Graph Theory

Prof. L. Sunil Chandran

Computer Science and Automation

Indian Institute of Science, Bangalore

Lecture No. # 34

More on Circulations and Tensions, Flow Number and Tutte's Flow Conjectures

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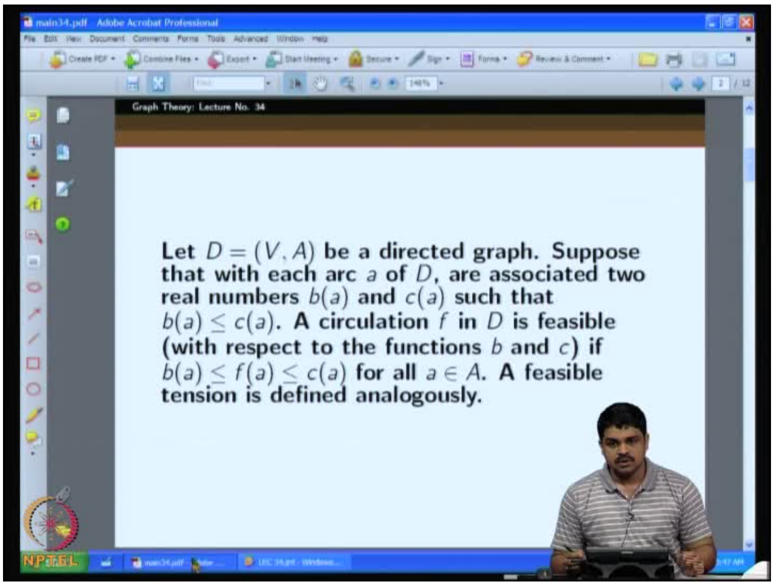

Welcome to the thirty forth lecture of graph theory. Today we will talk about feasible circulations, so the here feasible when I say feasible, we have a directed graph and then with each arc we are associating two real numbers. So, let us a b of a and c of a; b of a is a lower bound, c of a is a upper bound which means that lower bound for the circulation value on that arc, and of case b of a has to be less than equal to c of a.

Now a circulation f in d is called a feasible circulation with respect to these functions f b and d b and c, if b of a is less than equal to f of a is less than equal to c of a for all a element of a. A feasible tension is defined analogously for instance we are given upper bound b, and lower sorry lower bound b, and upper bound c.

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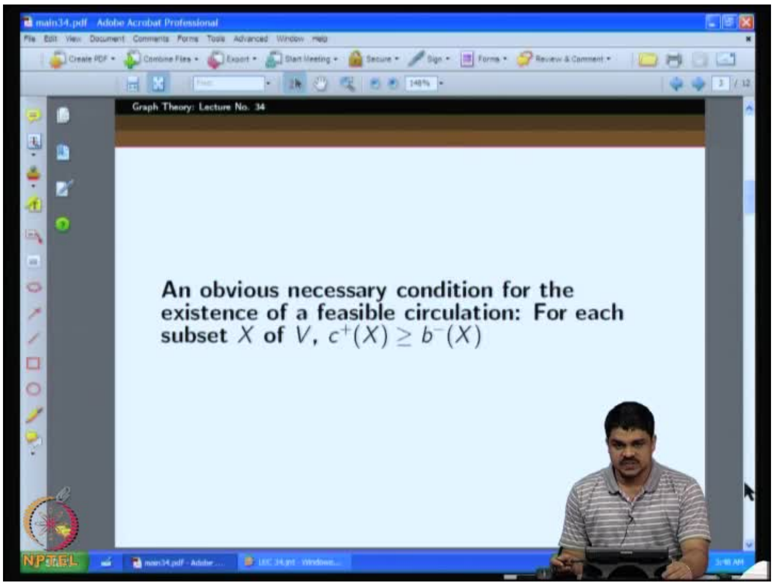

Now, the main topic that we will consider in this class is a necessary, and sufficient condition for the existence of a feasible circulation in a network, given the lower bound function b and the upper bound function c.

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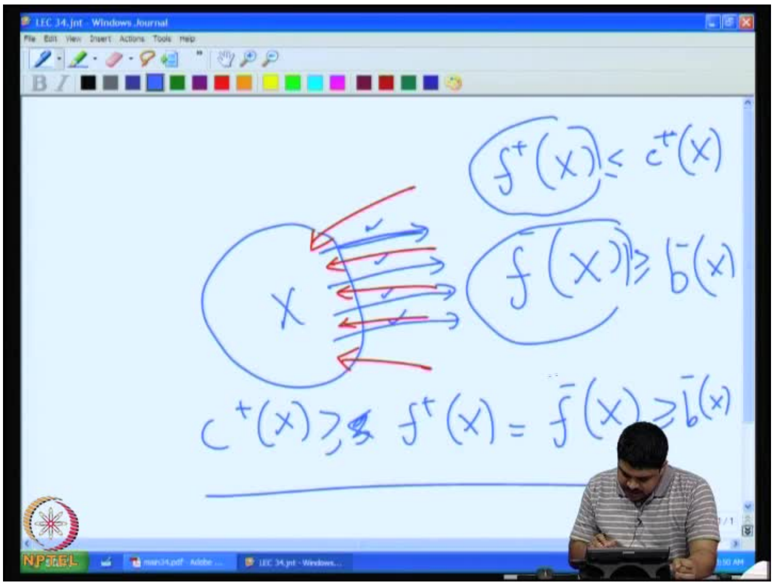

What one can immediately see is this for instance as far as a necessary conditions is like this, for instance if we are if we take consider suppose we have this graph and this is a subset x. Now, we know that the because this is a value of the flow value here on this thing, so what what can I tell about f plus of x; that means, the value of the flows on these arcs the outgoing arcs; that has to be less than equal to definitely c of x this is the c plus of x right.

Now, on the other hand we can say that f minus of x; that means say if you consider the incoming arcs, these are the incoming arcs for the x right. The some other flow values on these incoming arcs has to be greater than or equal to b plus sorry b minus of x, because these are the incoming the whatever the lower bound, it has to be greater than this is an obvious two obvious things, but the interesting thing is that this quantity, and this quantity are equal that means we have f plus x equal to f minus x.

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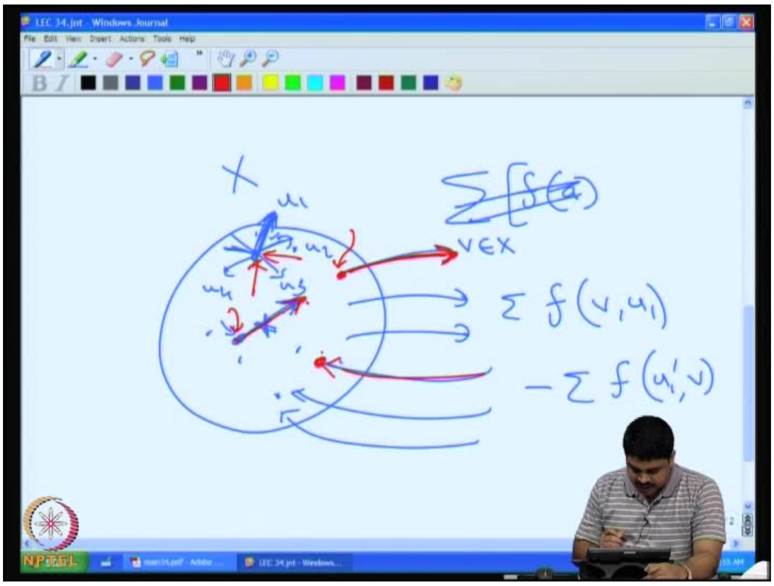

So therefore, we will sorry we will get c plus x greater than or equal to f plus x b minus x why is it, so why this equal that is because if you consider this set x, and then if you consider the vertices on this thing, so these are the vertices. Now, if you sum up the flows on the every arc adjacent on each vertex right, then what we get is right this if we if we take this sum, then what we will get is equal to the flow value on the outgoing arcs minus the flow value on the incoming arcs why is it. So far instance when I some these things, so we have so we can say for for each of these arcs we have the value of this one, and value of this one right.

And this will contribute to for instance, if I take an edge here right, now this will contribute to the outgoing flow on this vertex, and this will contribute to the incoming flow on the on this for this thing.

So finally, when I consider this sum right, for instance sum over all vertices this edge if edge effect will cancel of in other words, so for each v element of x if I sum over f of… So, let us say these are will will do like this, suppose this is u 1, u 2, u 3, u 4. So, if I for a vertex what I am summing up is the flow value on v u 1, all the outgoing arcs minus sum over, and then minus the sum over flow value on say u 1 dash comma v right; these are the contribution due to the incoming arcs.

In other words we know that for because it is a circulation, this sum should be zero for a vertex, so we sum up over a a vertex the all the flow values on the incoming arcs, and minus the flow values on the outgoing arcs this being a being zero. The total has to be zero here, because each each vertex will contribute to contribute as zero, but on the other hand if I take a edge in a such that two vertices, both the vertices both the end points are inside x. Then this edge will contribute to the positive term of this vertex, and negative term of this vertex; therefore, this will this effect of this edge will be canceled in the total sum over all sum.

But on the other hand if I consider this edge this will be in the positive term of this vertex, but because this vertex is not there in x the negative, this is not added in the negative term; therefore, the this will remain; similarly, if I consider this incoming edge this will be in the negative term.

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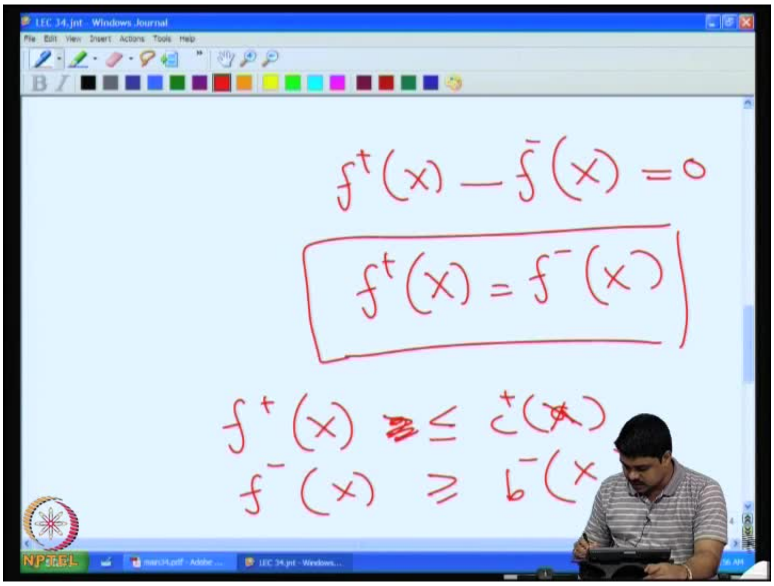

So finally, when I sum up some over all vertices what remain is the f plus of x minus f minus of x this, what will what will remain. But this, but this has to be zero right, because the overall value is zero. So, we get f plus of x is equal to f minus of x, but now we know that we already seen that f plus of x has to be greater than or equal to sorry has to be less than equal to c of a, and f minus of x has to be greater than or equal to sorry c of c plus of x, and this b minus of x right.

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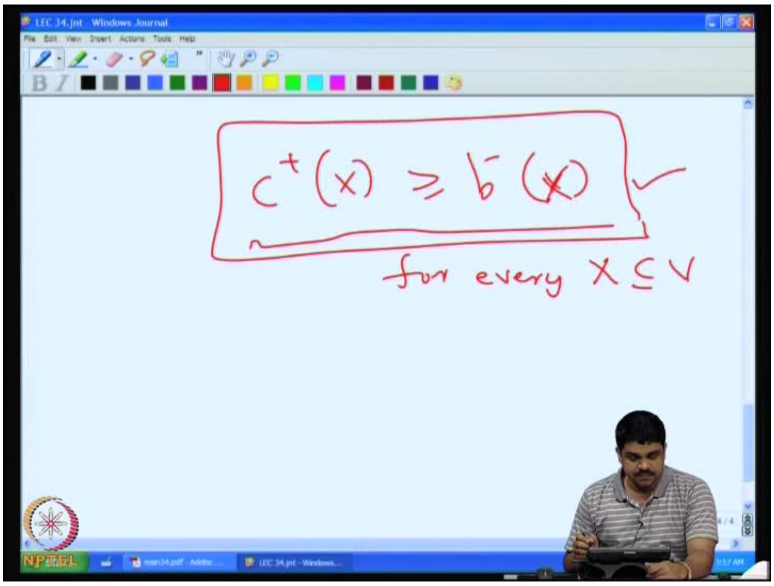

So, what we get as c plus of x has to be greater than or equal to b minus of x, for every x subset of v, this is a necessary condition for a flow, this is a necessary condition for a feasible circulation. Right if if the circulation is feasible, because we know each flow value has to be less than equal to c of a for each edge and great than equal to b of a for each edge.

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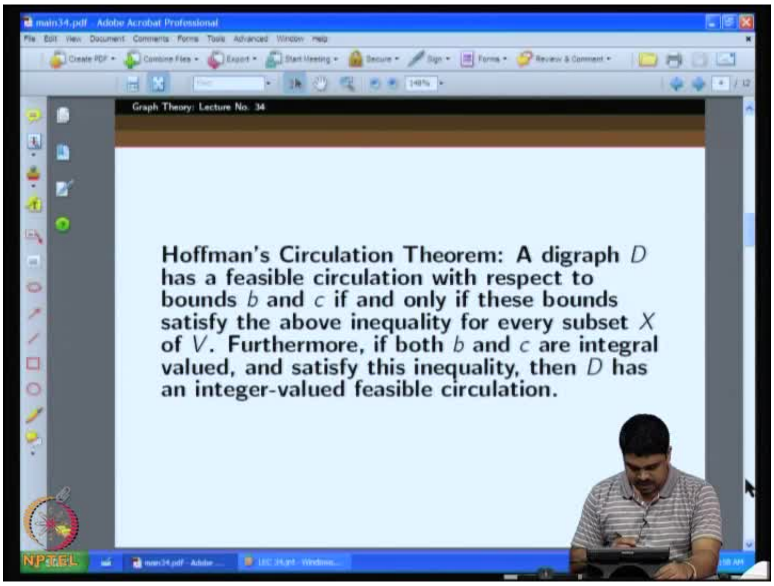

Now, we are going to prove that if for each subset x, this condition is satisfied, then there x is a feasible flow in the directed graph. How do we show this thing? So, this is to show this thing, so we also this is the this theorem due to Hoffman's Hoffman, it is known as Hoffman's circulation theorem, and this is about the necessary and sufficient condition for the existence of a feasible flow in a digraph given lower bound function b, and upper bound function c. It says that e this condition that is for each subset x of v our upper bound c plus x has to be greater than equal to the lower bound b minus of x, this is indeed the condition for the existence of a feasible flow in this graph.

What we have just now seen is that if this condition is satisfied, then there is sorry if this condition has to be if this condition is not satisfied we cannot have a feasible flow, in other words if there is a feasible flow, this condition will be satisfied. On the other hand, we will show that if this condition is satisfied, then we can indeed get a feasible how to get a feasible flow we will show. The other part of this theorem is that if b and c are integral valued, and satisfying this inequality this condition is satisfied, then d has an integer valued feasible circulation not just a circulation it has an integer valued feasible circulation.

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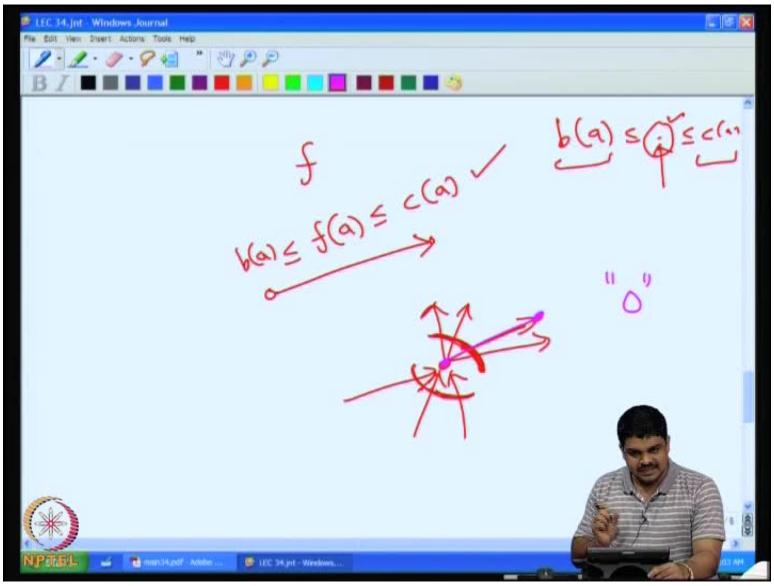

This is what? We are going to do now. The the proof is a done like this, first we consider a function f we consider a function f, and we rather than making it a circulation we would rather make it make it feasible first. In other words for each arc a, we will define the value of f of a to be in between c of a and b of a, this is quite easy because you know by definition b of a less than equal to c of a, there is something here, we can always place get a value in between, and then we place it here.

Suppose if this and this are integers, we can always take an integer value itself which is in between this. For each arc we will select the value of f to be something in between b of a and c of a, now we have made it feasible, but it need not be a circulation; for a circulation what is the condition for each vertex the some of the flow on the incoming arcs has to be equal to the some flow on the outgoing arcs right, this should be equal to this.

Now, what we can see is that each vertex, if you if you each vertex there this incoming outgoing arcs for each vertex right, so the we can sum up the values on this arcs, we can sum up the values on this arcs and this minus this we can consider.

Now, if this is equal to zero that means as far as this vertex is concerned, it this function f behave like a circulation as far as this vertex concerned, but it need not be zero; it can be more than zero, it can be less than zero - it can be positive or negative. If all are zeros for all vertices this value is zero, then we can say that it is indeed a circulation, and then we have go to a feasible circulation, because we are selecting function values to be such that it is in between lower bound, and upper bound for each arc.

Now, we can assume that, so it is not like that there are some vertices with positive excess we can say which is excess right, and some with negative excess right. So, is it possible that all of them only positive excess what whenever there is an excess is it only positive, it cannot be show, because if you sum up over all vertices as we have already seen. Each edge will contribute say for instance this edge if I consider, this will contribute to the positive term of this vertex, and the negative term of this vertex. Therefore, it will cancel of finally, we have to when you sum up over all vertices the in the flow on the outgoing edges minus the flow on the incoming edges, this total sum has to be zero, because every edge contributes - once to a positive term, once to a negatives term; therefore, the cancel off right positive.

So, therefore total has to be zero any way, therefore it is not possible for all for some vertices have only positive excesses will no negative excess, so if there is a vertex with positive excess that may be incoming total some of the incoming the flow on the outgoing edges minus the some of the flow on the incoming edges is positive for some vertex, then they should be some other vertices with this value negative.

Now, whatever is the excess we can consider as a measure of the overall deviation from what we aim, what we aim is every vertex to have zero excess right. That means it is a circulation, so this deviation whether negative or positive, we can consider its positive part that mean absolute value. And we can call it as the and some of this absolute values can be taken as the total excess now.

Now what we do is we are going to demonstrate a method by which we can modify the value of the flow on the arcs of this network or may be on some arcs of this network, in such a way that we will get a new flow, but when we doing this we will make sure that the new flow values. So, the new new value of the functions we cannot still call it as a circulations over flow, so we will say that for the new function value is again feasible; that means this for every arcs it is in between the bounds, that means in between the lower bound and the upper bound while the progress, we will make is that for with respect to this new function the total excess the sum of the absolute values of those excesses on each vertex will bill us will reduce to will come down, every time there is an excess. We will show that we can make a make some progress like this, so until the excess becomes zero; that means we end up with the circulation feasible circulation, this is the aim.

So, we just after show how suppose there is a positive excess total excess, then how we can make a progress such that after changing the function with respect to the new function the total excess is strictly less than the old total excess to do this thing. What we do is? We consider any vertex with say negative excess, say this is a vertex u, and when I consider the some of the flow values the values the function values on the outgoing arcs minus the some of the values function values on the incoming arcs for this vertex it is a negative value let say.

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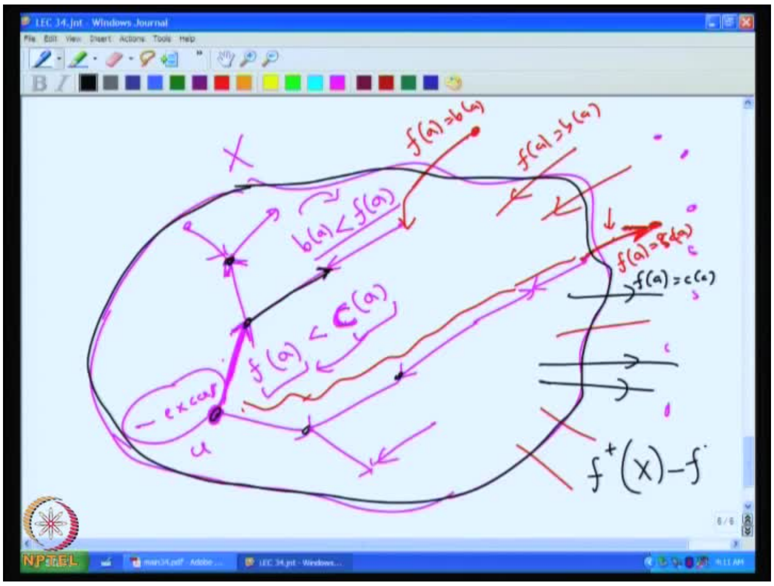

Now, we will define something called an improvable path from u, this is like the max flow min cut the prove for the max flow min cut theorem that we are done earlier. So, what we do we will consider any path, such that suppose if it is a in a in the path starting from u, if it is a forward arc what we will see is whether the current value f f of this a is strictly less than the upper bound; that means c of a. Either the value of the function now is strictly less than then its its its its admits your arc, and other hand suppose if it is like this if the arc is like this. What we will see is the value of the function is strictly great than b of a, this is what we look whether this is strictly less than c of a or this is strictly great than b of a right.

So, now for in other words we will collect we will collect starting from u see remember, this is negative excess here it is a negative excess, then we will find out all the vertices which can be reached from u via some feasible sorry via some improvable path this called improvable improvable path I too repeat what is an improvable path? Improvable is a path starting from u, and each edge on this path is such that if it is a forward arc, in the path its going out toward from when we are traversing from u, if it is a forward arc. Then the value of the current function on this arc will be strictly less than the upper bound for that arc, and if it is a reverse arc the value of this the function will be strictly more than the strictly more than the current arc in this function

Now, what can be infer infer about suppose x is the set of vertices, which are reachable from u by some improvable arcs, these other vertices there can be some vertices outside also right. Now, there can be some arcs which are going from x to outside right, what can I tell about it if it is a forward arc, if it is an arc going from x to outside then what can I say about the function value here. So, then this f of a equal to c of a right, otherwise if f of a was strictly less than c of a, then we could have got this vertex also in x right, because this vertex is reachable from u via an improbable path to here. And then we could have use these two get in, so that means if it is a forward arc if it is an arc going out of x, then it is clear that the current function value at this edge is equal to c of a.

On the other hand, if you have a arc like this incoming arc from too then what can we say then f of a has to be equal to b of a that is why we cannot use this arc to go out, and capture this vertex right. So, this we are seen in the case of max flow the proof of in the proof of max flow min cut slightly different, but then more or less the same idea right, so these are the improvable paths.

So therefore, we have this setup, so whenever it is an incoming arc f of a value is equal to b of a there, whenever it is a outgoing arc, whenever it is an outgoing arc the function value is equal to c of a there.

Now, as we known if you sum up the excesses over all these vertices; that means the sum of the function values on the outgoing arcs minus there, sum of the function values on the incoming arcs over all these vertices right, inside x what will we get as we have seen the edges which are both its end points inside as will not contribute its sum, because one one for one vertex it will contribute to the positive term. And to the other vertex, it will contribute to the negative term therefore, we know that the only contributes are the the edges which are going out of x or come in to x namely the f f plus x, and f minus x and this total value will be f plus x minus f minus x.

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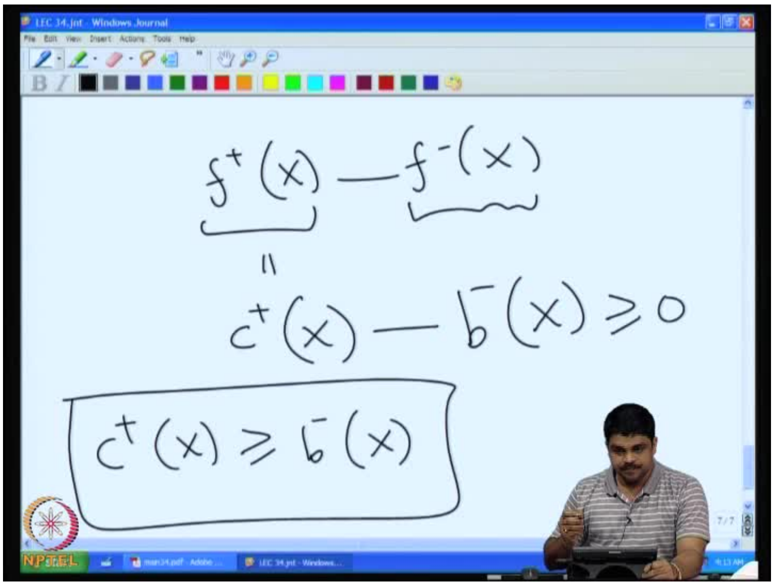

So, if you sum up the excess of over all the vertices inside x alone what will we get f plus x minus f minus x, but then the f plus x is equal to c plus x, because for all the outgoing edges the function value is equal to c of a there right for that particular arc edges are going to out, we know that it is it has touched the upper bound. Therefore, this will be equal to c plus x, and this will be b minus x, because each of the incoming arcs the functions value is touching the lower bound. And because of the condition that c plus x is great than or equal to b minus x, for each subset x, this has to be greater than equal to zero right.

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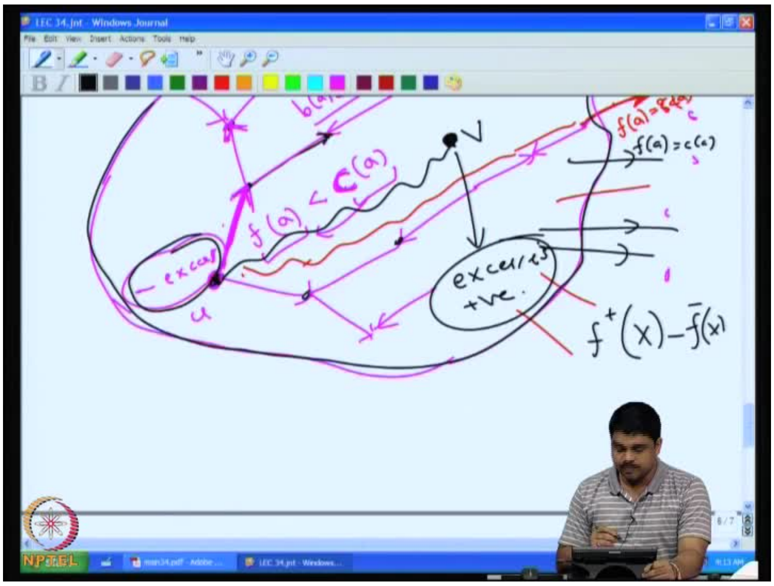

So, the greater than or equal to zero is the when you sum up the excess over all vertices in x, what we get is going to be value greater than or equal to zero, it can positive it can be zero, non negative not negative, but can it be zero no it cannot be I know it can be positive or zero. It can be zero, it can be positive or zero, it cannot be negative. But on the other hand within x we know we have this u, and its excess is negative, so one of them is negative, but the overall the thing the total some of excess has to zero or positive.

So, that means they should be at least one, somewhere in x may be this one say v such that its excess is positive positive right, it should be there; otherwise how can the total be non negative, so because there is one negative, there should be one positive at least. So, let us take u and v, and also v is part of x, so there is in improvable path from u to v, there is in improvable path from u to v.

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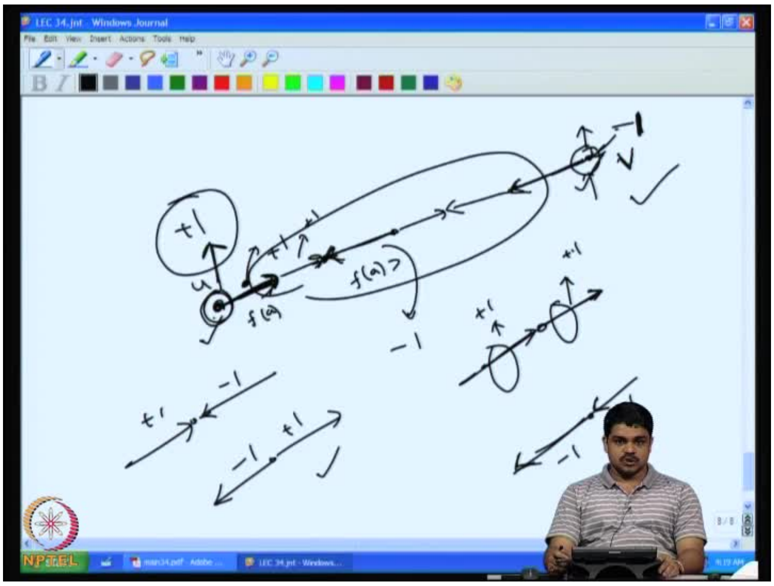

Let us consider this path, so this is u and here is v, and let us consider this improvable path it can be something like this some are positive, some are negative right. So, it can be that. Now, what we do is this is an improbable path, the function values at each of these things is if it is positive, it is strictly less than its upper bound, if it is reverse arc then function value is strictly greater than its lower bound.

What we do is? Whenever it is forward arc, we will increase the function value by one plus one, here also plus one see, but whenever it is a reverse arc we will decrease it by minus one. See, remember this f of a is also is an integer value, b of a and c of a is also integer value, see we can always do the things without integers also, but it the two arc explain the idea we can assume that its b c everything is integer, and then started with an integer valued f.

So, we will increase plus one for the forward arcs decrease one from reverse arcs, and all other arcs we will leave it as such. Now, we can easily check that for these vertices the the interior vertices the increase and decrease over all balances, because there is no change for the excess. The excess will not change why? For instance if a vertex sees one here, one here, then here both cases we have increased plus one plus one, but then this is an incoming arc this is an outgoing arc. So from the outgoing arc also, there is a for the someone the outgoing arcs also we have a plus one increase for this incoming arcs also we have a plus one increase, so the total is going to remain same.

On the other similarly, if it is like this same situation both these negative right, both minus, so far incoming arcs also one is reduced for the outgoing arc also one is reduced. And in this case, if it is like this for instance, then here is a plus one, here is a minus one; so you can see that both are incoming arcs here, so one plus one one minus one, so it is like incoming a total someone incoming as changed by zero right. Similar situation if it is like this, both are outgoing arcs, so this will be minus this will be plus one, so total it should be like the total someone the outgoing arcs of increase by zero nothing right, one plus one and one minus one.

So therefore, the interior vertices of the paths will not have any change in the excesses, but here we know that clearly there is a plus one increase, here there is a minus one decrease, so plus one increase or one decrease right. So, this add as minus one and this is plus one, because this is an outgoing arc here, only outgoing arc is there it is all incoming arc is here, but this was in negative excess already plus one increase will only the absolute value will only reduce here; that is the important thing. Because this is the already negative when you add plus one to excess absolute value reduces.

Similarly here this is positive, this is positive one and when I decrease the absolute values reduced, here also the absolute value of the excess reduce absolute value of the excess reduce. The total absolute value of the values of the excesses reduces there. So therefore, we know that the excess reduced, so let us the key point here was to start with a vertex with negative excess which is guaranteed, because the total has to be zero, total some of the excesses has to be zero, it is not possible to have a only positive excess. If there is no negative, then there is no positive also everything will be zero, then we do not have to do anything.

So, now once we selected u with a negative excess, we we just could wanted to select another vertex v with the positive excess, but also reachable by a improvable path; that means a path where each edges such that if it is a forward edge, it is strictly less than the upper bound, if it is a reverse edge its value the function value is strictly greater than its lower bound, and we have shown that it possible.

So therefore, this procedure can be repeated, and then finally the total sum of the absolute values of the excess will come to zero at that time we will get the feasible feasible function which is also a circulation. So, this is this explains as why that condition that c plus of x greater than equal to b minus of x for every subset x of v is not only a necessary condition, but also a sufficient condition for the existence of a feasible flow right. So, if such that condition is satisfied for every subset a feasible flow indeed a excess we have demonstrated how to get a feasible flow.

Now, that is we will now there is an equivalent theorem in the case of tension also, we would not repeat all the things again, but we can see that in the in the case of tension the the place of cuts is taken by cycles. So, we can consider a cycle, and we can show that the if you consider a cycle and associate a certain direction to this, c minus is the reverse arcs with respect to the sense of direction we gave on that cycle, and b of c minus has to be less than equal to c of c plus, this is a necessary condition right.

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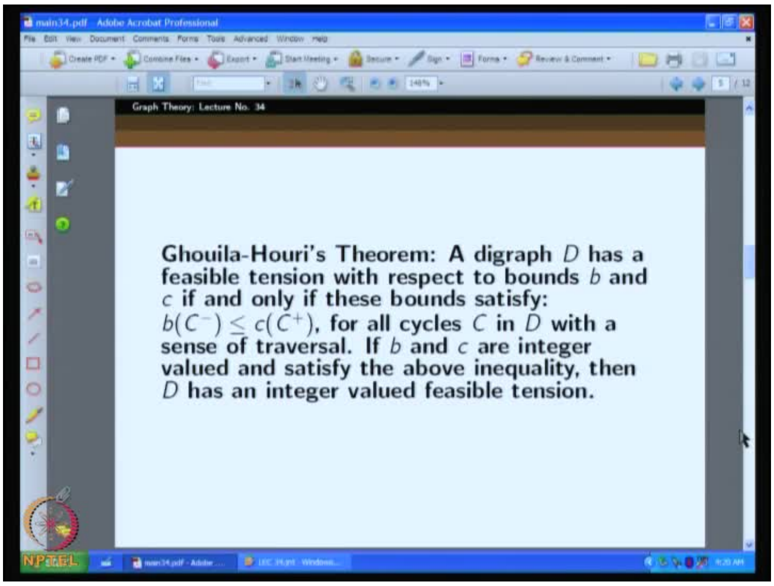

Now, this necessary condition is also sufficient it was proved by… And that is also proved and this is this theorem, so but we will we will not get in to the details of this, any way that is very much similar to the the theorem that we have same. Now, we will go to the to a different topic here in the flows, because in this class we have to finish flows; therefore, we will quickly cover some more topics, so that even if your not doing all that proves at least two see some concepts.

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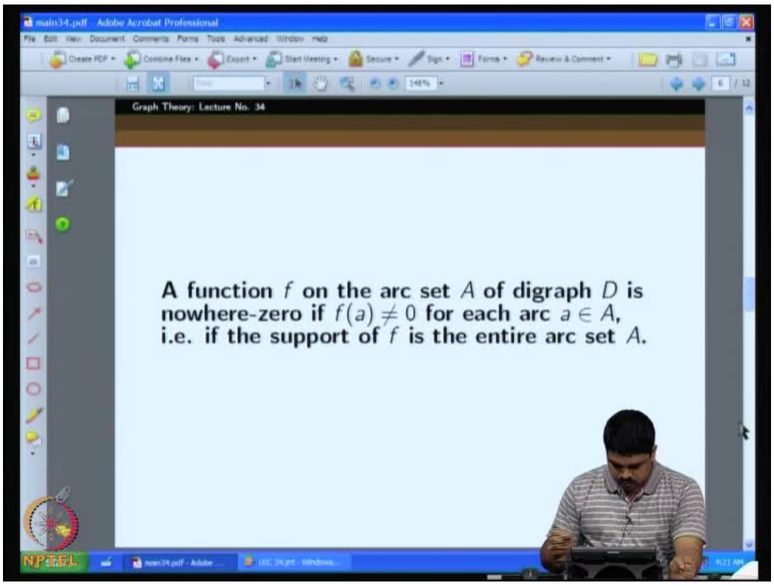

Now, let us consider what is a nowhere zero flow? Nowhere zero circulation, so consider f or maybe we can in general consider function defined on the arc set of a, it is called the nowhere zero, if for each arc f of a is not equal to zero; that means if I consider as a support of this function, then that is the entire arc set itself there is no arc for which f of a is equal to zero, that is that is a very intuitive definition.

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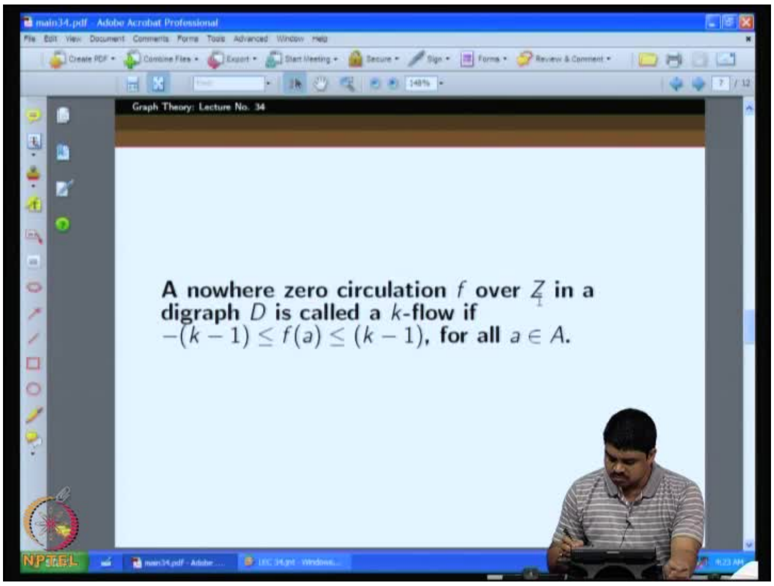

Now, we will define something called a k flow, here circulation is called a k flow, if for each arc its values in between minus of k minus one and plus k positive k minus one in between minus of k minus one and k minus one. So far a this case, let us we will take it as k great than or equal to two flow, so the why is it is it has to be come zero, it you put one it has to be come zero, so to onwards right.

So, in between because remember this is a nowhere zero circulation, it has to be first important thing, it has to be nowhere zero circulation, so one it say it say historically it is called k flow though though it is a actually circulation, so but its let us conveniently use this flow k flow to need not this thing. For instance if you are talking about the two flows, you want the values of the function this circulation to be minus one or plus one, of case the values are integer integers, we are not allowed to use fraction of the values or real values

So, so all the values are fraction values are to be from set; for instance if it is k equal to three that is minus to minus one one or two; these are the low values right. For each arc a value of the circulations should be minus two minus one one or two, so this is what a k flow is…

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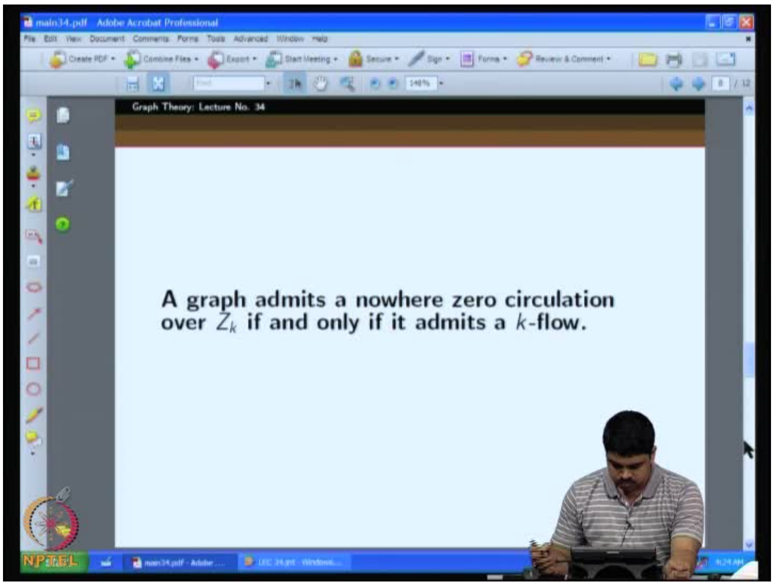

Now, very interesting thing is you can define a circulation over set k, what do you what do you mean by that. So, the values are zero sorry nowhere zero, so 1, 2, 3, 4 up to k minus one set k right, the summation is now done modules module mod k there right; it is not the useful summation, we will sum and take the remainder with respect to the mod k.

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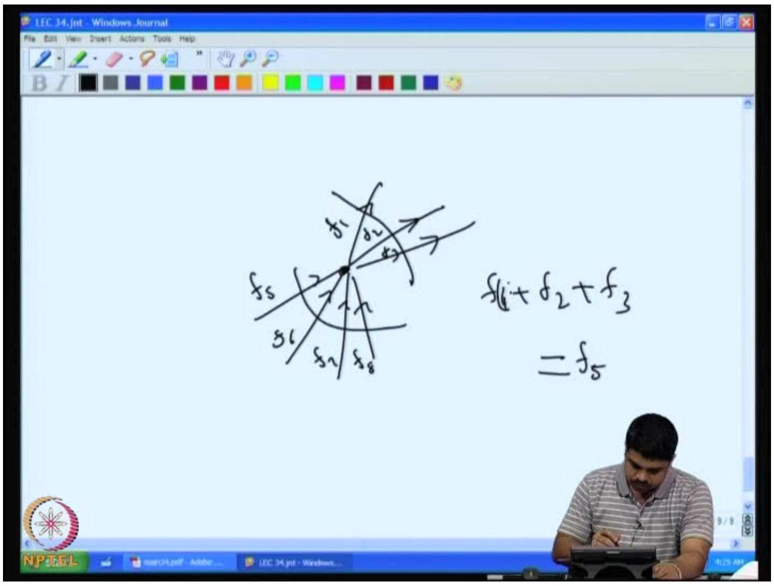

So, this way also we can defined, so this is somewhat different, because you know now the only the main condition for the circulation is was that the some of the flow values - the circulation values on the outgoing edges should be equal to the some of the flow values on the incoming edges, But you see here the actual value need not be same, but it need not it need to be same this some of the values say suppose this is k f 1, f 2, f 3, so f 1 plus f 2 plus f 3 has to be equal to say this is f 5 f 6 f 7 f 8.

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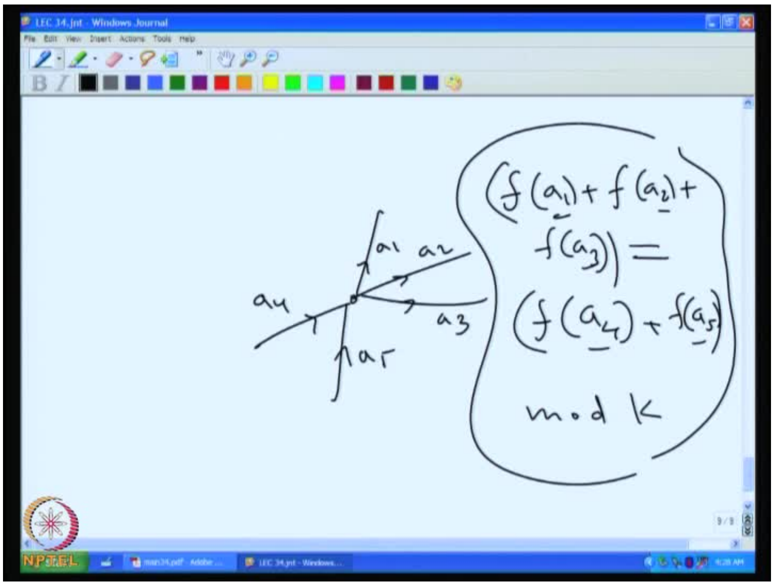

So, f 5 may be we consistent, we can say it can be like let say this is a one, this is a two, this is a three, and these are say a four and a five. We know usually we want f of a one plus f of a two plus f of a three equal to f of a four plus f of a five, why because this a four and a five are the incoming arcs a 1 a 2 and a 3 are the outgoing arcs.

Now, we do not want this we just need this mod k only, so when you this may be different values, but when you divide by k take the reminder has to be same right. Therefore, somewhat different this flow, but the interesting fact is that a graph admits a nowhere zero circulation over z k, if and only if it admits a k flow its one side should be same, because for instance if I have a k flow then I can show that there is a set k flow also why is it.

So, so one way of seeing it is, so for instance we have after all we can we can interpret each of this values plus the integer values in between minus minus k of minus one two plus k minus one, as the values with respect to in that z k in z k. And then if you sum up that will, because when when forever flow its equal to zero, then naturally this will with when you take with respect to mod k that will be equal. So, one side is definitely same for instance, if you let us definitely see for instance if you have a k flow, then we do have a nowhere zero circulation over z k also, but on the other hand that is more non trivial for instance how can we say that suppose this condition is satisfied; that means with respect to this mod k equality is satisfied.

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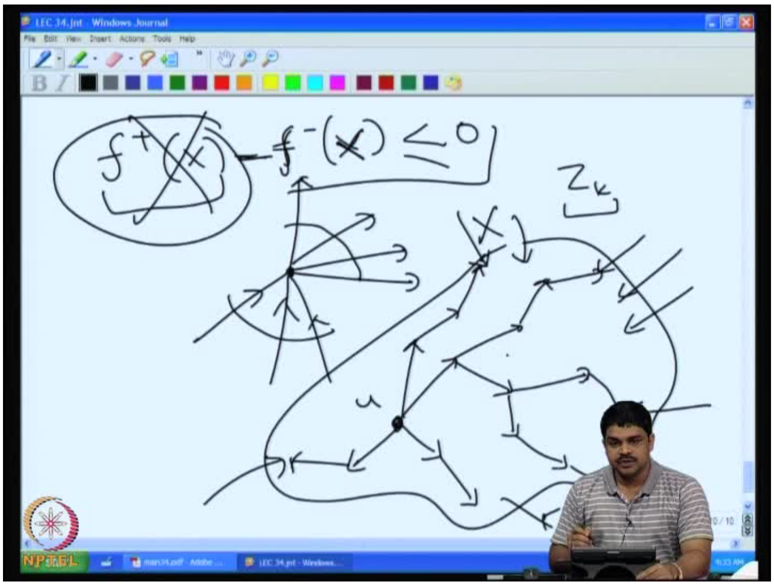

The actual equality can be obtained for some slightly modified flow, so the we can to show this thing we can use a similar trick that we have done in the earlier case, suppose let us assume that we have a flow in the z k world right. The values are the function it is a circulation in z k, and then now we can again define this notion of excess right.

So, you know every excess is a multiple of k now when you take this, if you consider it as usual integers though not in the mod z k world. So, if you take the sum here, and minus the sum here like the some on the incoming edges minus, the some on the we can again define the excess here, this excess is definitely divisible by k that right is when I take the difference it is should be zero mod k; therefore, divisible by k.

So, but as before we can consider it has a positive excess and negative excess, and zero excess and then we can try to reduce the total excess, so that finally the actual the total excess will become zero like, we did in the other case, so far instance we can see if all are not zero already in that case we are already done.

So, if if there is a positive suppose we can if there is a positive, then they should be negative all cannot be positive along, because again if you take the the total sum of the excess again has to be zero, like we explained in the earlier case right, because every edge contributes to one positive term one negative term.

Now, so we can consider start with u which as positive excess this term say, and then we can consider all the vertices which are reachable from u by a directed path this time, we are not allowing any this notion of path. We are taking is different, we are taking a directed path not allowing any reverse edges, so all the vertices which can be captured using directed paths starting from u. Now, we collect all the vertices, let it be x now you know any edge which is going, so which some interaction with outside world should be incoming edges; only that is why we cannot go and capture the remaining once right, it should be like this it should be everything should be like this incoming edges.

Now, we know if we sum up the excesses over all these excess that as earlier it has to be f plus of x minus f minus of x, you are not repeating the argument here, and this f plus of x is not there at all, because there is no outgoing edge. Because if there is an outgoing edge, we can capture the the end point of that outgoing edge also from u, you can reach to that vertex also by a directed path.

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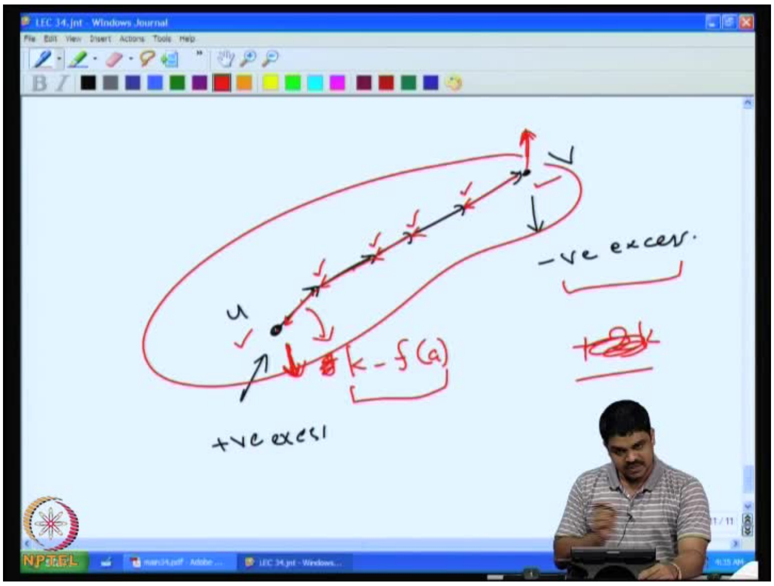

So, this is not there, so minus of f minus x only is there, so this has to be a negative value right this is less than equal to zero. Now, u has a positive excess the total sum of the excess on the vertices for the vertices of x is less then equal to zero. So they should be one with the negative excess. So, let call it v call it v, this is u this is v this as positive excess and this is negative excess, and then there is a directed path and also there is a directed path from here to here.

What will you do, so we consider like this, so this is the directed path from this thing we consider changing the direction of the edges here, and then we can assign the value k minus f of a for each of them. So, if f of a is the value here put k minus f of a here for each of them right, the corresponding f of a value k minus this thing, and now for all other edges we will keep it same.

Now, we can see that excess for all these things are not changing while this thing we have a decrease of minus k here, this thing the excess will increase by decrease by k here. And increase by k here, because this is the negative excess increase will cause the absolute value to decrease.

So, total increase in that some of the total absolute values will be now by two k right plus two k sorry total decrease, so here so this is the positive excess here is a decrease, here is an increase. So, totally there will be a decrease in the total some of the absolute values of the excesses, so I just leave it to the a student's to figure out that is indeed happens.

So, why I am leaving it, because it is more or less the same tricks that we have done in the prove before, so the when we considered the excess of a feasible flow. So, just that the only concept was there, we were using the kind of improvable path specially defined here, we are just taking directed paths. All the other ideas are more or less same, so and then the just bringing the total sum of the absolute value of the excess to zero.

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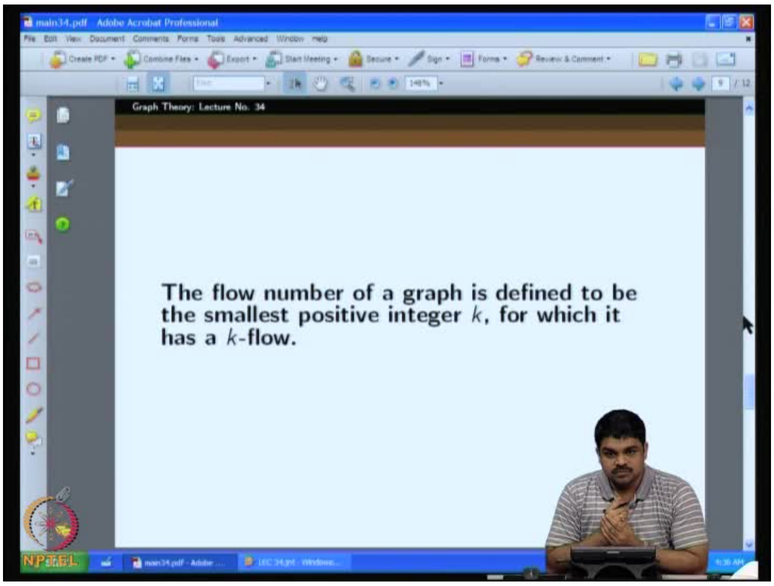

So therefore, we can do this thing and finally, by repeating this procedure we will come up with a flow where all the excess are zero which means that it corresponds to a usual k flow. Now, when we come to k flows the next main concept is the flow number that question is… So, given a graph what is the smallest positive integer k for which it has a k flow.

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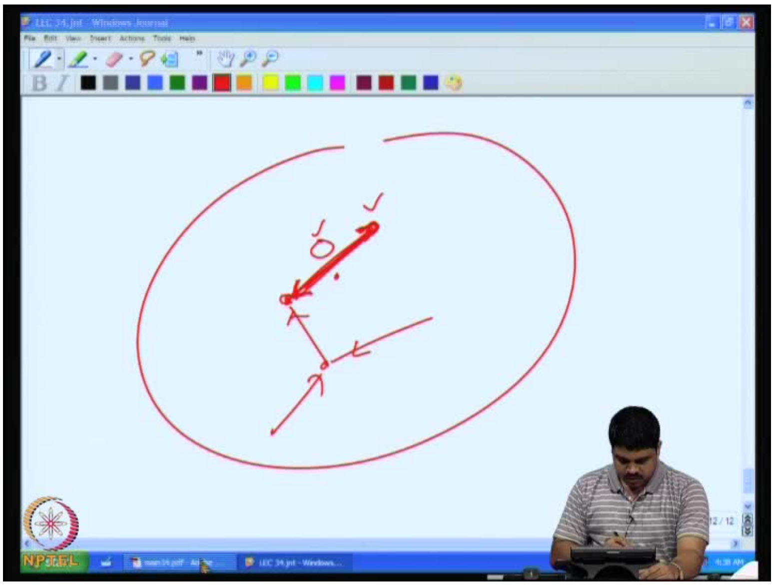

So, why I am talking about a graph not a digraph, now because the orientation does not matter any more, so for example suppose you have an edge here, so you have a graph, and then each edges given a direction. And then there is a k flow, then we can show that for any different orientation also we have a k flow the reason is we can just suppose for this edge. If I reverse the direction and put the negate the flow value here, then still a flow right, because for this these are the only two vertices which are affected here earlier it was a outgoing edge, but now this an incoming edge right.

So therefore, when you negate the thing the total incoming value by will be reducing by that quantity, because here is the negative value here while earlier there was a positive outgoing edge with corresponding positive value is now not there; therefore, it balances similarly for this edge.

So therefore, it is still a circulation, so if we have a k flow for a given orientation for any other orientation. We can get a k flow different k flow with I adjust my negating the flow values right in corresponding, whenever we change the orientation we negative immediate.

So, the orientations are not very important when you it when we talk about the existence of the k flow, we can if you say that for this. We can simply say that for this graph there x is a k flow, so in other words we say that whichever is the orientation we can find a k flow.

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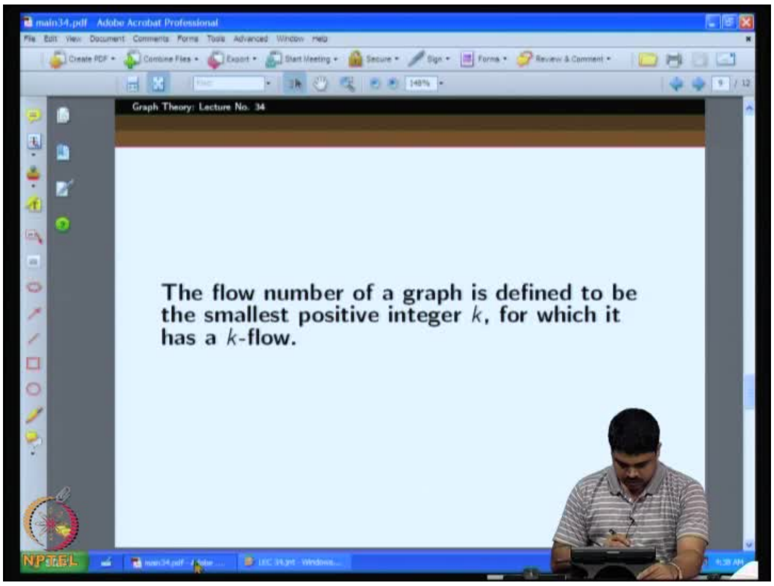

So therefore, the we can talk about given a graph this smallest integer k for which when a orientation of that has a k flow, so one some simple examples we can consider the a graph when this graph admits at two flow, this simplest the smallest possible value is two So, it is that is if and only if it is even for instance, if it is an even graph even graph means every vertices degree has it degree equal to an even number. Then it is very easy to see that you can given if can get you can get it two flow for that two flow, means the values allowed are just one or minus one how do you get two flow two flow for that.

So, just first give a even orientation for that even orientation means for every vertex, we we have to give incoming arcs, and outgoing arcs numbers are to be is equal how do you do that? You can consider Euler toward of the graph, then give the orientations with respect to the arc; that means as we traverse the Euler path in the direction right.

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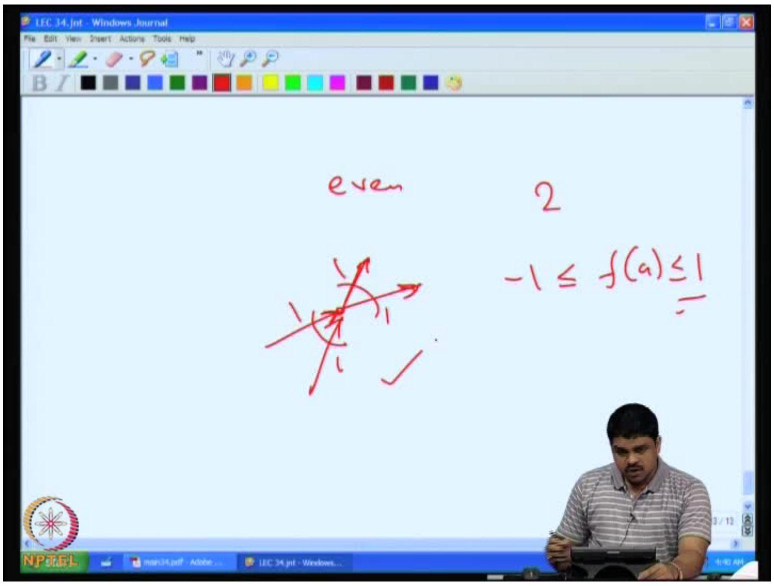

So, every vertex will equal number of incoming edges and outgoing edges, so then if you give one one one for every every edge, then indeed the total on the incoming edges will be equal to the total on the outgoing edges. Therefore, it will be a circulation, and also it will be a also it will be a yeah that also it will be a feasible sorry k two flow, because the one one is in between minus one flow the values will be in between minus, and one so one is an allowed value.

On the other hand, we can also show that the the suppose there is a two flow it has to be an even graph, so you can consider any orientation for which that excess a two flow, and then what we can do is whenever the flow value is negative one we reverse the orientation of that edge and then make it positive one.

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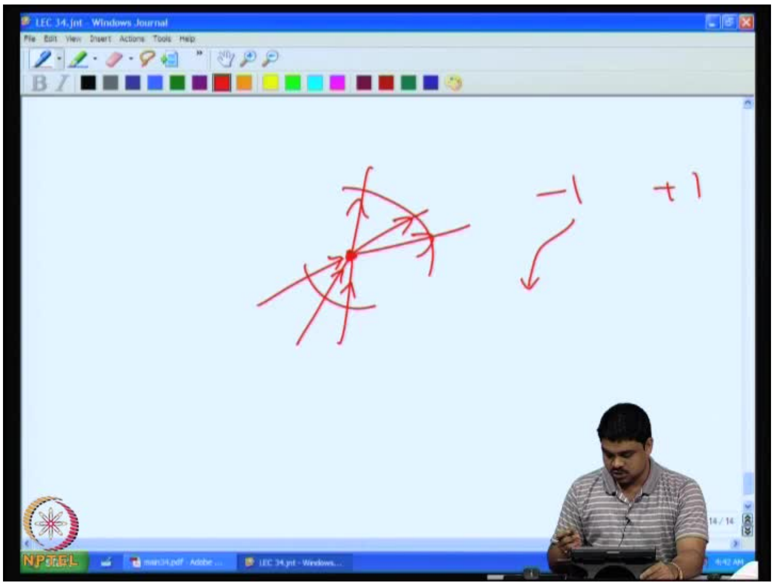

So, all the flow values can be considered one, now now it is easy to see that for every vertex all are one right, so it is very easy to see that the number of incoming edges is equal to number of outgoing edges otherwise how is it possible that the total on the sum of the values on the outgoing edges equal to the sum of the equal to the values on the sum of the values on the incoming edges.

So far that the only way is the number of incoming edges equal to the number of outgoing edges, if all values are one right we can, because only allow values are minus one and plus one, whenever there is the minus one. We just reverse the orientation of the edge and make it plus one, we know that this is this can be done this is still a circulation right that is that is what I explained just before…

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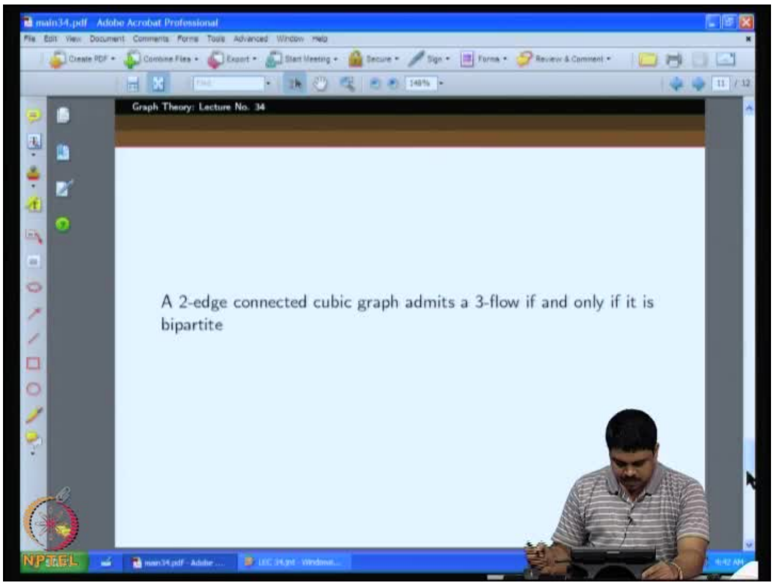

So therefore, this total will remain circulation, so we see that there is a two flow incoming degree has to equal to outgoing degree which means that when I remove the directions the sum the degree of the each vertex has to be an even number. Similarly, another statement we can make about it this thing is any two edge connected cubic graph; so two edge connected is required, because we know if there is a bridge as we as we have seen it is a cut its one edge it has to be zero right, because we are talking about nowhere zero flows. Therefore, two edges connectivity is a must…

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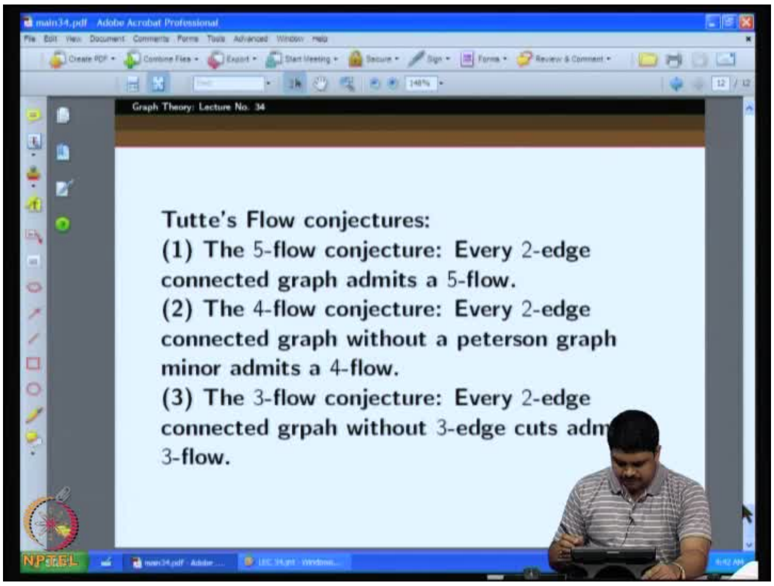

So, two edge connected cubic graph admits three flow, if and only if it is bipartite say now we do not have time to do the prove; therefore, we will keep it keep it for the student to work out. And finally, we conclude with the three famous conjectures from tutte about the flow number of graphs, he says every two edge connected graph admits a five flow, this is the five flow conjecture it is known that every two edge connected graph admits a six flow, whether it admits a five flow not still upon.

The four flow conjecture say every two edge connected graph without a peterson graph minor admits a four flow, and the final third conjecture the three flow conjecture every two edge connected graph without three edge cuts admits a three flow. So, these are very significant open problems in graph theory, so this thing we will leave the topic of flows, and in the next class we will consider the application of probability in graph theory. Thank you.