Graph Theory

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Lecture No. # 38

Probabilistic Method: Second moment method, Lovasz local lemma

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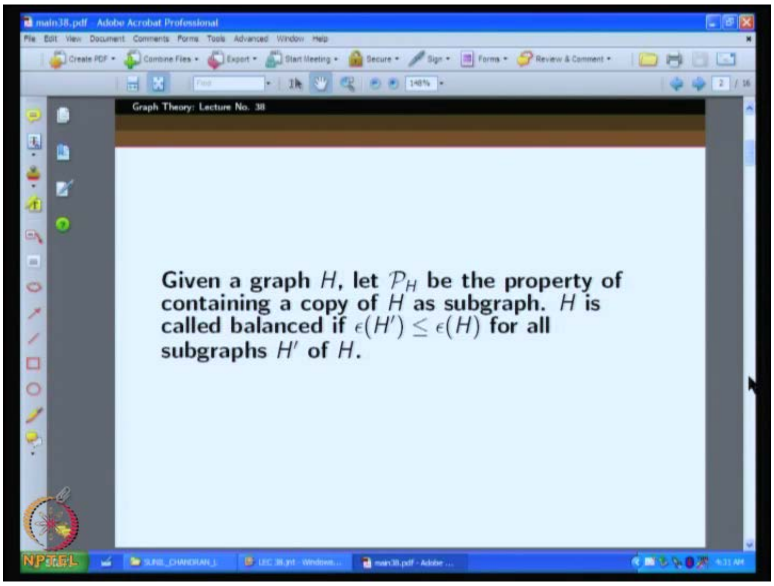

In the last class, we were describing a technique of proving threshold function for a given property. So, this technique was based on the second moment method. So, here we have an example we look at it; so here, we have this property P H - H is a given graph, it is a fixed graph. The property P H is the property of containing a copy of H as sub graph; that means, you have from the n vertices. So you have this GNP model, and you take a random graph based on that probability base, and then the question is do we have in the random graph, do we have copy of H as a sub graph; for this given H is a fix sub graph outstand. So, the question is what is the probability? So is there a threshold function for this property.

So, our second movement method usually consider as a random variable, namely here what is the correct correct random variable namely the number of isomorphic copy's of H, its very clear that H is there - this number has to be greater than equal to 1, if H is not there in g then it has to be 0. It cannot be negative value right 0, 1, 2, 3 like that; these are the only values this random variable can take. As we discussed yesterday, in the in the last class its very clear that proving that when the probability P of n in the GNP, the P function P, P of n is less than threshold; that less than the threshold means when P by t tends to 0 a sentence infinity right; if the function P is such that P by t, t is a threshold function, P of n by t of n tends to 0 as tends to infinity.

That means the probability function is below the threshold. In that case, we want show that this property almost surely will not be there; that means they wont be any isomorphic copy's of H and g. For another words if a counting, the random variable will b equal to 0 for this particular which our with most of the cases, almost cases. So, we want to show so that x is a random variable, x equal to 0 with almost surely with a probability attending to one as tends to infinity or in other words probability of x greater than equal to 1 will tends to 0, as tends to infinity.

This is what we want to proof; on one side, the other side we will want to prove, if P by t tends to infinity sorry tends to 1 sorry P by t tends to infinity as an tends to infinity, then the then the probability, then the probability probability of having property P H that means, at least one copy H in the random graph should be also attending to 1. That means the probability of x greater than equal to 1, should be equal to attending 1 has (( )) this this are the two things we want to do. So the first thing to prove that when the the the probability P of n is below the threshold that probability that x greater than 1, will tend to 0. It easy to show in this way what we will do is, we will consider the expectation of x and definitely if x is grater probability of x greater than equal to 1 will be less than expectation of x by 1 with by mark of in equality.

Therefore, if expectation itself is shown to be less than 1, it attending to 0. Then of course, it means that it will it will the the with with a probability trending to 1, x will be equal to 0. This is the first thing other thing we will have to use the second moment right. So, lets say, lets me take about this problem once again; here H is a fix graph - not only it is a fix graph, it is a balanced graph. What you mean by balanced graph? The the we are interested in the number of edges divided by number of vertices; this is epsilon. So, if for every sub graph of H also, we have number of edges divided by number of vertices is less than equal to that of epsilon of original H; that means, that edge concentration edge density will be always smaller or equal. For each sub graph compare to the original graph; its its not that somehow the total big graph is sparse while there are some induced sub graph of it which is dens, that that situation will not araise.

So, for every sub graph we will have its edge density less than equal to edge density are original what; this is what? A balanced graph is and as we had seen in the last class, this balanced graph model will capture many interesting graphs, for instance cycle, park, trees, and complete graphs - all this things are balanced graph. Therefore, we already show showed how to our theorem about this property, the threshold function for this property immediately implies for cycles, trees, and complete graph is extra. So, it is a very useful theorem, we will to prove it general theorem, because we interesting special ceases immediately comes out right.

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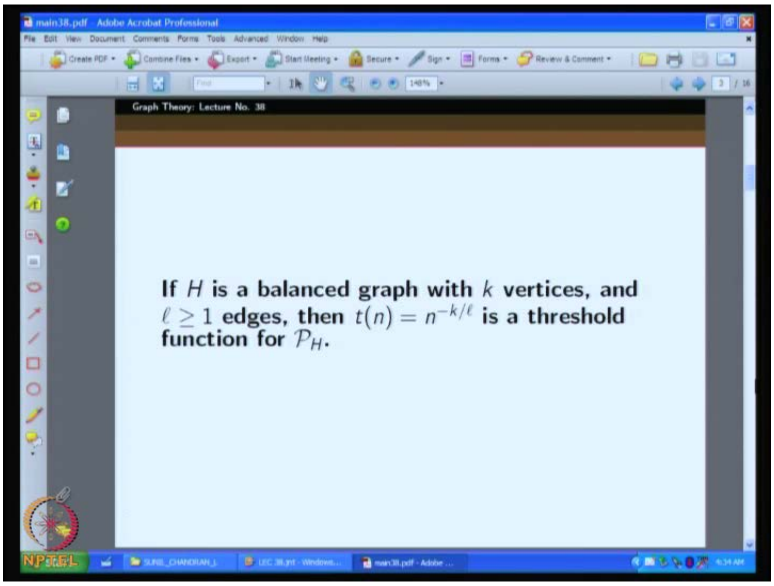

Now, lets a try to prove this things right; so this is what we want to prove. The threshold function is always n to the power minus k by l; k being the number of vertices, l being the number of edges k by l, that is 1 by epsilon right. So, this is what we want show; if H is a balanced graph with k vertices and l greater than equal to 1 edges right.

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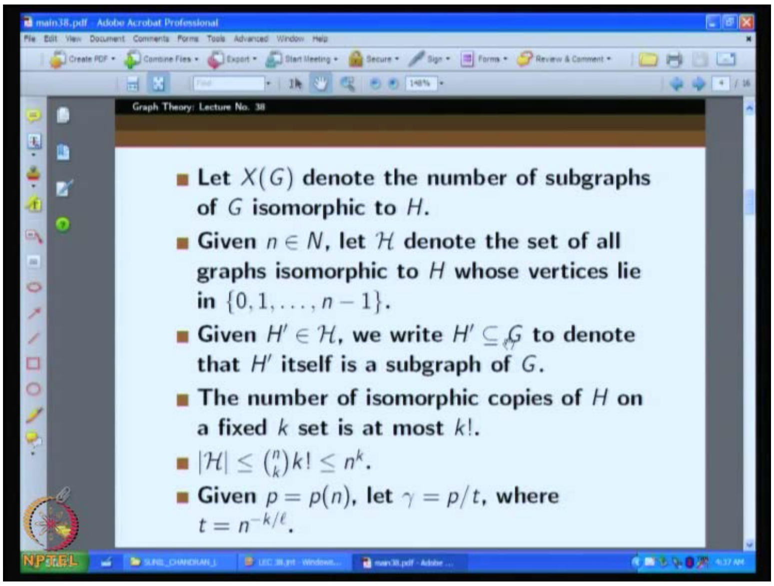

So to show this thing, let as as we mention lets consider this random variable which a denote X of G denote which denotes the number of sub graphs of G isomorphic to H. Now, the for n element of N; lets define this class of graphs H; so that is let let it denote - the set of all graphs isomorphic toH whose vertices lie in 0,1 to n minus 1. See look 0, 1 to n minus 1 is the setup vertices on which we are considering the random graph G, and b model that the n vertices are fix their right; so that n vertices 0, 1 to n minus 1. Now, H they can be any graph which is isomorphic to H, and with vertex set from this collection right. So, since H as only k vertices, any k subset of this thing can be vertex set, and the it should (( )) isomorphic to H. So all such a graphs, the collection of all such graph is denoted by H the cap cal H.

And now see given H dash element of H we right, so we have a notation here. H dash is a sub set of G, to denote that H dash itself f is a sub graph of G. Not that it isomorphic copy is there is the for instance if you for instance this number, vertices number 10, 20, 30, 40 like that; till ten times k right, k time ten. So, then so if you look at the corresponding vertices - same vertices vertices number ten from this 1 to 0 to 1 to n, 20 0 to 1 to n like that; and if you see the correct graph H itself their, then we will say we will say that, that particular graph is a subset of this, that is where it is not just that we have this graph some isomorphic copy of its there. The number of isomorphic copy of H on a fixed k set is at most k factorial, because all possible permutation of that that is a maximum we can get.

So now, the cardinality of H will be defiantly n choose k, because select n choose k in to k factorial, because we can select each k k sets of vertices from n set n choose k is, and in that k set we can have at most k factorial copy's - isomorphic copy's of H not more than that. So, less than equal to n raise to k; this is n factorial n into n minus 1 k terms divided by k factorial, that k factorial is canceled and each of this terms is replayed by n; so n less than equal to n to the power k.

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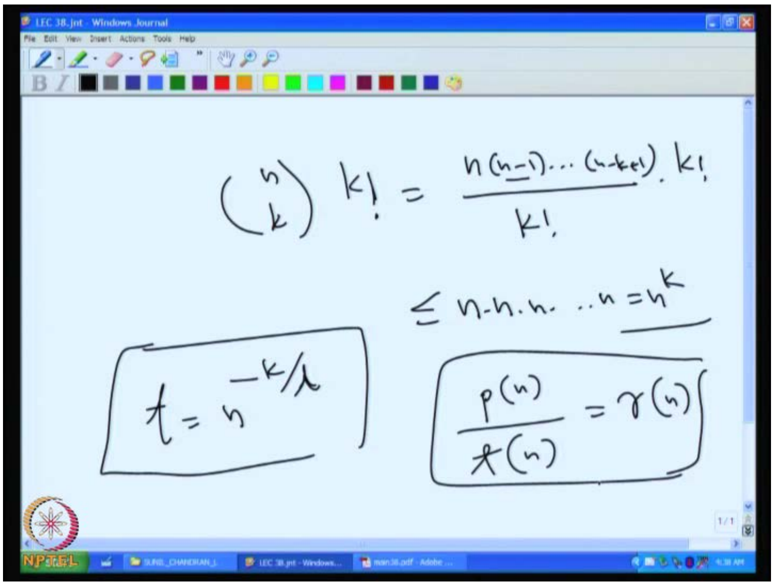

Now, the given P equal to P of n. So, we can (( )). So I hope you understand choose k in to k factorial, and I say choose k into k factorial - this is n in to n minus 1 into n minus k plus 1 divided by k factorial into k factorial, this is definitely less than equal to n into this is this can be replace by another n n into n into so n, so that is n raise to k. So that is are we substituted here with n raise to k; so H the cardinality of H is at most n raise to k. Now, given P equal to P of n P equal to P of n, let gamma equal to P by t; so we say this is the t is the threshold. What was the threshold value the t, t essentially equal to n raise to minus k by l that you have that remember remember k by l, this is the threshold; so P of n is some probability function. So this this is what it denote by gamma of n right, gamma right. Now, so we know that equal to n raise to minus k by n right.

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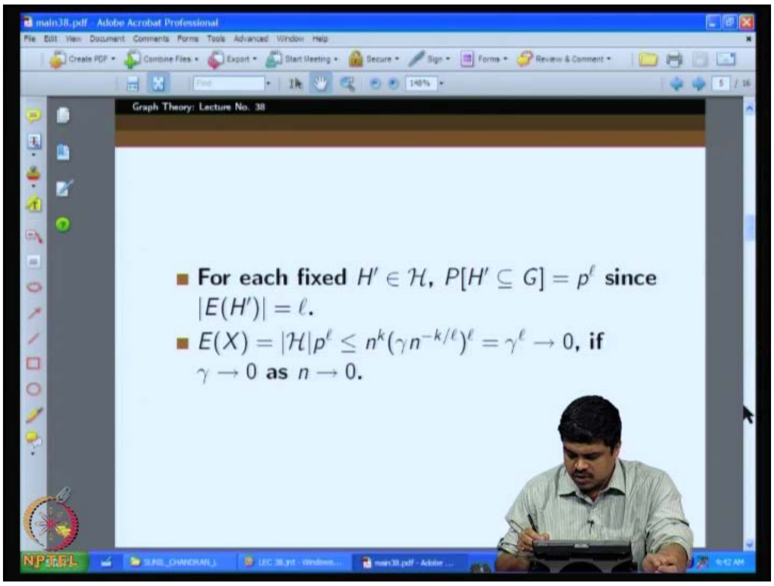

Now, let us look what is the probability of for a certain H dash in H to be in G, probability of H dash subset of G, because H dash as l edges in it. So, the probability defiantly P to the power l right, because each edge has to appear and that happens probability P there all independent; therefore, you can multiply it out P to the power l is the probability. Now, what will be the expectation of x then, so for each graph in this set H, you have probability P raise to l, you submit up; so cardinality of H into P l what is that is less than equal to n to the power k, because H - this capital H is n to the power k, and at most n to the power k; so we can substitute by n to the power k and P to the power l can be, because P p power l; see remember here, we have defined gamma is equal to P t - so P by t. So P equal to gamma t; so gamma into n to the power gamma into n to the power k by l right;

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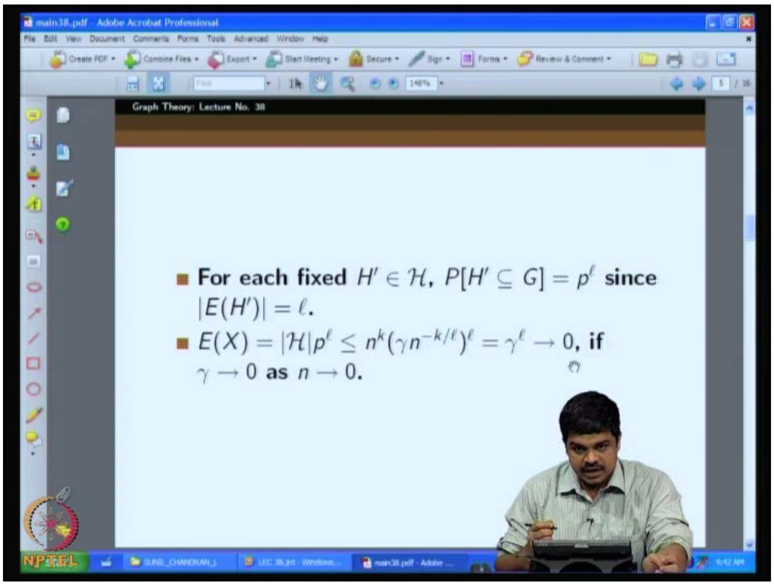

so the now, we can substituted gamma n to the power minus k by l whole power l; so then, because this is l. Now, we will cancel this l n to the power minus k, n to the power k will get cancel and we have gamma, gamma to the power l left. And thinks, we now that gamma tends to 0 as tends to 0. So, gamma (( )) also tends to 0 that is all; so, let us this fixed value that number of edges in H. So, gamma to the power also will tends to 0. When when P of n is such that P of n by t of n tends to 0 has sentence to infinity, that is gamma of n tends to 0 a sentence to infinity. For such P of n definitely this expectation will tends to 0 a sentence to infinity; if it tends to 0 what does it mean, so for a large enough n expectation will be less than 1, so which means that x has to be equal to 0 right; and the probability will be defiantly less than equal to their expectation by 1, probability of x greater than equal to 1 will by by Markovin equality. So this, since expectation tends to 0, the probability also will tends to 0. This what we are using is Markovin equality here.

This is the easier part, there we now we look at the more difficult part, namely to show that if gamma tends to gamma of n tends to infinity, and then when the gamma P of n by t of n is such that it sentence infinity, this also tends to infinity; then then our probability that x greater than equal to 1 will also tend to 1 or an others probability where x equal to 0 will tend to 0 right the other way. That means, with high probability we will have a copy of H in the graph, this is what we need to prove.

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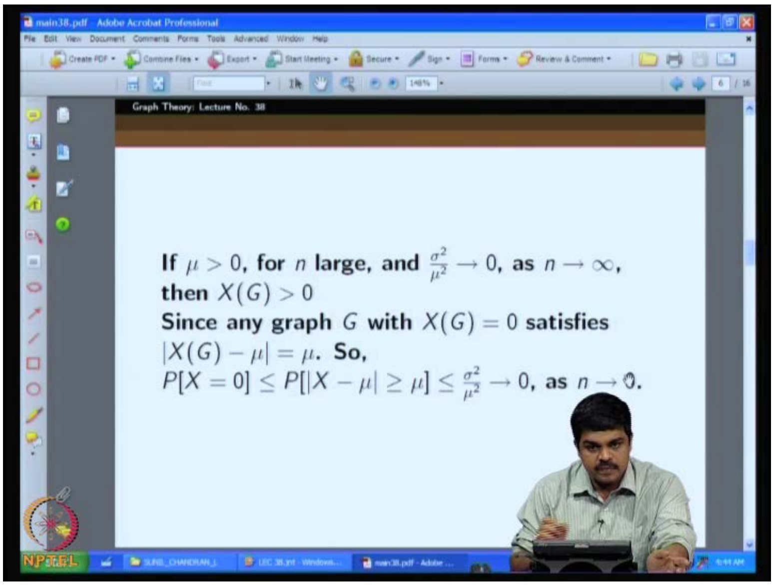

So, to prove this we had a seen in the last class that, a tool can be used lama can be used (( )) the name; so what the lama told is if new greater than 0, that means if expectationis greater than 0 which is always, so for our, because all the values are 0 than then only mu will be 0, because our values x takes values 0, 1, 2, 3 onwards. So, if all the values is not 0 all out graphs does not have 0 value, then of course, there are mu will be grater then 0; and for n large sig if this variants divided by mu square sigma square by mu square tends to 0 a sentence infinity, then chi of g will be greater than 0, then chi of g will be greater than 0. This is what it it else right sorry not chi of g; X of G will be greater than 0.

The X be more variable, so so the how how with high probability X X. So the probability that X greater than 0 will tend to 0; so the proof we had seen, because whenever X equal to 0, when you consider this other random variable x by mu x minus mu it will be equal to mu; so if you this is you can say the this probability x equal to 0 probability will be less than equal to probability that x minus mu greater than equal to 0; and then we can usually equality by this is less than equal to sigma square by mu square, and if this tends to 0 definitely this probability has been tends to 0; that means, this probability has to be 0, which means that the probability that this happen that x greater than 0 happens tends to 1 right.

This is this is the idea we are using now now that we have this lemma, we just have to show that this sigma square by mu square tends to 0 as n tends to 0 sorry n tends to infinity n tends to infinity. So, this is wrong; so, but we know that a sentence infinity our gamma tends to infinity; So therefore, as gamma tends to infinity the sigma square by mu square has to tends to 0, that is all we have to show.

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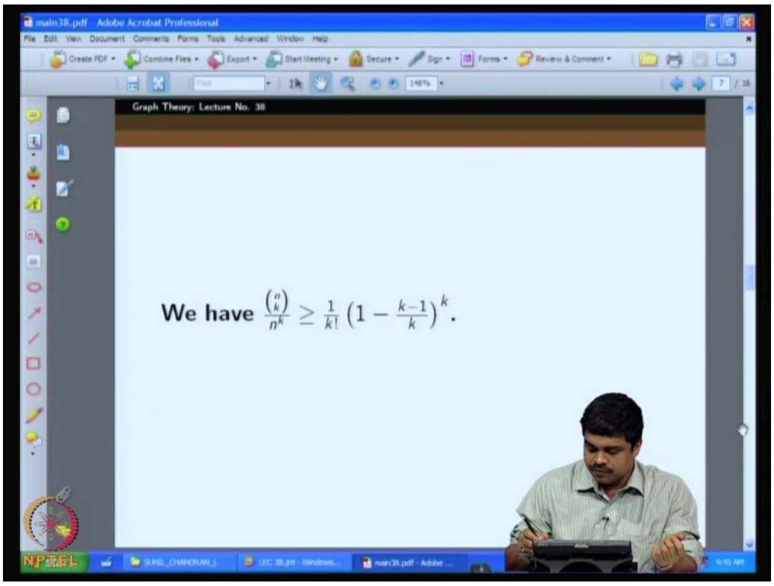

Now, (( )) we just at to do calculation for this things, its not very difficult, just that calculations are little little more involve them usual; so we need this particular observation. So this is, this will help. So what is the suppose consider these two numbers n choose k and n to the power k; defiantly we know the n n to the power k will be bigger, but how bigger will it depend on how bigger that factor n raise to k by n, n choose to k are n choose k by n raise to k will you depend on n. So, its this says that it it will not depend on n, because this is what this we can express that, this is only a function of k. Therefore, n to the power k will be greater than n choose k by only a function of k, it want to be to much more than n n choose k.

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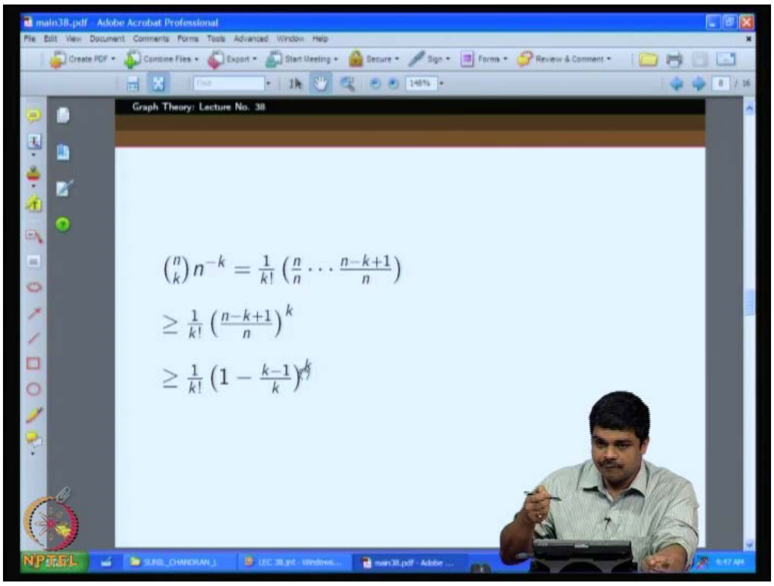

So therefore, so n's contribution will not be there in there; that is what this is proof is very easy, you consider this n choose k by n by k. So, this n choose k 1 by k factorial into n into n minus 1 into n minus k. So, n below we have n to the power k, we can distribute 1 and 1 and for each of things; so this will be 1 by k factorial in to n minus k; the last term is the smallest k, we can replays with smallest term. So, this raise to k, because the the k times, and now so this can be written as 1 by k factorial into 1 minus k minus 1 by n, and then because we can we are ready to take a lower bound, we can replace n by k here. Because we are only making it this is what we are subtracting, we coming a little bit bigger; so overall quantity will become become a smaller only.

So therefore, see what we are shown now is n choose k by n raise to k can be low bound by a function of k alone, so n choose k will not be much more than n by sorry n raise to k will not be much more than n by k. So, it will be more than n choose k sorry n choose k by a only by factor of sorry only by a factor of f of k, where f is a function of k that is out detail.

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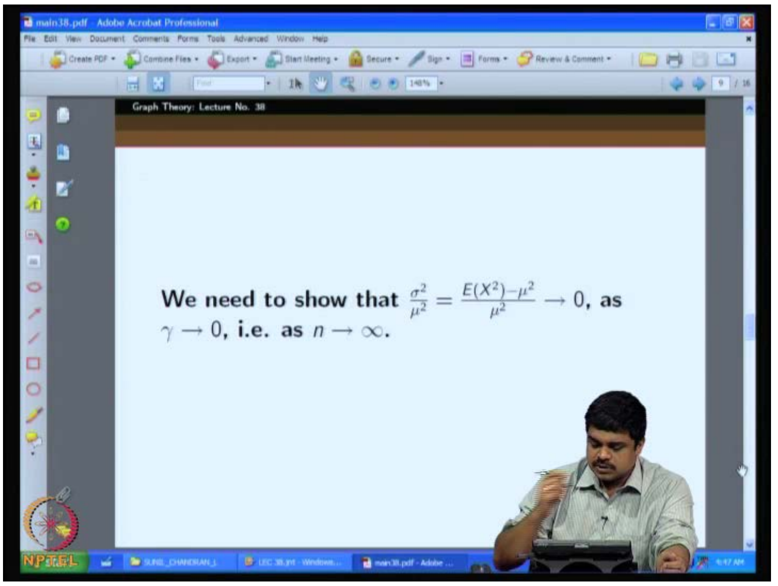

So this will be required latter. Now, to prove that the what we want, then the threshold probability P of n is a above the threshold P of n by t of n that is gamma of n tends to infinity; our probability that is x greater than 0 tends to 1.

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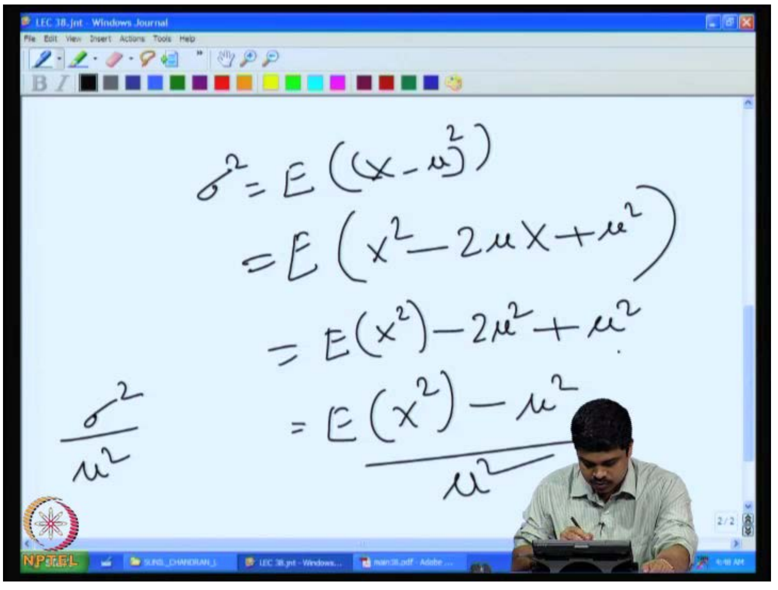

We need what we need to show is sigma square by mu square will tend to 0, that means what is sigma square? Sigma square is essentially, if you remember this sigma square is essentially the expectation of x minus mu whole square right. So this is what expectation of x square minus 2 mu x plus mu square right, because so the linearity of the expectation- this becomes expectation of x square minus 2 mu is constant and expectation of x mu another mu, so we get t mu square plus mu square, expectation of a constant again constant; so this will become expectation of x square minus mu square right. this is (( )).

Now, if you want if we are interested in sigma square by mu square, we will dived by mu square here. So, this what we taking here. So, this is we have to show this terms to 0. So, expectation of x square by mu square minus 1 is what we have. So, now we just have to calculate what is expectation of x square, that is the only interesting parameter mu right. So, now how will you calculated.

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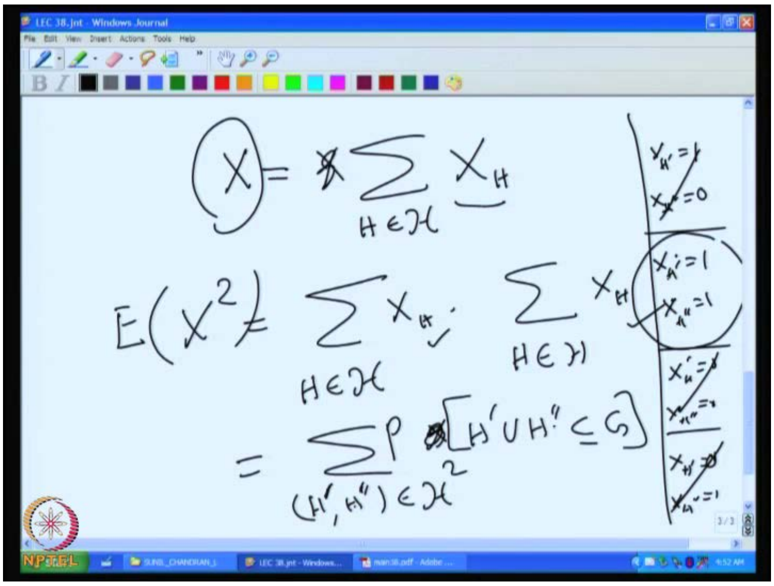

So, first of all we have to understand what is the expectation of x square – what is expectation of x square. To understand what is expectation of x square? See remember that X was in the last class, we show that X can be represented as the sum of several indicator random variables, namely so we can righted as sigma H for H element of this set right, because you know for every H in H, we can we can desired whether that H belongs to G or not, if it belongs to G this random variable will be b 1, otherwise b 0; if some over all such X such as, we will get this some some of copy is of H in G right.

Now, if you want to find the if you want to find x square what will happen, if we will be coming up this there is an H element of H here, X H into another H right, H element of H here X H. So, that means very possible H from this hand every possible H from this; so that will that will be like H comma, we can lets say H dash and H double dash right; this pair belongs to this H square we can see. Any 2 1 H dash 1 H another H double dash one

H; they can be even equal. So, we are just the prob, because this edge the the essentially when will this b 1, because some cases that will be 0. So, there are various possibilities X H dash can be 1, see there are this this many possibilities here. X H dash can be 1 x H double dash can be 0, this is one possibility - X H dash can be 1, X H double dash can be 1, and X H dash can be 0 and X H double dash also can be 0 for X H dash can be 0, and X H double dash can be 1.

So, in all this cases the product will becomes 0; so if only this case interesting for us when you some of this things right; and when will this happen both H dash should be present, and H double dash should present. That means H dash union H double dash should be a sub set of G. This probability what we are interesting, because this probability will give the expectation of that right. So then, because when you when you interested in the expectation of X – expectation will be for each value what is the probability? Probability that, because 1 either 0 or 1 right; this indicator random variables are…So therefore, will we can put probability of H dash union H double dash is a subset of G right this is what will come.

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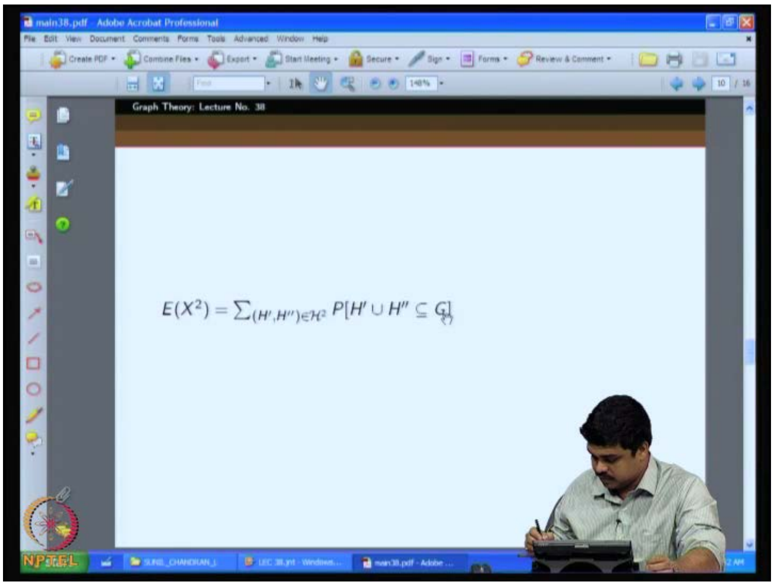

So then lets go back here, so that is what we are return here; expectation of X squares is essentially overall pairs H dash and H double dash from H square, we under some of the probability that H dash union H double dash is sub set of G.

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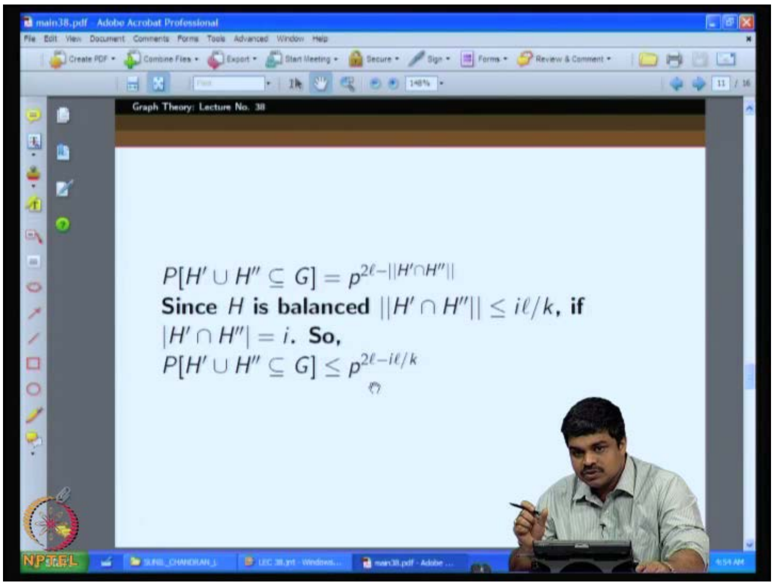

Now, what is this probability? Defiantly, because if they have a disjoint there are l plus l 2 l edges will be there, but then there are there are common edges here. So, we have to minus of that right. P 2 the power 2 l minus H dash intersection H double dash. The common edges this is this is the common edges; So, it should be clear in, because we are just looking at how many edges are there in this H dash union H double dash; that exactly that is 2 l minus number of edges in the intersection of number of edges common to H dash and H double dash.

Now, we know we have to get a value for this thing, some we want to put a upper bound for this thing; and there is easy, because H dash intersection is H double dash is the defiantly sub graph of H. So that is it edge density is defiantly less than that of H, that means less than equal to the that of H; that means, epsilon is at most l by k, because l by k is the edge density right. Now, we can if you assume that H dash intersection H dash, that is I vertices in it in that intersection. Then the number of edges is less then equal to i l by k i l by k. So this is a subset of G is less than equal to P to the power 2 l minus i l by k; that is what we will we will get right.

So, just that we estimated it, we got a upper bound for this thing and substitute that is i l by k here not l by k is essentially parameters of H, l b is the number of edges k b in the edge density - here the edge density of this thing at most that, and I we are assuming the number of vertices in the intersection I right. So, it can be 0 to k, it can be full, it can be non right; no vertices are its same same thing; same set of vertices. So, both are possible. So, zero to k is this possible values of i, now we should understand that depending on the pair we selected H dash from the H double dash - is intersection the cardinality of the number of vertices in the inter section will be different.

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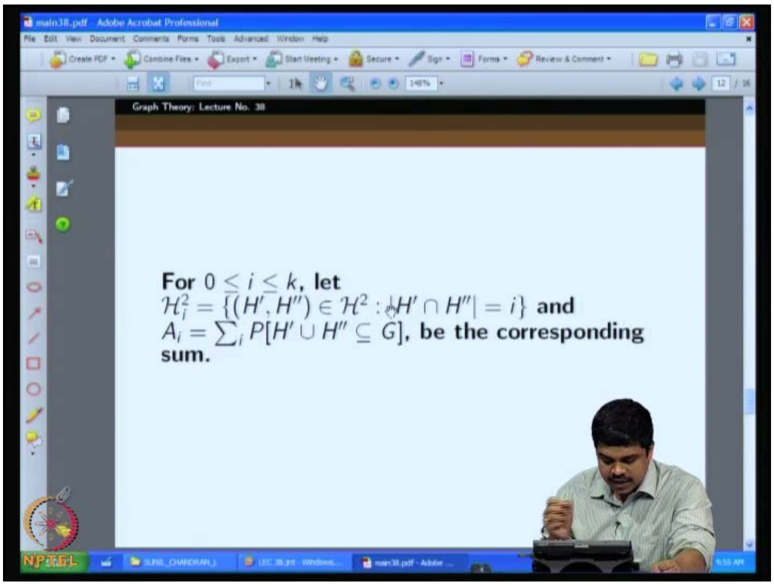

So, let as categorized them according to that. So far i is 0 less than i equal to i less than equal to k, let H i square, we are putting a subset of H. So H dash comma H double dash element of H square, such that there are intersection is equal to i. And the corresponding some of this probability, that means this am writing A i for the some, such that H dash union H double dash subset of G, the probability of that for such pairs where H dash and H double dash intersection is equal to i such pairs. We are finding the sum. Now for i equal to, now how do we find it out, this we are interested in this sum, because if I sum a 0 plus a 1 plus a 2 plus up to a k, we are done.

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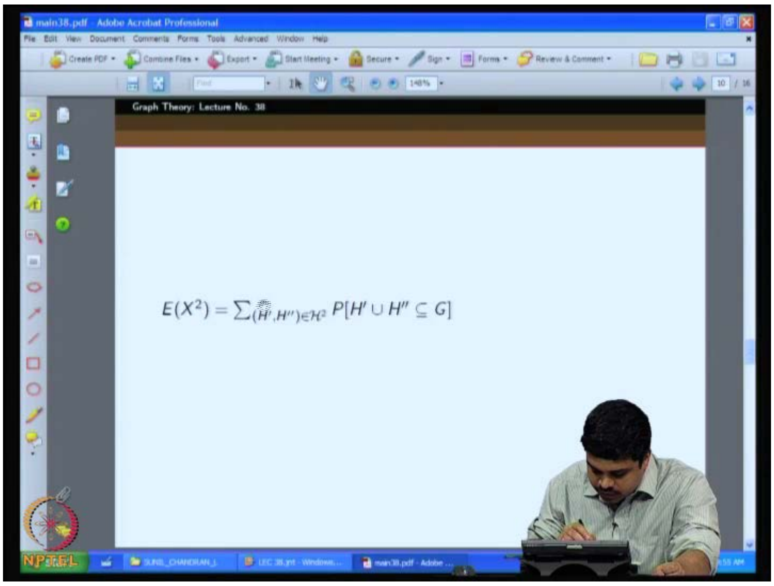

We are got we are looking for we got the expectation right, this this sum. This is what we to find out? Instated of directly summing this - we are summing different subs subs subs sums here, infact instated of summing over all the pairs directly, we will we put the pairs in several groups depending on their value of their intersection, and then we are summing over all such sets right. And this is what we want to do now which is balanced, so now what we do? So, so now we want to we want to find the sum.

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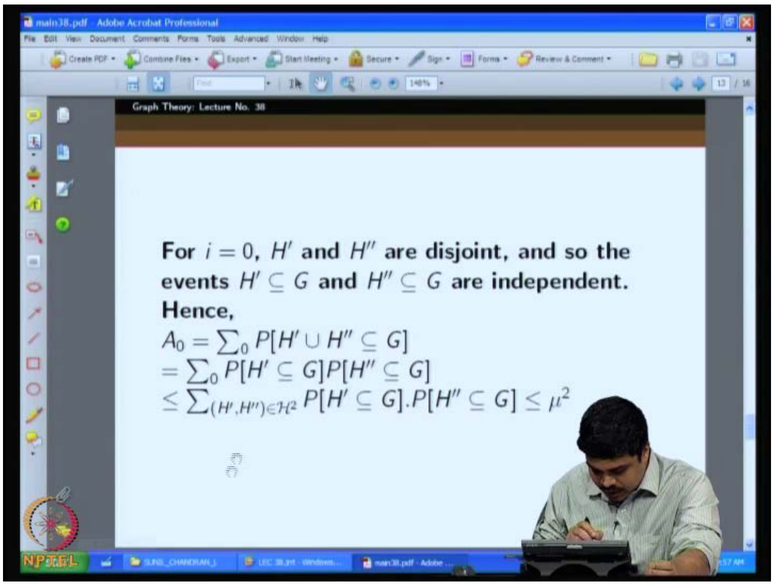

First we start with i equal to 0, that means A 0 - A 0 is particularly easy, because in this case H dash and H double dash does not have, we have one vertex in the intersection, which means there are in independent right, there are independent. So, if there independent, then there is nothing much to worry, the probability of this H dash union H double dash is G is essentially the probability that H dash there in G, and H dash double dash is there in G.

And there are independent, we can multiply probabilities together. And then our all the H dash H dash H double dash pairs such that their intersection is 0, we have to do this summation. But then we can do a little more than that, we sum it our all H dash double dash pair. It will only increased a little bit, but it is only a is an upper bound. So, we we take all pairs H dash and H double dash from H square, and then this just sum this thing.

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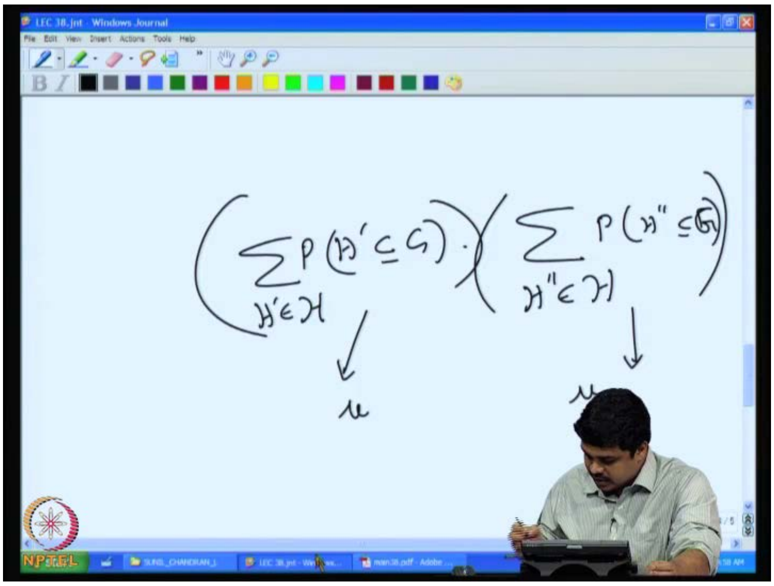

This can be a little more than looking what we have looking for, but its not problem, because see for instance then this what will happen this will turn out be, the probability that H dash element of H (( )) probability that H dash sub set of G, and so into H double dash subset of H probability that H double dash subset of G. So this is of case, what will you get this mu, and this also mu right. Because this is essentially that definition of mu, because for each H here this will give probability into 1, here in 0 into the corresponding probability would giving the expectation right, this this when you some over all the edges this will give mu this will give mu give mu.

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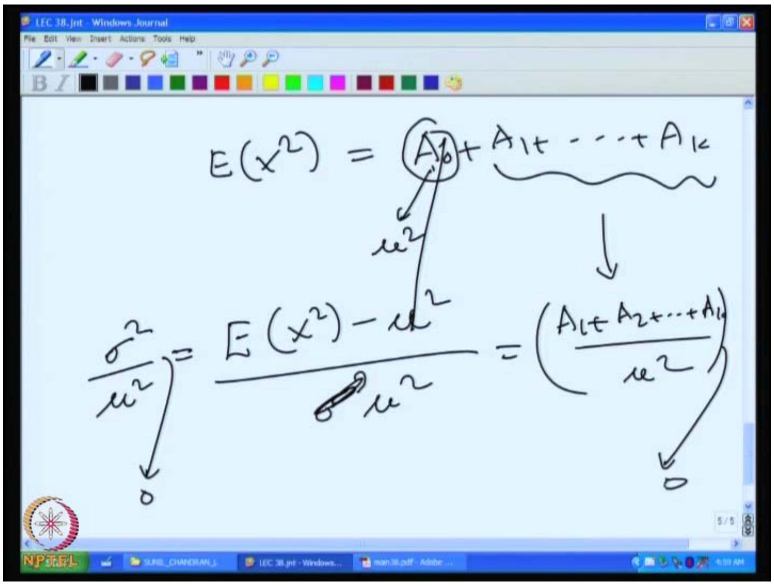

So therefore, what we get now is that this is less than equal to mu square right, (( )) zero less than equal to mu square, if x 0 is less than equal to mu square the good thing is when we when we want finally, the expectation of X square, we know that this is A 0 plus A 1 plus up to A k, when we already told this is less than equal to mu square, that means we can supplies replace it by mu square. So then, this will be upper an bound here; then finally, we want to find sigma square by mu square which is essentially expectation of x square minus mu square by sigma square sorry by mu square right.

Here what will happens, so here this mu square will go way; so this mu square and this mu square will go way. So, we will be left with things divided mu square mu square that is A 1 plus A 2 plus up to A k divided by mu square will be what will what will you look. So, if you want to show that these turns to 0 which (( )) these turns to 0. How will you show that, show the both of them turns to 0.

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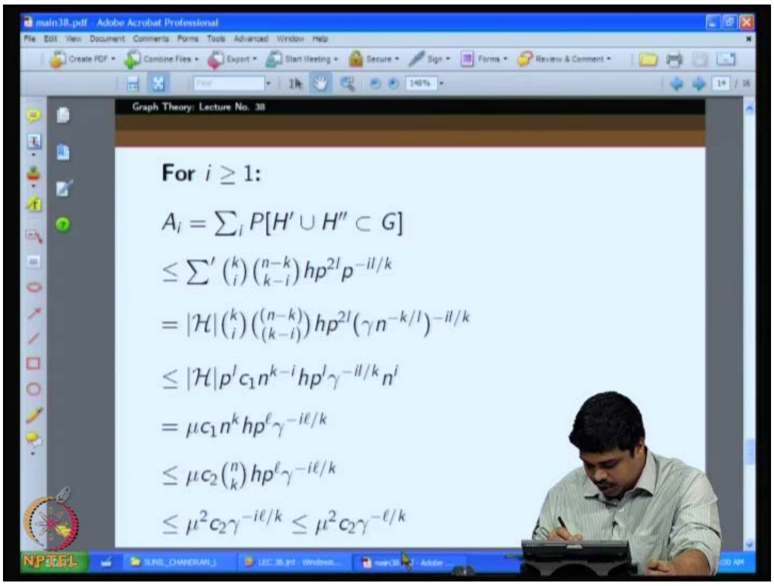

So the as well as take some A i for some i greater than equal to 1. So, this is what we want (( )) H dash in a H double dash subset of equal to G, we have to some over all. We know first, let us take one parameter out that is over all edge, so what will you say we will fix 1 x dash, and then we will some overall such x dash right fixing fixing 1 H.

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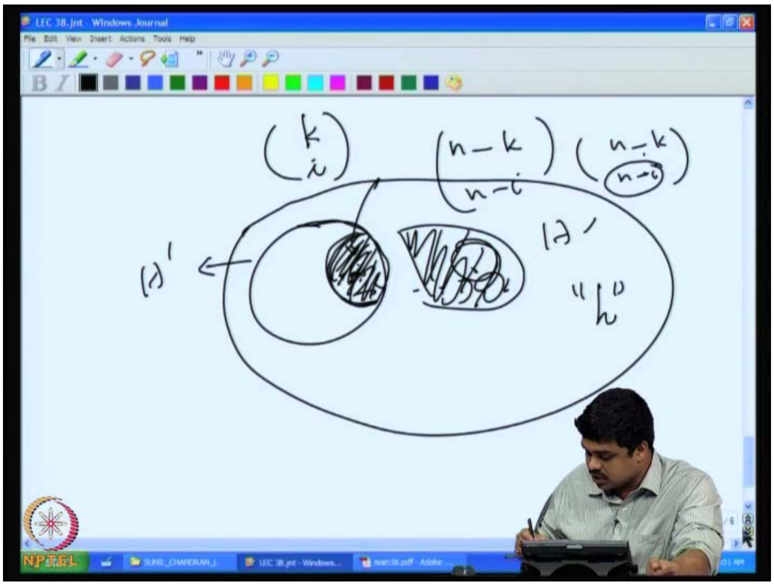

Now then once you fix it, because if k vertex - there are k vertices in it. So what we can do is. Once you fix k vertices, this is k vertices. We can take, because intersection is high, we can take any i set. So there are k choose i ways of selecting i’s this is H dash -

from H dash we can select a collection of i vertices k choose in k choose in i ways. And now there are n minus k vertices outside is it not, n minus k vertices outside. From n minus k vertices, we have to select the remaining n minus i - n minus k choose n minus i vertices. So, along with this an this if we put this, then we will get a subset, we can say that that is H dash. Suppose if I if I collect it here, H dash this and this, together will make a H dash.

In such that and also the intersection will be will be indeed i right, and also of course, that once we select this subset we can get H on it in different ways, say let say in a H ways right; so the total can be taken as k choose i into n minus k choose k minus i H, and this probability it has we know P raise to n minus i l by k right. Because there are that we have already estimate that this is the probability, but we have to some over all these collections first - this some corresponds to fixing of an H, and this should some should run through all possible H dash in H, and then once you fix that this is the number of ways, these three terms k choose i into n minus k choose k n minus i into H is the number of ways in which we can select H double dash. Such that there is a intersection of i vertices with the fixed H, H dash. Because first we can select i vertices from k, then the remaining k minus i can be select from the outside n minus k vertices, and this H is the number of ways in which you can get isomorphic copy of H, in that vertices that can we know that at most k factorial this H right. P to the power 2 2 l minus i here just split in to rot here. The 2 l minus i l by k right.

This is probability of this, because there are so many edges it has to be already seen. Now, because this this summing can be just replaced for every H. So therefore, there are so many H's. So, we cardinality of H will be surprised placed. Then here this is k choose i and n minus k choose k n k minus i into H into P to the power 12 into gamma. So, here P can be again replace by gamma into n raise to minus k by l into minus i l by k right. So here, as you show we can see that the when we canceled this here, this k will go and then we have l also we go way, and we will just have plus i here n raise to come from here gamma raise to minus i l by k will come from here.

And now from this side if you look - H into 1 out of this P raise to 2 l, we can give 1 P raise to l to this H into P raise to l will already become a mu that is expected value, because P raise to l is the probability of particular H to occur. So therefore, H into P l is mu is 1 mu and then why did i put c 1 here - c 1 was, because k choose i is a only of function of k at most we can say get to the power k of something. It want (( )) therefore, just placing by constant. And here, we just substituting by upper bound here - n raise to k minus l upper bound, when I combined n power n minus k n raise to k minus I, and n raise to i will get n power k here.

So, here we have H into P to the power l into gamma raise to minus i l by k here. Now, you see the, but then we now we can you know n raise to k is not much more than n choose k as a as a it is only a function of k more. So, we can take that function out here and joint with c 1 and make this constant c 2. c 1 is replaced by a constant c 2 now, contains k of k it is a it is a function of k, when I say constant it does not we only mean that it does not depend on n mu into c 2, and n choose k into H into P raise to l into gamma raise to minus. And this n choose k into H is already the cardinally of H once again, and P raise to l multiplied we will get this 3 terms together, we will get mu once again. So that becomes mu square into c 2 into gamma raise to i l by k. And here gamma this is minus i l by k gamma being greater than 1 now, because gamma is trending to infinity. So we can discard that i, we can just write minus i minus l by k right. Because so it is only making it a little bigger, so but then gamma tends to infinity. So this gamma raise to minus l by k will tend to 0 then right; this entire thing will can to 0, let we see.

So this is the reason why our sigma square by mu square tends to 0, sigma square by mu square tends to 0 n sigma square by mu square tends to 0 as would imply as we have seen that the probability that x greater than 0 tends to 1. And therefore, we will have with probability tending to 1, the copy of H in G - this vertex is. This is application, how we apply the second moment method. So with this thing we will case this second moment method idea, we will before winding up the probabilistic method and the random graphs - we will consider one more interesting tool from this from this topic. So namely the Lovasz Local Lemma.

So I will just very quickly go through this, because it is a we useful tool the interested students - so though and may not be able to make it to very clear here, the interested students is advice to read up this material, because this is very powerful and use full tool. So, now see how do we understand this thing. So, so here lets a look at this situation. So, usually as we had seen earlier what we want is to avoid to say that the certain structures occur; so that means certain events occur, that mean this events occur and then we this events can be interpreted as the complement of several bad events, that means the intersection of the compliment several bad events.

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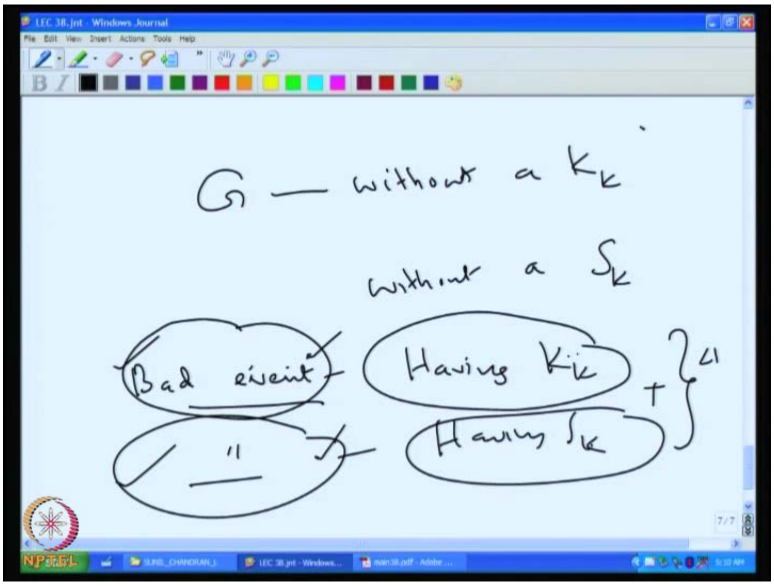

For instance in our first example of the ram c problem. We wanted a graph without a K k having a K k was a bad event here and without a S k independent set on k that means the bad event 1 bad event we wanted a avoid was having a K k is a sub graph and another bad event was a having a S k is a sub graph S k is a sub graph right, and what we did their was estimated the probability of having a this bad event and we estimated the probability of this bad event, and summed up and showed that when the number of vertices is so certain below certain function of K then, we still have after summing up we still have this probability together will be less than 1 therefore, there is a probability that neither this bad event no this bad event occur.

So there is a probability that we will have a good event I mean good event means not bad in either way right so, the in the good event will be the intersection of the compliment of this and this intersection of the complement this 2 bad events right that is what we did then, the problem they was that our tool was very week, because we found the probability of this this bad event we found the probability of this bad event, and summed up and we have to show that that is sum is less than 1.

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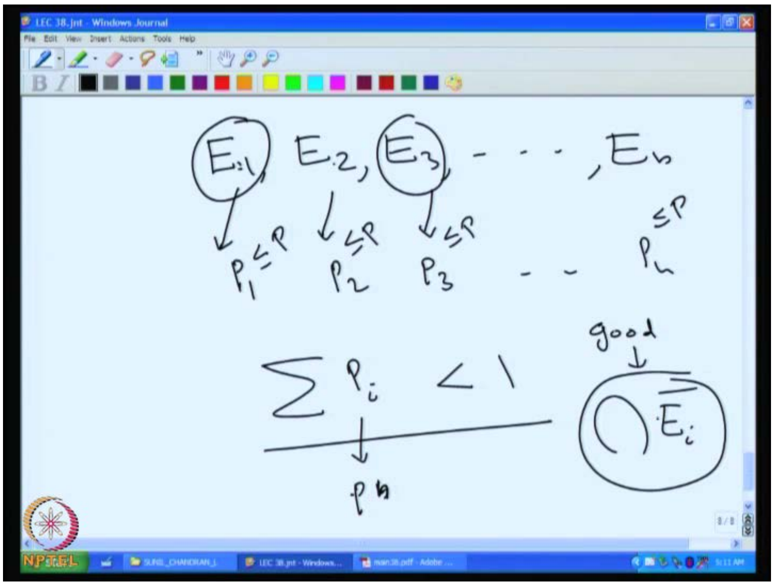

So, but what if this because there are just 2 event may see that what is the problem but suppose there are several bad events say E 1, E 2, E 3, and so there are n bad events right. Now, suppose the probability of this thing is P 1 the probability of this thing is P 2, and a probability of this thing is P 3, and so on. This is so, we will have to sum up these P i's and show that; this is still less than 1 for the good event to happen good event being good event is define to be something. In the intersection are the compliment of all these E i's that means, so should not be bad in the even way should not be bad in the E 2 way, it should not to be bad in the E 3 way, and it should not be bad in the E n way right.

The good means in the intersection of the complements of all these things right. So that is we want so, this may not be so, easy because they are to multiply for instance if this all these things are up upper bound by certain P; and then, this will be P n right at most P n. So, if n is big it may so happen that P n is already to large, so when their several events to handle so this may not be very feasible this approach. So, but still in many situations the number of events may be large, but it may look like this event - and this event are many of this events are mutually independent independent of each other. That means, won influence each other if suppose, you are all mutually independent the problem would be very easy mutually when I say mutually independent.

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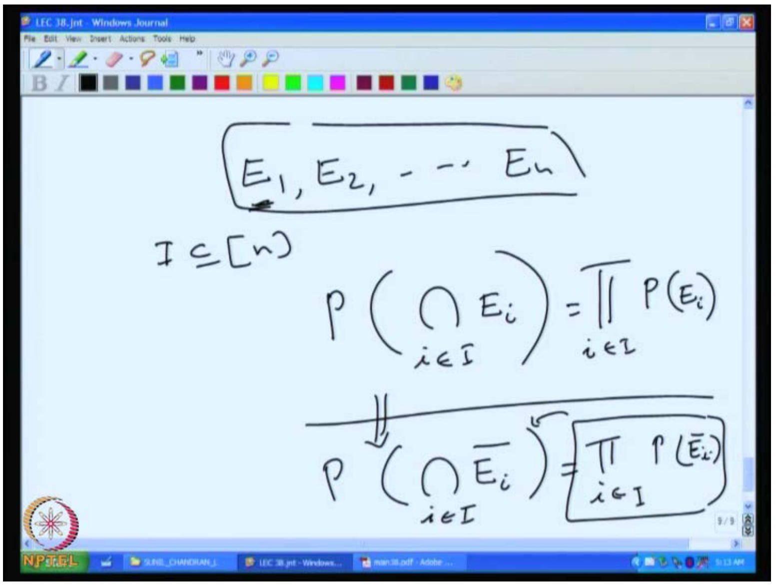

What do you mean by seen when I say E 1, E 2, E n are mutually independent events which means, that the for any subset so it take any subset I of n right. Then if I find the probability of the intersection of all these events right that will be the product of the corresponding probabilities, that is what you mean right, for any subsets not just for entire thing for any subset saw element of I right. So now, so and if this happens it is easy to show that the complements also will be mutually independent. That means, if you are interested in this event right for any subset 5 that will be again the product of the corresponding probabilities right i element of I right, for the this is the probability the complement event.

And you see, if each of this bad events had probabilities strictly (( )) that has to be for expecting that a good event - any event will be there which is not bad in that way. So, the probability of each of this event has to be less than 1, then each of these probabilities is greater than 0, and you take the product that will still be greater than 0. So therefore, if them mutually independent, then definitely we have a choice so the problem is when they are not mutually independent. And if they have to much dependents, we cannot do much, because we will have to some them up, but then there is this intermediate situation where it is not mutually independent, but still the dependency among this events is quite low so how do we capture this concept to capture this concept.

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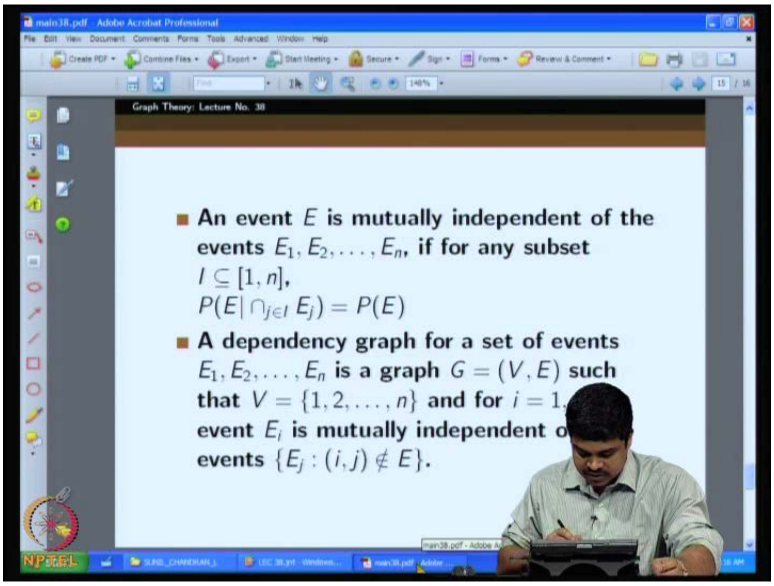

We introduce a concept called dependency graph. So, when to defined this dependency graph want to defined something called mutually when is an event? E is mutually independent of a set of events E 1, E 2, E n. So, we say that an event E is mutually independent of the events E 1, E 2, E n. If for any subset I subset of 1 to n or n; so, the probability that E occurs conditioned on that all these events in i occurred right, together occurred simultaneously occurred is again. It is not it is not the probability values not changed is condition probability same as P of E.

In that case, we will say that the they are is mutually independent of the events E 1, E 2, E n. Now, what is a dependency graph? We will draw a graph such that each of this events E 1, E 2, E n will become the vertex set. So will say vertex set is 1 to n, essentially E 1 will be the 1 of the vertices, E 2 will be the another vertex, E 3 will be another vertex, and E n will be n like that n vertex graph to the vertices capture the bad events.

And we will have to make for each vertex, we will have to make all those vertices neighbors. In such a way that neighbors should be defined in such a way that non neighbors. So, from starting looking from a certain event E i, it is non neighbors with respect to is non neighbors; it should be mutually independent that means for i equal to 1 to n. The event E i is mutually independent of the events E j, such there i, j is not an edge

(( )) the mutually independent of the non neighbors of it i, E i is mutually independent of the events corresponding to the non neighbors of i.

So slightly, because rather than saying that I look at to vertex 2 vertices i and j, and then if E i j, and E i and E j are mutually independent sorry independent then I I do not put edge between them or if E i and E j are dependent. Then I put an edge that is not a way it is defined, it defined that after constructing the graph. If you look at the non neighbors for a particular E i particular vertex I, the events corresponding to the non neighbors let it be certain collection. So our E i should be mutually independent of that right. So, so from problems that will be very clear, how it happens? So, not that we will keep we have to kept construct a (( )) such a way that it happens, but it will be a chromatic many questions so, if this happens then what have you same.

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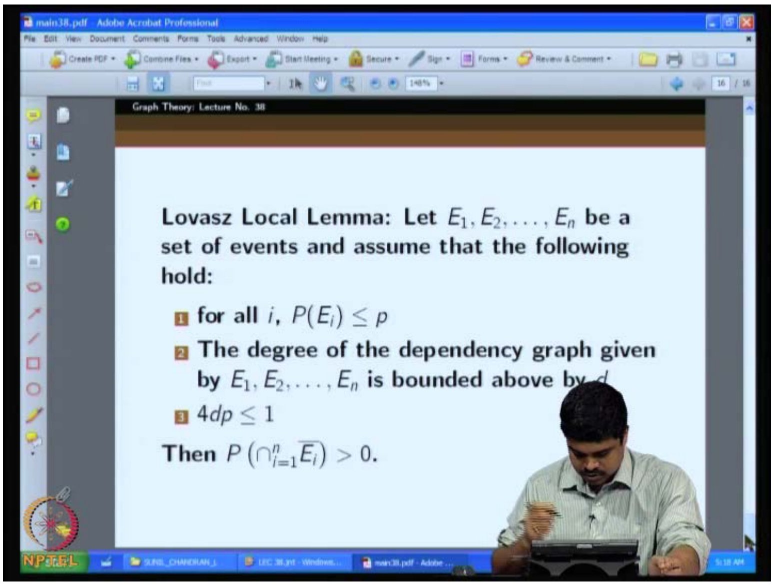

Then we say that we we have this 2 which is call Lovasz Local Lemma this is call the symmetric version of the Lovasz Local Lemma or the simple version that is a more complicated or general version a symmetric version. We want look at it, because we just want to mention this tool. So, that the students can be cup up that later. So, E 1, E 2, E n be the set of events this so called bad events, and assume that the following holds. So, each event the probability of this the corresponding bad event E i is at most p the degree of the dependency graph. After we construct the dependency graph as we described earlier. E 1, E 2, E n.

The dependency graph on this events is bounded by t. So, this is a bound on the degree of each 1 that means look at E i there are at least n minus 1 minus d events. Such that, this E i is mutually dependent of, and then 4 d into p d p in the degree of the dependency graph P being the upper bound for the probability of E i that should be total let be less than equal to 1, then the probability that i equal to 1 to n E i bar will be greater than then that means, we have a non 0 probability for the good event to happen - good event means it is in the compliment of the intersection of all the bad events.

That means it is not bad in any of the this ways E 1 way or E 2 way or E n way. It is not bad right. So the probability that will be a probability therefore, there should be some point in the sample space which will satisfy a property. So, looking for a property of that type right, this is what it says so now quickly to illustrate the point we will we will take up small question here the question is this.

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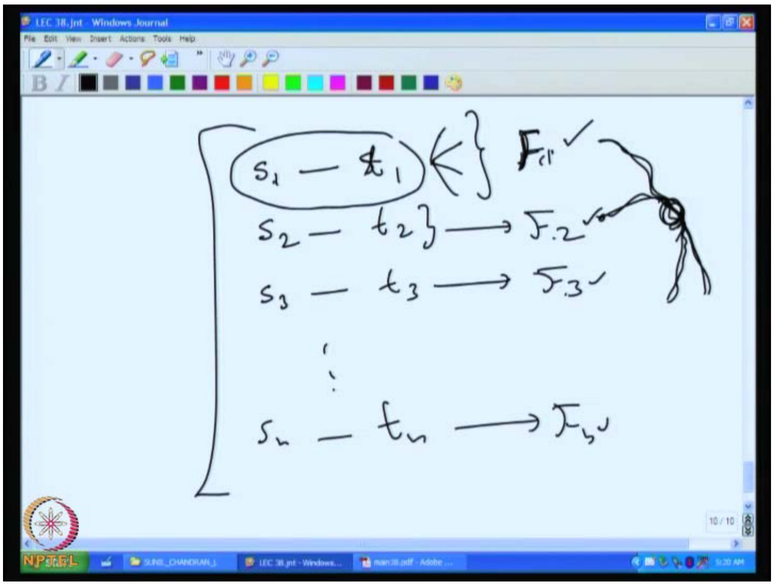

Now, we have so this I hope this simple example will illustrate the point. So, suppose we have this in a graph in a network we have several source destination pairs S 1, t 1; so S 2, t 2, S 3, t 3 and we have s n t n. So, many source destination pairs, and the our intension is to give a communicating path for them that means, that should be a path from S 1 to t 1. The only this we have a constrain due to some practical reasons, that they should pick up their paths from a set of given paths which is called a f 1 f 1 right. Similarly, they should pick up there, a given path from a set of given paths from f 2, and here from f 3 and this 1 f n.

So, each of this will contains several paths. We can pick one from so they have they known have more choices any any 1 from this they can pick, but the problem is if you this is the path here another path here, may intersect one some edges of this thing, but they can be some edge intersection that is not allowed, because we are not suppose to you, take if I take path here for S 1 t 1. Then I should not path take a path S 2 t 2, such that it goes some touches this thing. So, we want to take disjoint path for each of them, but it the way f 1, f 2, f 3 are given they have this kind of intersection. So, the question is it possible to find, is it possible to find the paths in such a way that there all disjoint. So, what we are going to do here is that, suppose we have a condition that any path in f 5.

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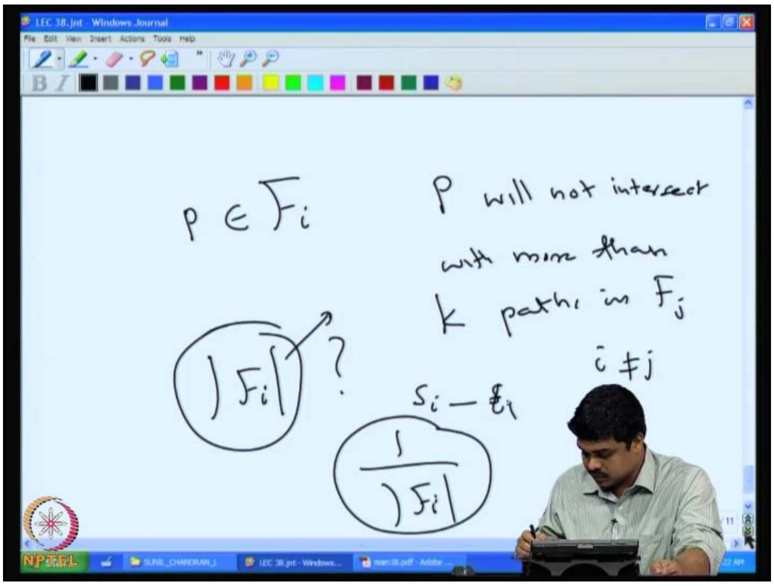

If it take a path from f 5, some path P from f i then we are grantee that P will not intersect P will not intersect with more than k paths. In another set say f j (( )) i not equal to j here right. In this case, we can pick up with the, pick up paths individual paths for S i t i. Such that, the disjoint for each S i, t i we get a disjoint path, non intersecting paths, non touching paths. So, we need the condition for defiantly we can see that the question will be the estimate, it how big should be f 5 for each f 5 right, because each f 5 are very small then it may not be possible, because if there are 1 path to selecting each f 5.

And we will have to select that only there is no grantee that, they not touching each other right. So, we should have some choice they should be certain number of parts in f 5, then only we will able to do that how big should be. So the idea is to select let say, we will fix it in the end; we will give equal probability for each path in f 5 for when I want to say I will select a path for S i t i, we will we will not give any preference to any path in it, each of them will be given equal probability, and randomly we will select one. So, that means each path will give the probability of 1 by cardinality of f 5 right. So, we know that that will be probability with will select it right.

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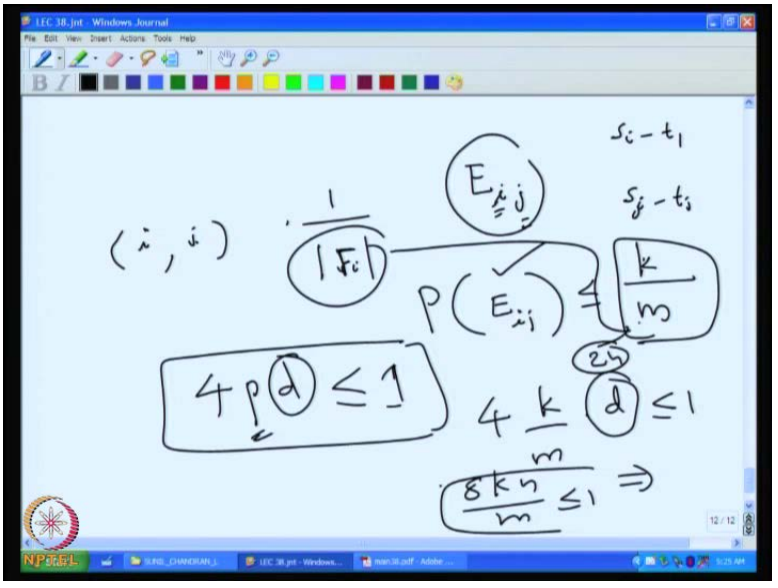

1 by cardinality of f 5 will be the probability with which we select the the path. So, now the the question is this event bad event is what we will define a bad event E i j, where E i j means the path selected for the i th pair S i t i intersect with the path selected for the j th pair s j t j. What is the probability - that probability of E i j then definitely that is less then equal to, because one path is selected for j and this path will intersect with at most k path in j. Therefore, it is k by m. If you say m is the at least m paths are there in this right, this is so let say right k by n.

Now, we know what we need is 4 p times d should be less than that is what we want to right. So, 4 p d should be less than equal to 1 so that is for the Lovasz Lemma less than equal to 1 if this happens where d is d is the degree of the dependency graph d is (( )) this number that is 4 into k by m into d is less than equal to 1. Then we know that we we can get the required property that means none of this E i j's happen, that means none of the path for none of the pairs i, j the path selected for i will intersect with the path selected for j. So, now what should be the degree of the dependency graph, it is so you know the dependency graph, because 1 event degree will be definitely only 2 n right, because we know when when will the dependency is come for this i, and j either i or j should be in that right.

So we can substitute by 2 n here. So, I I if you not following, because we do not have time. So, the students can think about it find out the 2 n. So, we get if 8 k n by m is less than equal to 1, this will happen. That means m is greater than at least 8 k, this can happen. So, so we we will start with another topic in the next class. If this is not clear the student can think about it, thank you.