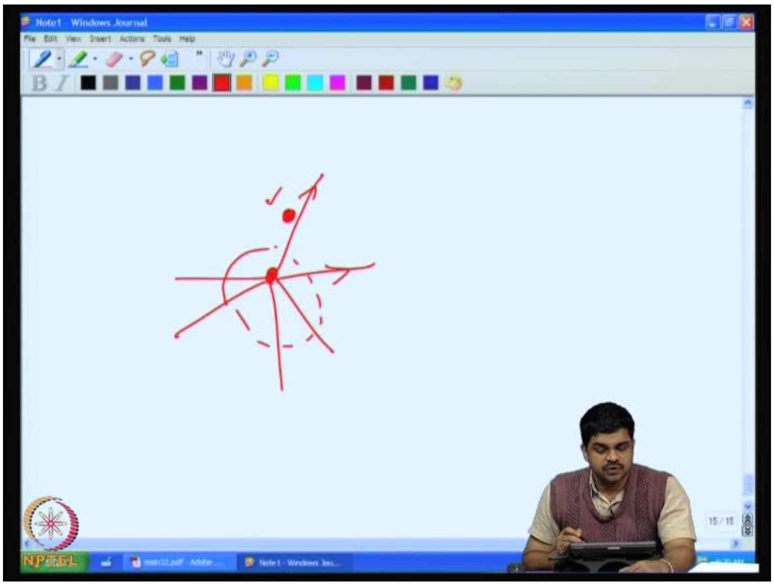

So why is it so? So, if you consider the support of f, and you take any vertex, is it possible that there is only one, with respect to the circulation there is exactly one incident edge of this vertex as non-zero value. That is absolutely not possible. Suppose if there is one non-zero value how can they the incoming arcs equal sum up to the outgoing arcs right. They cannot add up to zero, right. They cannot add up to zero if there is only one, because if you sum up all of them that that value will only come, right. It is a non-zero value, and it is not possible to have just one non-zero edge incident on one vertex. There should be at least one more.

So, if so now if you take the sub graph corresponding to the support of other thing, this means those, if you collect those edges which have non-zero values of this circulation and then we will see that every vertex have degree at least two and naturally therefore, they should be a cycle in the sub graph right, because if you start moving out of a vertex we can always keep moving at least once. And only way to stop is to comeback, right, comeback and form a cycle. So this we have seen several times. Therefore, if every vertex have degree at least two with respect to that sub-graph sub-graph corresponding to the arcs where the flow, the circulation values are non-zero.

Then it is very clear that with respect to this arcs the sub-graphs should contain a cycle, because the degrees of each vertex at least two. We can trace this cycle, and then the first time it comes back and revisits the vertex the circle is formed. It is not possible to keep going without revisiting cycle, because once you enter a vertex we can always go out, because there are two edges incident in it.

So here we did not consider the directions on the edges. We were just talking about the undirected, the underline undirected graph of the support: corresponding to the support we will have a cycle. But we can suppose all values of the circulation were non-negative; so then, we can also see that the support- the arcs corresponding to the support will have a directed cycle. Why is it so? Because for every vertex if there is a, is not possible to have, only all the all the non-zero values the the the on all all the non zero edges incident on it or all in coming or all outgoing is not possible. If there is one outgoing edge which is non-zero, then they should be at least one incoming edge also which is not zero otherwise how can they together sum up to zero, that means, how can they incoming values equal to the outgoing values, because all are non-zero, right, all are non-negative.

If the negative and positive, it could have been possible. Even if all the outgoing edges are zero incoming edges themselves with some negative and some positive they could added to zero. But if all are non-negatives it is not possible, some of the incoming has to be present if there are some out some of the outgoing edges have non-zero values.

So we will get, indeed a directed cycle. We can follow the direction the edges you can enter and go out. Therefore, there will be directed cycle if all the values of the circulations are non-negative. And now, the next claim is that if so before that we have to define some special kind of circulations, circulations. So you can consider particular, so you take any cycle and then you give a particular direction, sense of traversal on that cycle, right then say it can be like this.

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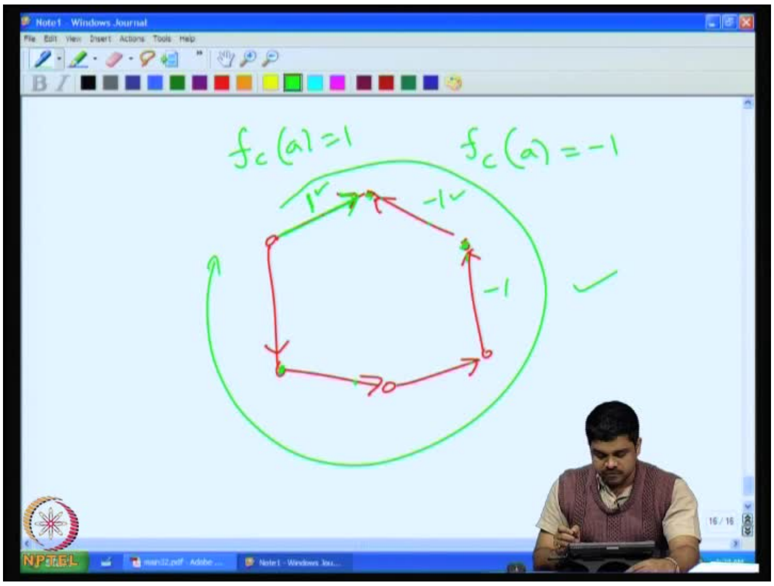

So this is a cycle, so cycle suppose this is a directions. Now we can traverse, we can decide to traverse this cycle like this. Now is because we traversing this way so this is a reverse arc, this is a reverse arc, so this is reverse arc, this is a reverse arc, this is also a reverse arc but this a forward arc, right. So because in the direction of the traversal. Now, we will say that, if it is for the forward arcs will define one, f of f c of this arc is equal to one, and for the reverse arc f c of this arc will be equal to minus one, right.

So now, you can see that this is indeed as circulation, because if both are reverse then this is minus 1, this is minus 1, you can see that they are balancing in each other, so outgoing edges and incoming edge here. Similarly, here it is an incoming edge, it is also an incoming edge, this is 1; this is minus 1. Therefore, they adapt a zero. So therefore, is this indeed a circulation, but easily verified for an each of this thing. This is called a circulation associated with the cycle. We will use the circulations associated with cycle to study circulations in general in the next class.

Thank you.